Wavelets and Filter Banks on Graphs

Pierre Vandergheynst

Signal Processing Lab, EPFL

Joint work with David Shuman

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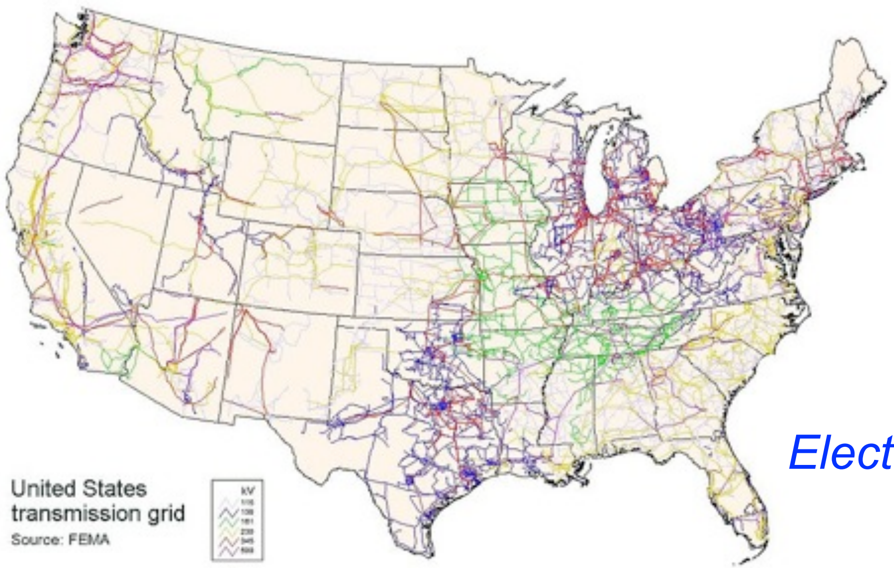
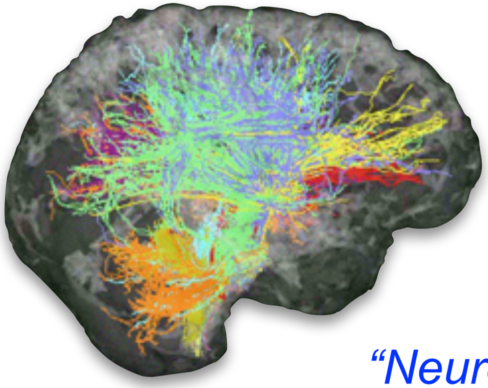
Processing Signals on Graphs

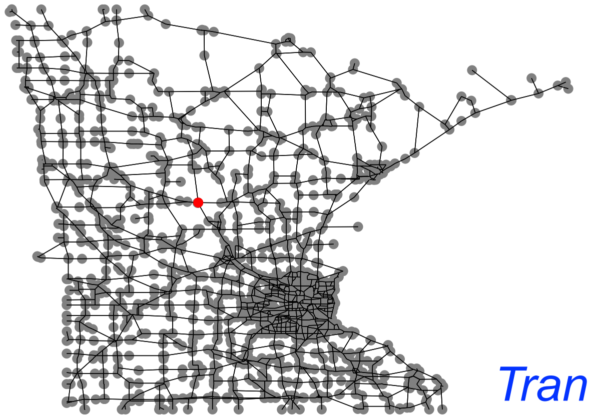

United States transmission grid
Source: FEMA

Electrical Network

Social Network



"Neuronal" Network



Transportation Network

Short outline

- Summary of one wavelet construction on graphs
    - multiscale, filtering, sparsity, implementation
- Pyramidal algorithms
    - polyphase components and downsampling
    - the Laplacian Pyramid
    - 2-channels, critically sampled filter banks ?

Spectral Graph Wavelets

Remember good old Euclidean case:

$$(T^s f)(x) = \frac{1}{2\pi} \int e^{i\omega x} \hat{\psi}^*(s\omega) \hat{f}(\omega) d\omega$$

$$(T^s \delta_a)(x) = \frac{1}{s} \psi^* \left(\frac{x-a}{s} \right)$$

Spectral Graph Wavelets

$G=(E,V)$ a weighted undirected graph, with Laplacian $\mathcal{L} = D - A$

Dilation operates through operator: $T_g^t = g(t\mathcal{L})$

Translation (localization):

Define $\psi_{t,j} = T_g^t \delta_j$ response to a delta at vertex j

$$\psi_{t,j}(i) = \sum_{\ell=0}^{N-1} g(t\lambda_\ell)\phi_\ell^*(j)\phi_\ell(i) \qquad \mathcal{L}\phi_\ell(j) = \lambda_\ell \phi_\ell(j)$$

$$\psi_{t,a}(u) = \int_{\mathbb{R}} d\omega\, \hat{\psi}(t\omega) e^{-j\omega a} e^{j\omega u}$$

And so formally define the graph wavelet coefficients of f:

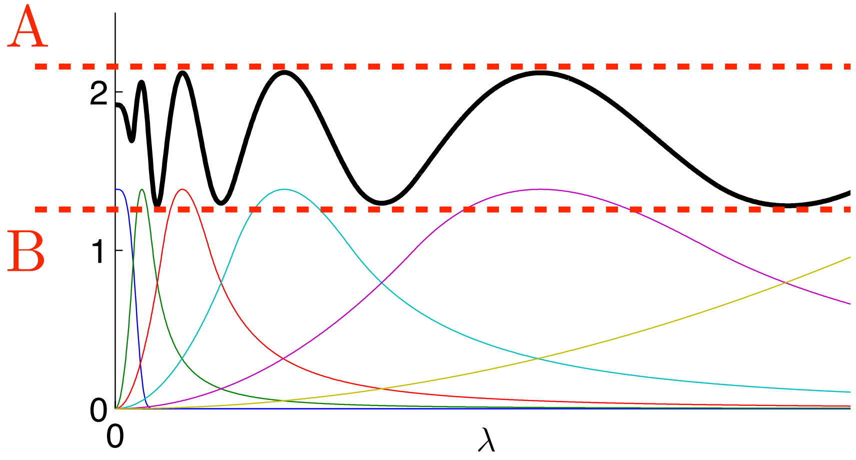
$$W_f(t,j) = \langle \psi_{t,j}, f \rangle \qquad W_f(t,j) = T_g^t f(j) = \sum_{\ell=0}^{N-1} g(t\lambda_\ell)\hat{f}(\ell)\phi_\ell(j)$$

Frames

$\exists\, A, B > O,\ \exists h : \mathbb{R}_+ \to \mathbb{R}_+$ (i.e. scaling function)

$$0 < A \leqslant \underbrace{h^2(u)}_{\text{scaling function}} + \sum_s \underbrace{g(t_s u)^2}_{\text{wavelets}} \leqslant B < \infty$$

$\phi_n = T_h \delta_n = h(\mathcal{L})\delta_n$



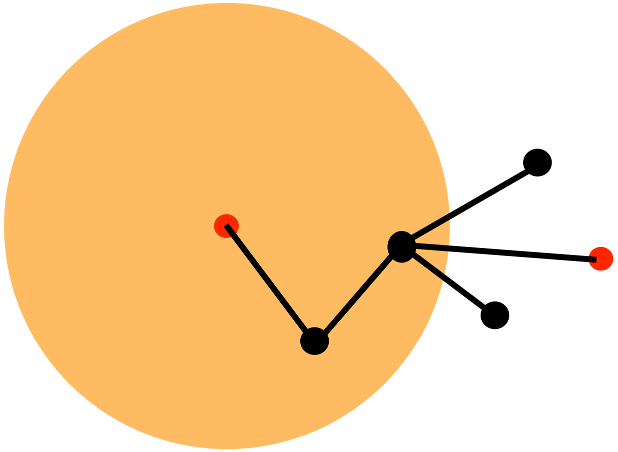
A simple way to get a tight frame:

$$\gamma(\lambda_\ell) = \int_{1/2}^{1} \frac{dt}{t}\, g^2(t\lambda_\ell) \Longrightarrow \tilde{g}(\lambda_\ell) = \sqrt{\gamma(\lambda_\ell) - \gamma(2\lambda_\ell)}$$

for any admissible kernel g

Scaling & Localization

Effect of operator dilation ?

