A novel sampling theorem on the sphere with implications for compressive sampling

Jason McEwen

http://www.jasonmcewen.org/

Department of Physics and Astronomy
University College London (UCL)

BASP Frontiers 2011 :: Villars, Switzerland

Outline

Spherical harmonics

- Consider the space of square integrable functions on the sphere $\mathrm{L}^2(\mathrm{S}^2)$, with the inner product of $f, g \in \mathrm{L}^2(\mathrm{S}^2)$ defined by

$$\langle f,\, g \rangle = \int_{\mathrm{S}^2} \mathrm{d}\Omega(\theta,\varphi)\, f(\theta,\varphi)\, g^*(\theta,\varphi)\ ,$$

  where $\mathrm{d}\Omega(\theta,\varphi) = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ is the usual invariant measure on the sphere and (θ,φ) define spherical coordinates with colatitude $\theta \in [0,\pi]$ and longitude $\varphi \in [0, 2\pi)$. Complex conjugation is denoted by the superscript *.

- The scalar spherical harmonic functions form the canonical orthogonal basis for the space of $\mathrm{L}^2(\mathrm{S}^2)$ scalar functions on the sphere and are defined by

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, \mathrm{e}^{im\varphi}\ ,$$

  for natural $\ell \in \mathbb{N}$ and integer $m \in \mathbb{Z}$, $|m| \leq \ell$, where $P_\ell^m(x)$ are the associated Legendre functions.

- Eigenfunctions of the Laplacian on the sphere: $\Delta_{\mathrm{S}^2} Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$.

- Orthogonality relation: $\langle Y_{\ell m},\, Y_{\ell' m'} \rangle = \delta_{\ell\ell'}\delta_{mm'}$, where δ_{ij} is the Kronecker delta symbol.

- Completeness relation:

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y_{\ell m}^*(\theta',\varphi') = \delta(\cos\theta - \cos\theta')\, \delta(\varphi-\varphi')\ ,$$

  where $\delta(x)$ is the Dirac delta function.

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Spherical harmonic transform

- Any square integrable scalar function on the sphere $f \in \mathrm{L}^2(\mathrm{S}^2)$ may be represented by its spherical harmonic expansion:

$$f(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m}\, Y_{\ell m}(\theta, \varphi) \; .$$

- The spherical harmonic coefficients are given by the usual projection onto each basis function:

$$f_{\ell m} = \langle f,\, Y_{\ell m} \rangle = \int_{\mathrm{S}^2} \mathrm{d}\Omega(\theta, \varphi)\, f(\theta, \varphi)\, Y_{\ell m}^*(\theta, \varphi) \; .$$

- We consider signals on the sphere band-limited at L, that is signals such that $f_{\ell m} = 0, \forall \ell \geq L$ $\Rightarrow$ summations may be truncated to $L-1$.

- Aside: Generalise to spin functions on the sphere.
  Square integrable spin functions on the sphere ${}_s f \in \mathrm{L}^2(\mathrm{S}^2)$, with integer spin $s \in \mathbb{Z}$, $|s| \leq \ell$, are defined by their behaviour under local rotations. By definition, a spin function transforms as

$${}_s f'(\theta, \varphi) = \mathrm{e}^{-\mathrm{i}s\chi}\; {}_s f(\theta, \varphi)$$

  under a local rotation by χ, where the prime denotes the rotated function.

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Sampling theorems on the sphere: state-of-the-art

- Inexact spherical harmonic transforms exist for a variety of pixelisations of the sphere, for example:
  - `HEALpix` (Gorski *et al.* 2005)
  - `IGLOO` (Crittenden & Turok 1998)

  