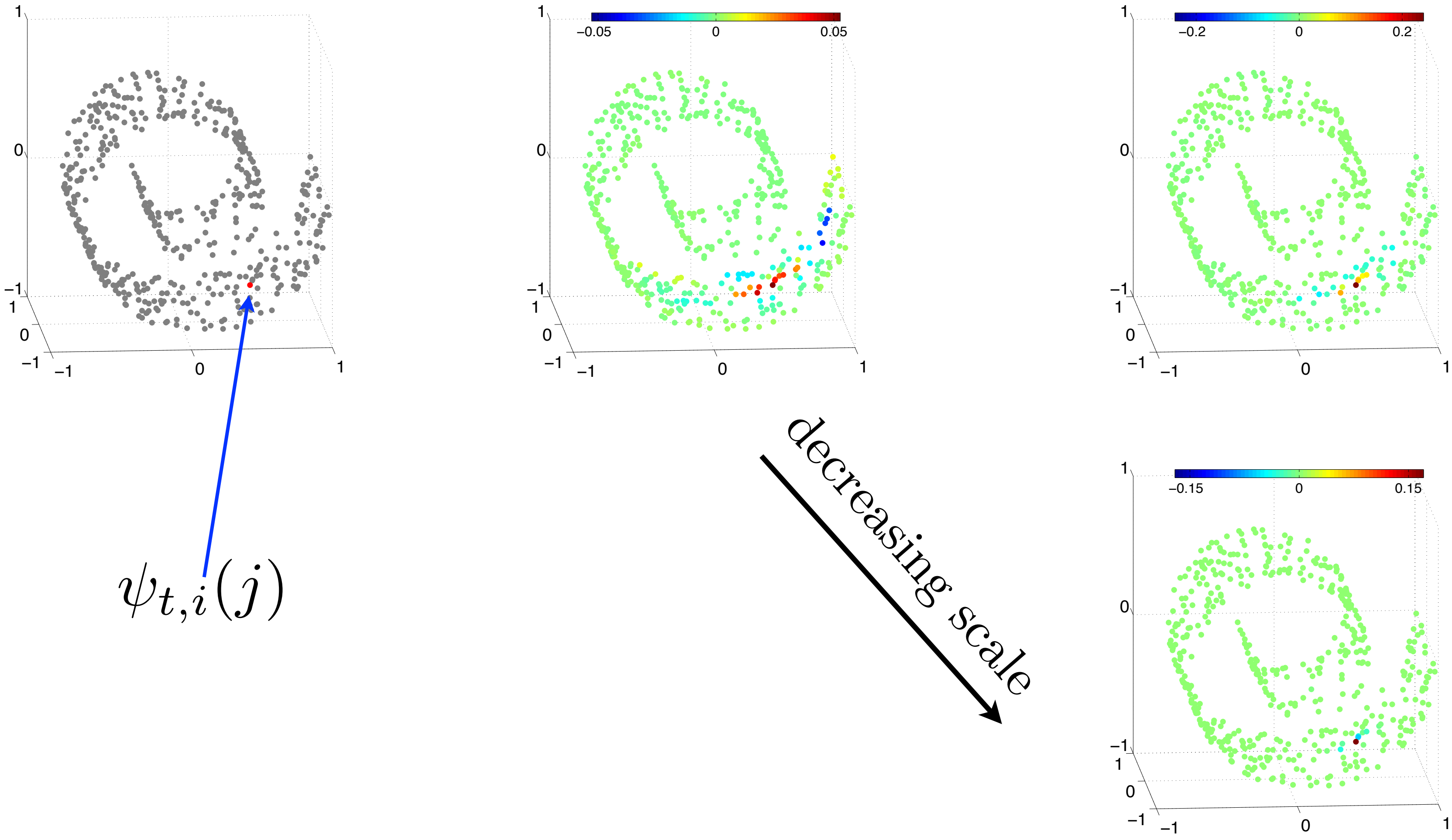


Theorem: $d_G(i,j) > K$ and g has K vanishing derivatives at *0*

$$\frac{\psi_{t,j}(i)}{\|\psi_{t,j}\|} \leq Dt$$ for any t smaller than a critical scale function of $d_G(i,j)$

Reason ? At small scale, wavelet operator behaves like power of Laplacian

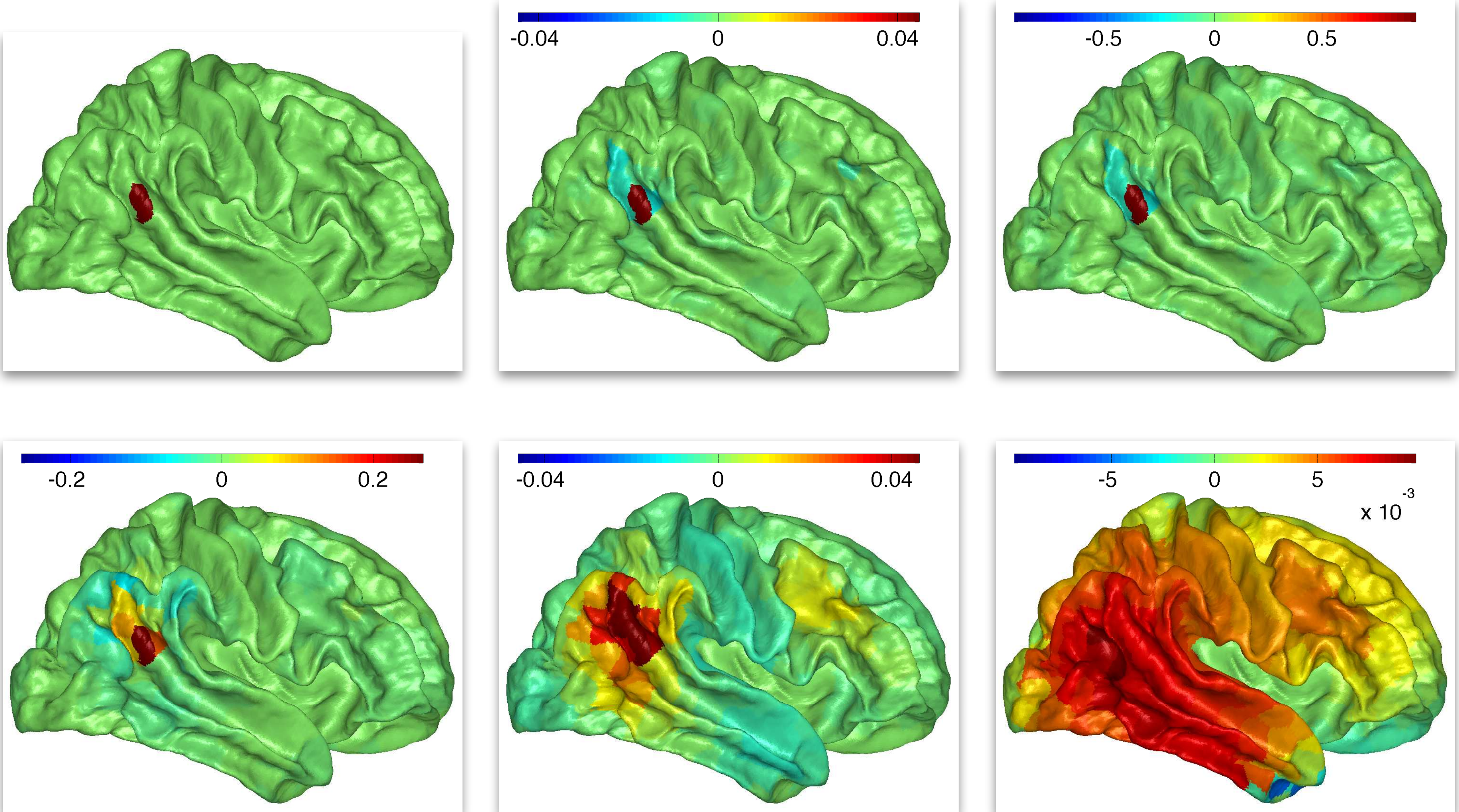
Scaling & Localization



$\psi_{t,i}(j)$

decreasing scale

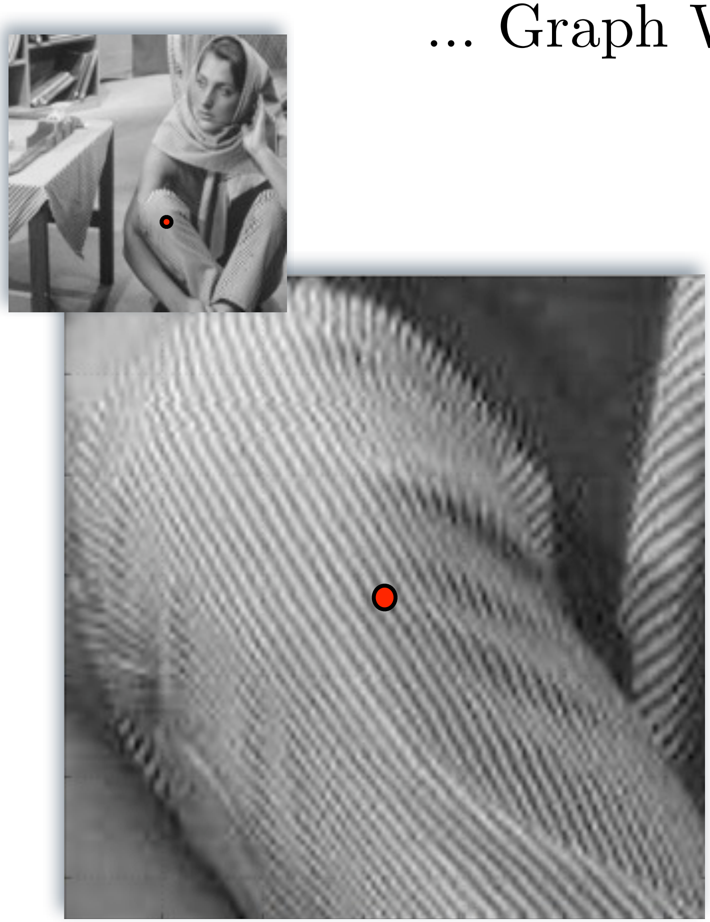
Example

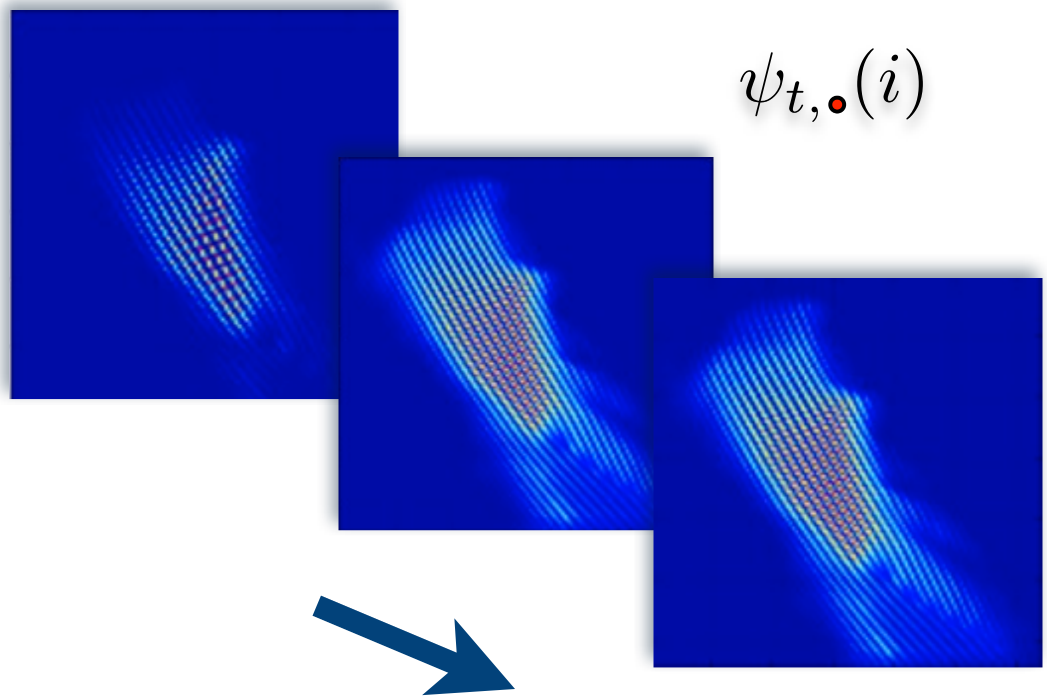


Non-local Wavelet Frame

- Non-local Wavelets are ...

... Graph Wavelets on Non-Local Graph





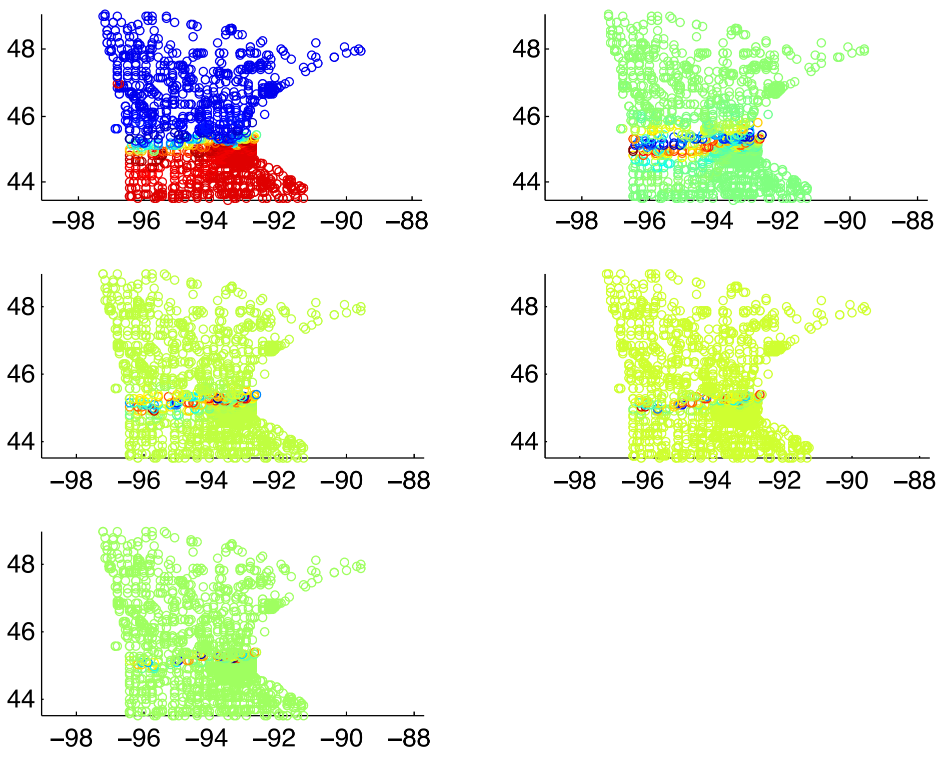
$\psi_{t,\bullet}(i)$

increasing scale

Interest: good *adaptive* sparsity basis

Sparsity and Smoothness on Graphs

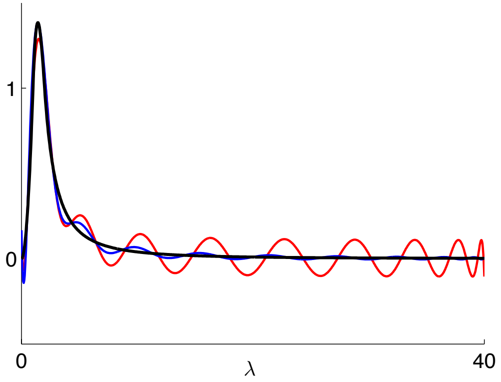
scaling functions coeffs



Remark on Implementation

Not necessary to compute spectral decomposition for filtering

Polynomial approximation : $g(t\omega) \simeq \sum_{k=0}^{K-1} a_k(t) p_k(\omega)$



ex: Chebyshev, minimax

Then wavelet operator expressed with powers of Laplacian:

$$T_g^t \simeq \sum_{k=0}^{K-1} a_k(t) \mathcal{L}^k$$

And use sparsity of Laplacian in an iterative way

Remark on Implementation

$$\tilde{W}_f(t,j) = \left(p(\mathcal{L})f^{\#}\right)_j \qquad |W_f(t,j) - \tilde{W}_f(t,j)| \le B\|f\|$$

sup norm control (minimax or Chebyshef)

$$\tilde{W}_f(t_n,j) = \left(\frac{1}{2}c_{n,0}f^{\#} + \sum_{k=1}^{M_n} c_{n,k}\overline{T}_k(\mathcal{L})f^{\#}\right)_j$$

$$\overline{T}_k(\mathcal{L})f = \frac{2}{a_1}(\mathcal{L} - a_2 I)\left(\overline{T}_{k-1}(\mathcal{L})f\right) - \overline{T}_{k-2}(\mathcal{L})f$$

Computational cost dominated by matrix-vector multiply with (sparse) Laplacian matrix.

In particular $O(\sum_{n=1} M_n|E|)$

http://wiki.epfl.ch/sgwt

Note: “same” algorithm for adjoint !

Distributed Computation

Scenario: Network of N nodes, each knows

- local data f(n)
- local neighbors
- M Chebyshev coefficients of wavelet kernel
- A global upper bound on largest eigenvalue of graph laplacian

To compute: $\left(\tilde{\boldsymbol{\Phi}} f\right)_{(j-1)N+n} = \left(\frac{1}{2} c_{j,0} f + \sum_{k=1}^{M} c_{j,k} \overline{T}_k(\mathcal{L}) f\right)_n$

$$\left(\overline{T}_1(\mathcal{L}) f\right)_n = \left(\frac{2}{\alpha}(\mathcal{L} - \alpha I) f\right)_n$$

sensor only needs f(n) from its neighbors

$$\left(\overline{T}_k(\mathcal{L}) f\right) = \frac{2}{\alpha}(\mathcal{L} - \alpha I)\left(\overline{T}_{k-1}(\mathcal{L}) f\right) - \overline{T}_{k-2}(\mathcal{L}) f$$

Computed by exchanging last computed values

Distributed Computation

Communication cost: 2M|E| messages of length 1 per node

Example: distributed denoising, or distributed regression, with Lasso

$$\arg\min_a \frac{1}{2}\|y - \mathbf{\Phi}^* a\|_2^2 + \|a\|_{1,\mu}$$

$$a_i^{(k)} = \mathcal{S}_{\mu_i,\tau}\left(\left[a^{k-1} + \tau\mathbf{\Phi}(y - \mathbf{\Phi}^* a^{k-1})\right]_i\right)$$

$$\mathcal{S}_{\mu_i\tau}(z) := \begin{cases} 0 & , \text{ if } \mid z \mid\leq \mu_i\tau \\ z - \mathrm{sgn}(z)\mu_i\tau & , \text{ o.w.} \end{cases}$$

Total communication cost:

Distributed Lasso [Mateos, Bazerque, Gianakis] Cost $\sim |E|N$

Chebyshev $\mathbf{\Phi} y$ 2M|E| messages of length 1

$\mathbf{\Phi\Phi}^* a$ 4M|E| messages of length J+1

Cost $\sim |E|$

Graph wavelets

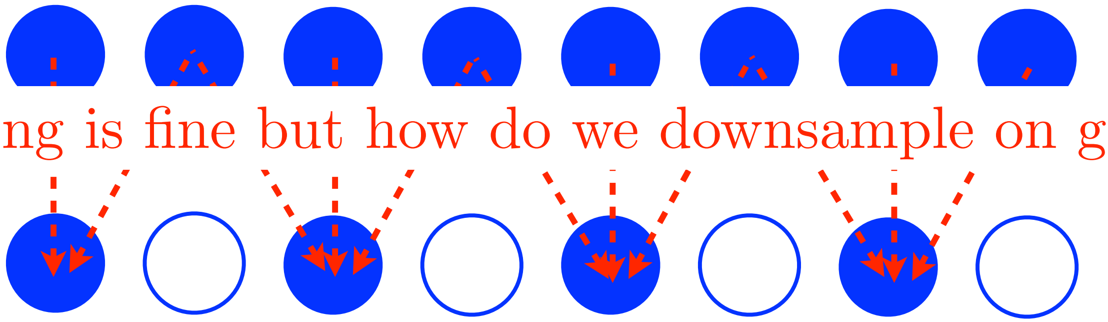
- Redundancy breaks sparsity
  - can we remove some or all of it ?
- Faster algorithms
  - traditional wavelets have fast filter banks implementation
  - whatever scale, you use the same filters
  - here: large scales -> more computations
- Goal: solve both problems at one

Basic Ingredients

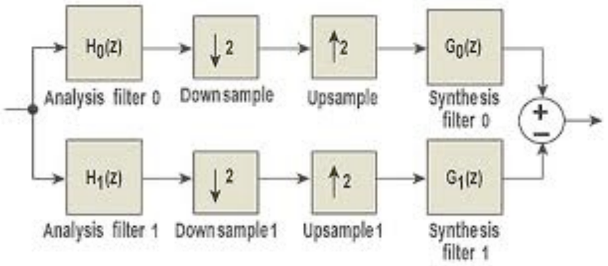
Euclidean multiresolution is based on two main operations

Filtering (typically low-pass and high-pass)

Down and Up sampling

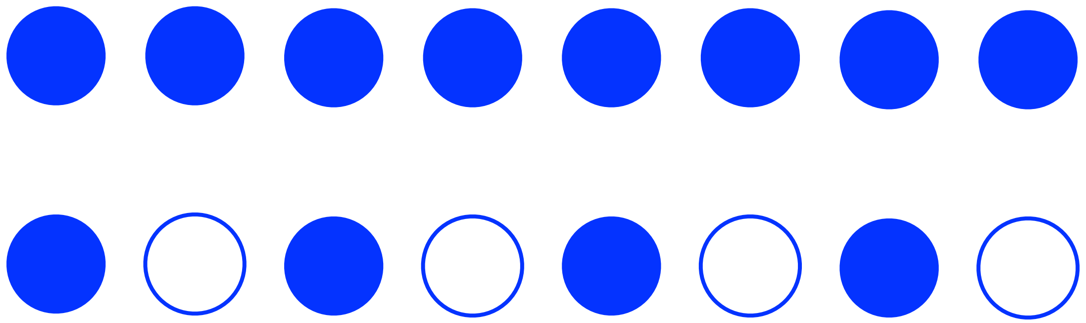


Filtering is fine but how do we downsample on graphs ???



Basic Ingredients

Subsampling is equivalent to splitting in two cosets (even, odd)



Questions: How do we partition a graph into meaningful cosets ?

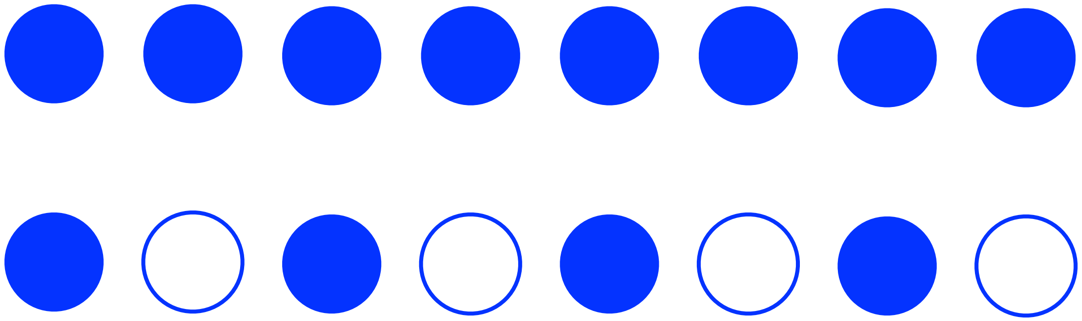
Are there efficient algorithms for these partitions ?

Are there theoretical guarantees ?

How do we define a new graph from the cosets ?

Cosets - A spectral view

Subsampling is equivalent to splitting in two cosets (even, odd)



Classically, selecting a coset can be interpreted easily in Fourier:

$$f_{\mathrm{sub}}(i) = \frac{1}{2} f(i) \big(1 + \cos(\pi i)\big)$$

eigenvector of
largest eigenvalue

Cosets and Nodal Domains

Nodal domain: maximally connected subgraph s.t. all vertices have same sign w.r.t a reference function

We would like to find a very large number of nodal domains, ideally $|V|$!

Nodal domains of Laplacian eigenvectors are special (and well studied)

Theorem: the number of nodal domains associated to the largest laplacian eigenvector of a connected graph is maximal,

$$\nu(\phi_{\max}) = \nu(G) = |V|$$

IFF G is bipartite

In general: $\nu(G) = |V| - \chi(G) + 2$ (extreme cases: bipartite and complete graphs)

Cosets and Nodal Domains

Nodal domain: maximally connected subgraph s.t. all vertices have same sign w.r.t a reference function

We would like to find a very large number of nodal domains, ideally $|V|$!

Nodal domains of Laplacian eigenvectors are special (and well studied)

For any connected graph we will thus naturally define cosets and their associated selection functions

$$V_+ = \{i \in V \text{ s.t. } \phi_{N-1}(i) \geq 0\} \qquad V_- = \{i \in V \text{ s.t. } \phi_{N-1}(i) < 0\}$$

$$M_+(i) = \frac{1}{2}\big(1 + \operatorname{sgn}(\phi_{N-1}(i))\big) \qquad M_-(i) = \frac{1}{2}\big(1 - \operatorname{sgn}(\phi_{N-1}(i))\big)$$

Examples of cosets

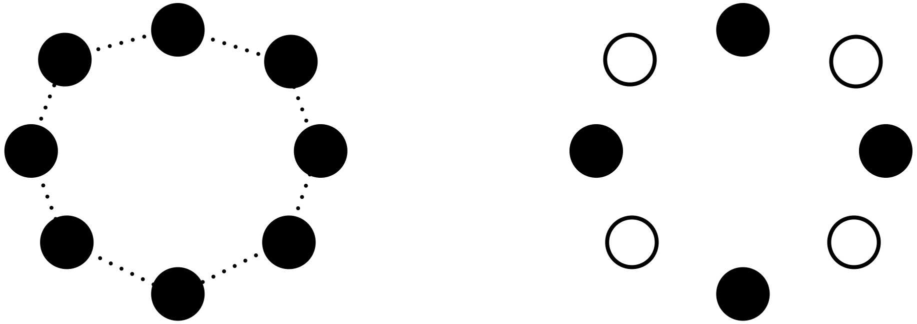
Simple line graph

$$\phi_k(u) = \sin(\pi k u/n + \pi/2n) \qquad \lambda_k = 2 - 2\cos(\pi k/n) \qquad 1 \leq k \leq n$$

Examples of cosets

Simple line graph

Simple ring graph



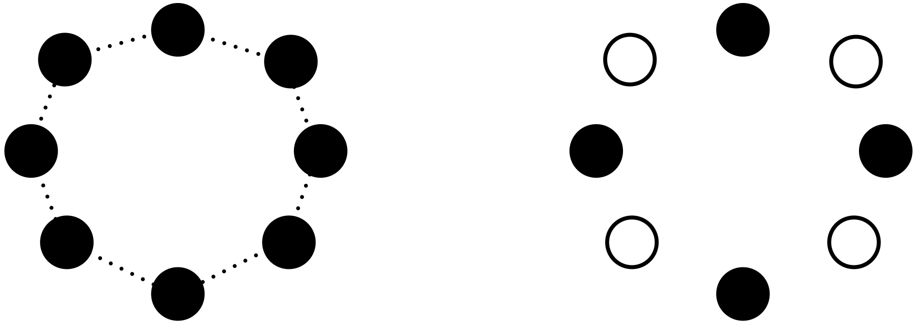
$$\phi_k^1(u) = \sin(2\pi k u/n) \qquad \phi_k^2(u) = \cos(2\pi k u/n) \quad 1 \leq k \leq n/2$$

$$\lambda_k = 2 - 2\cos(2\pi k/n)$$

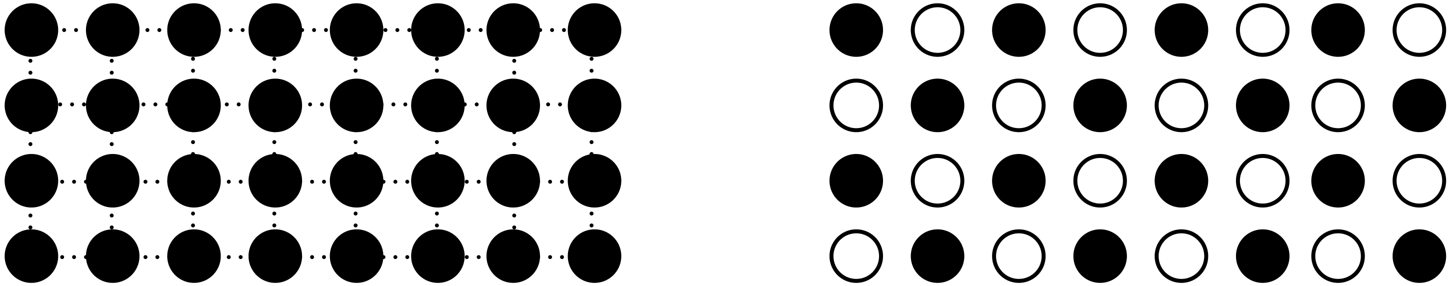
Examples of cosets

Simple line graph

Simple ring graph



Lattice



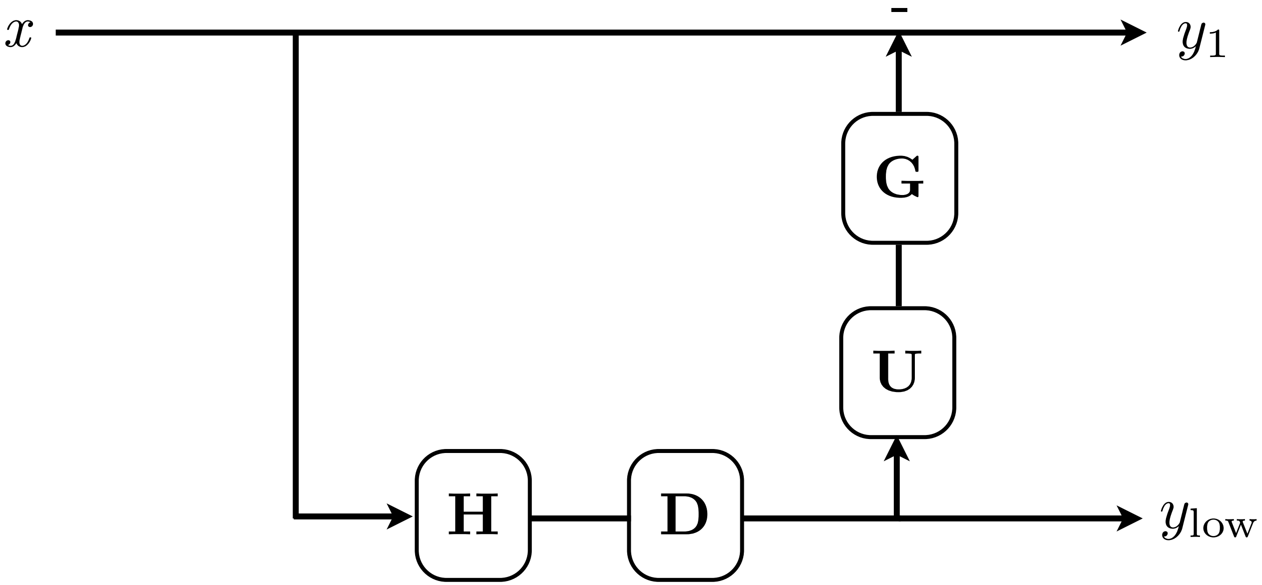
quincunx

The Agonizing Limits of Intuition

- Multiplicity of $\lambda_{\max}$
  - how do we choose the control vector in that subspace ?
  - even a prescription can be numerically ill-defined
  - graphs with “flat” spectrum in close to their spectral radius
- Laplacian eigenvectors do not always behave like global oscillations
  - seems to be true for random perturbations of simple graphs
  - true even for a class of trees [Saito2011]

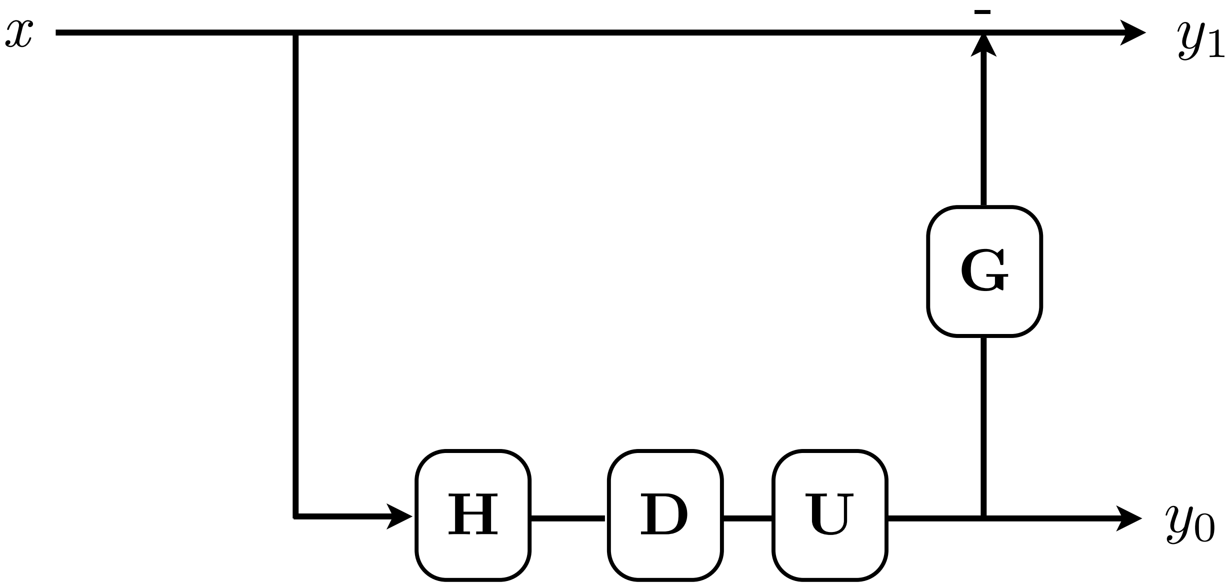
The Laplacian Pyramid

Analysis operator



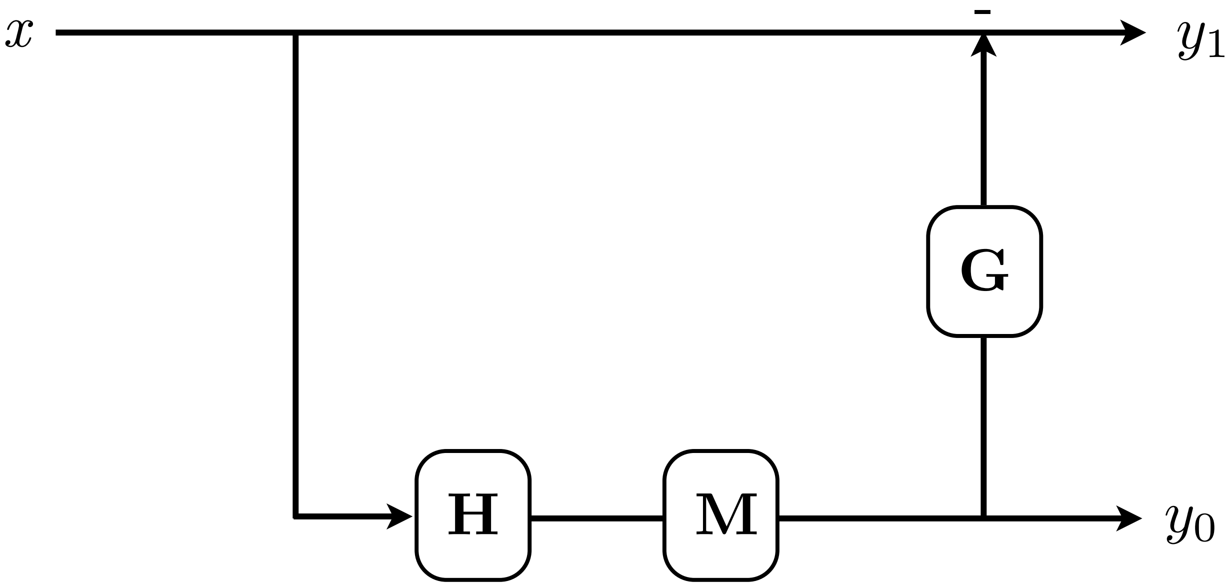
The Laplacian Pyramid

Analysis operator



The Laplacian Pyramid

Analysis operator



$$\begin{aligned} y_0 &= \mathbf{H_m} x \\ &= \mathbf{MH} x \end{aligned} \qquad \begin{aligned} y_1 &= x - \mathbf{G} y_0 \\ &= x - \mathbf{GH_m} x \end{aligned}$$

$$\underbrace{\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} \mathbf{H_m} \\ \mathbf{I} - \mathbf{GH_m} \end{pmatrix}}_{\mathbf{T_a}} x,$$

The Laplacian Pyramid

Analysis operator

$$\underbrace{\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} \mathbf{H_m} \\ \mathbf{I} - \mathbf{G}\mathbf{H_m} \end{pmatrix}}_{\mathbf{T_a}} x,$$

Simple (traditional) left inverse

$$\hat{x} = \underbrace{\begin{pmatrix} \mathbf{G} & \mathbf{I} \end{pmatrix}}_{\mathbf{T_s}} \underbrace{\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}}_{y}$$

$\mathbf{T_s}\mathbf{T_a} = \mathbf{I}$ with no conditions on $\mathbf{H}$ or $\mathbf{G}$

The Laplacian Pyramid

Pseudo Inverse ?

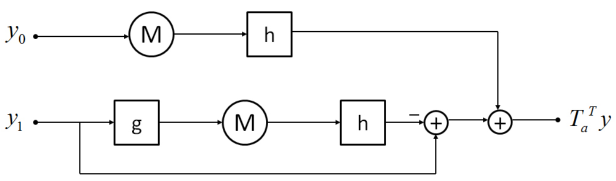
$$\mathbf{T_a}^\dagger = \left(\mathbf{T_a}^T\mathbf{T_a}\right)^{-1}\mathbf{T_a}^T$$

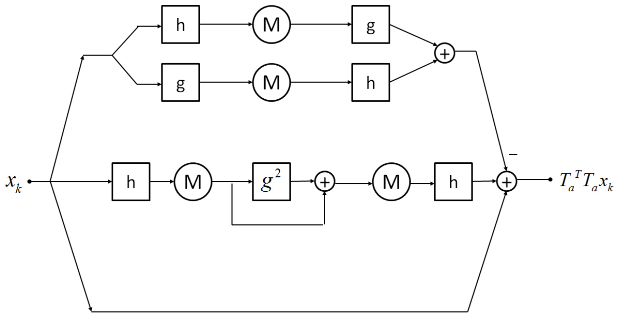
Let's try to use only filters

Define iteratively, through descent on LS:

$$\arg\min_x \|\mathbf{T_a}x - y\|_2^2 \xrightarrow{\qquad\qquad} \hat{x}_{k+1} = \hat{x}_k + \tau\mathbf{T_a}^T(y - \mathbf{T_a}\hat{x}_k)$$

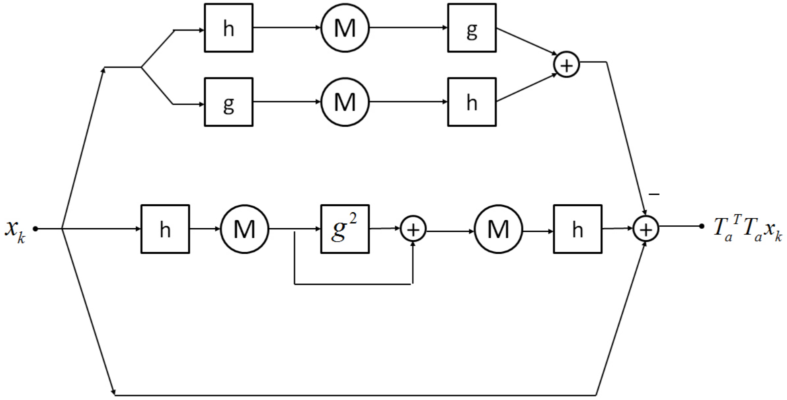
$$\mathbf{T_a}^T = (\mathbf{H_m}^T \quad \mathbf{I} - \mathbf{H_m}^T\mathbf{G}^T)$$





The Laplacian Pyramid

we can easily implement $\mathbf{T_a}^T\mathbf{T_a}$ with filters and masks:



With the real symmetric matrix $\mathbf{Q} = \mathbf{T_a}^T\mathbf{T_a}$ and $b = \mathbf{T_a}^T y$

$$x_N = \tau \sum_{j=0}^{N-1} (\mathbf{I} - \tau\mathbf{Q})^j b$$

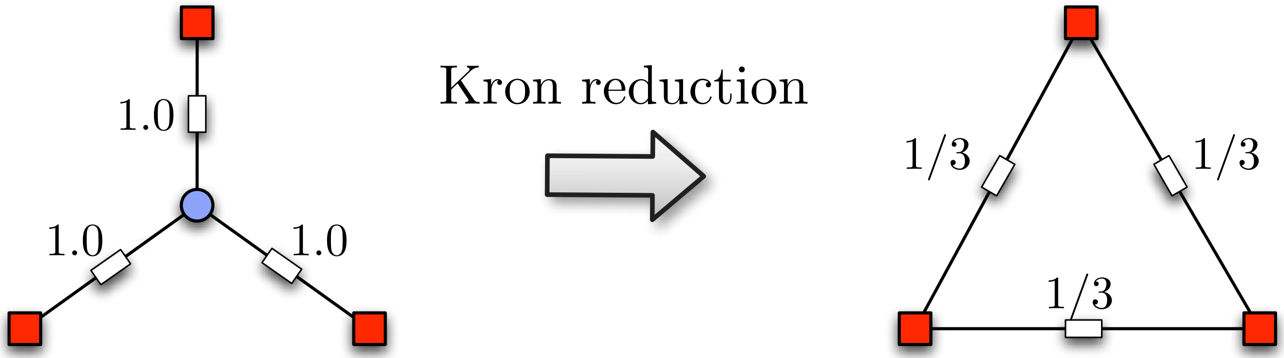
Use Chebyshev approximation of: $L(\omega) = \tau \sum_{j=0}^{N-1} (1 - \tau\omega)^j$

Kron Reduction

In order to iterate the construction, we need to construct a graph on the reduced vertex set.

$$\mathbf{A}_{\mathrm{r}} = \mathbf{A}[\alpha,\alpha] - \mathbf{A}[\alpha,\alpha)\mathbf{A}(\alpha,\alpha)^{-1}\mathbf{A}(\alpha,\alpha]$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}[\alpha,\alpha] & \mathbf{A}[\alpha,\alpha) \\ \mathbf{A}(\alpha,\alpha] & \mathbf{A}(\alpha,\alpha) \end{bmatrix}$$



Kron reduction

[Dorfler et al, 2011]

Kron Reduction

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$$\mathbf{A}_\mathrm{r} = \mathbf{A}[\alpha,\alpha] - \mathbf{A}[\alpha,\alpha)\mathbf{A}(\alpha,\alpha)^{-1}\mathbf{A}(\alpha,\alpha]$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}[\alpha,\alpha] & \mathbf{A}[\alpha,\alpha) \\ \mathbf{A}(\alpha,\alpha] & \mathbf{A}(\alpha,\alpha) \end{bmatrix}$$

Properties: maps a weighted undirected laplacian to a weighted undirected laplacian

spectral interlacing (spectrum does not degenerate)

$$\lambda_k(\mathbf{A}) \leq \lambda_k(\mathbf{A}_r) \leq \lambda_{k+n-|\alpha|}(\mathbf{A})$$

disconnected vertices linked in reduced graph IFF there is a path that runs only through eliminated nodes

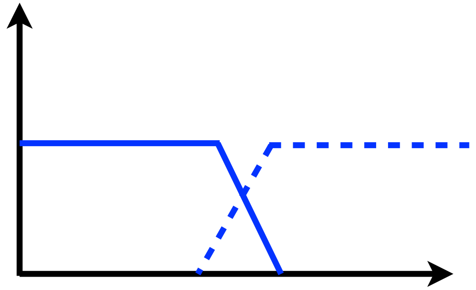
Example

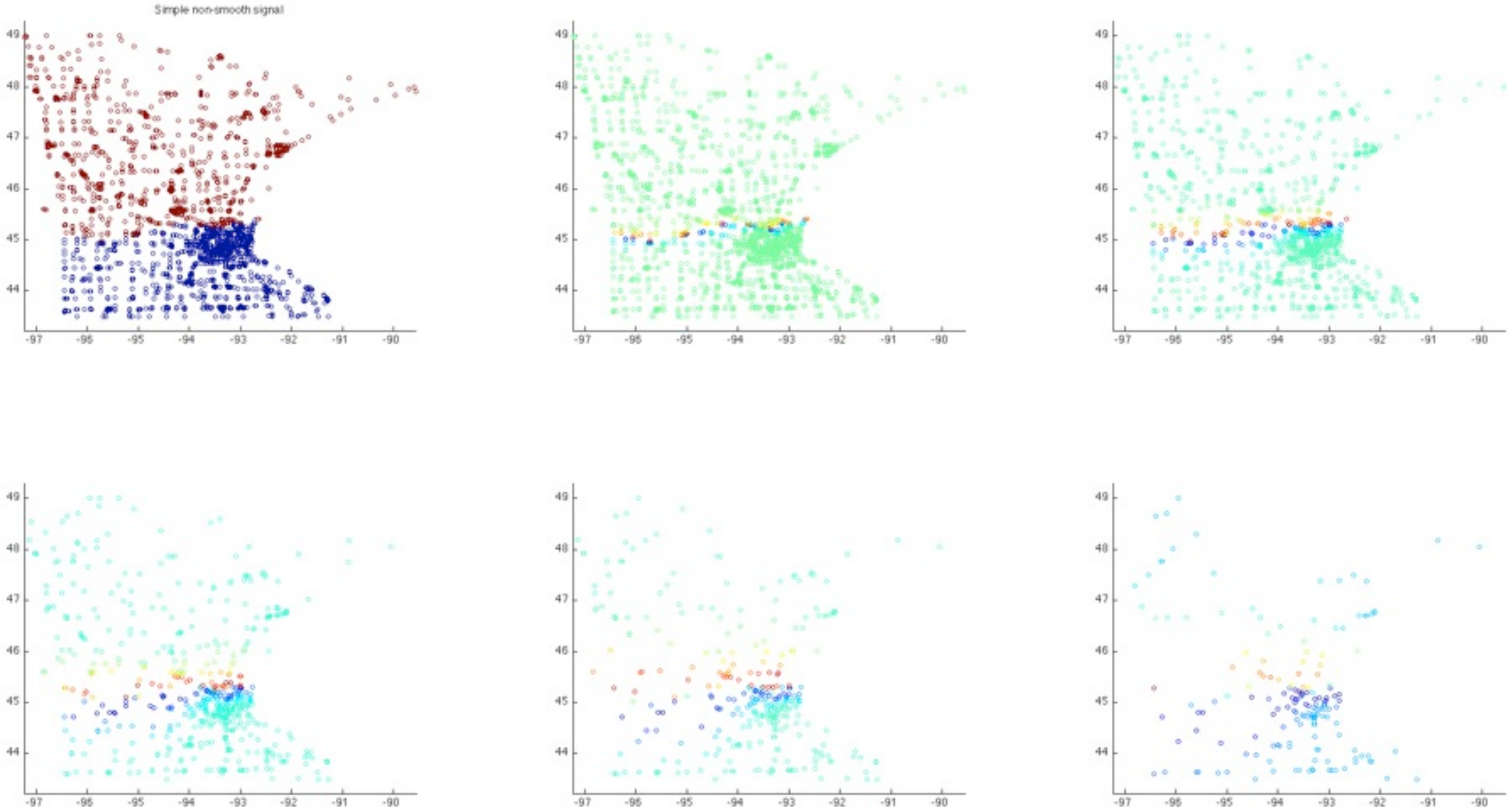
Note: For a k-regular bipartite graph

$$\mathbf{L} = \begin{bmatrix} k\mathbf{I}_n & -\mathbf{A} \\ -\mathbf{A}^T & k\mathbf{I}_n \end{bmatrix}$$

Kron-reduced Laplacian: $\mathbf{L}_r = k^2\mathbf{I}_n - \mathbf{A}\mathbf{A}^T$

$$\hat{f}_r(i) = \hat{f}(i) + \hat{f}(N-i) \quad i = 1, ..., N/2$$




Simple non-smooth signal
49
48
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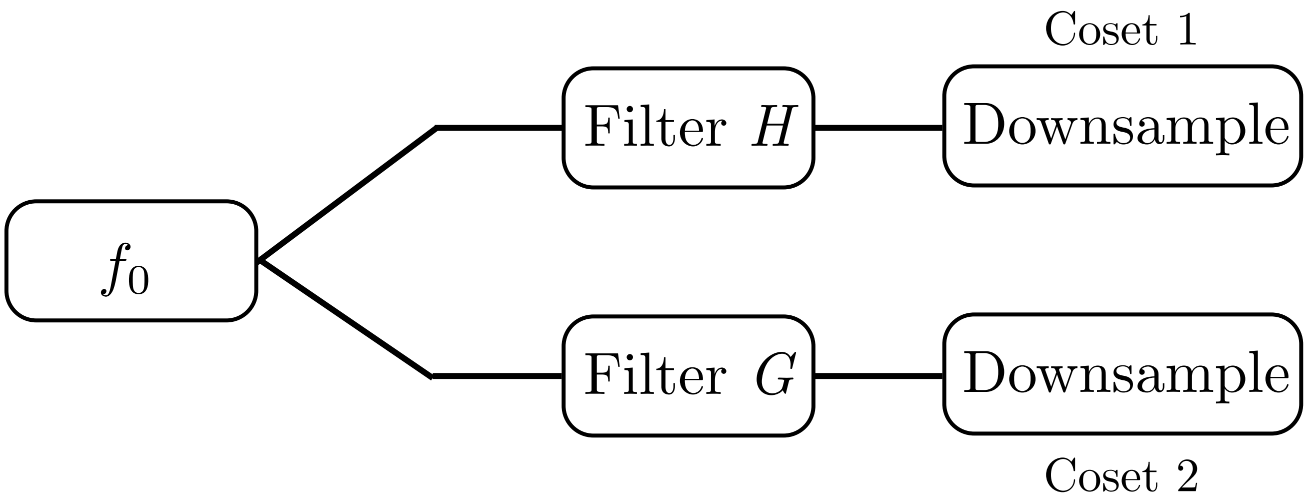
-90

Filter Banks

2 critically sampled channels



Theorem: For a k-RBG, the filter bank is perfect-reconstruction IFF

$$|H(i)|^2 + |G(i)|^2 = 2$$

$$H(i)G(N-i) + H(N-i)G(i) = 0$$

Conclusions

- Structured, data dependent dictionary of wavelets
  - sparsity and smoothness on graph are merged in simple and elegant fashion
  - fast algo, clean problem formulation
  - graph structure can be totally hidden in wavelets
- Filter banks based on nodal domains or coloring
  - Universal algo based on filtering and Kron reduction
  - Efficient IFF *some* structure in the graph
  - Unfortunately no closed form theory in general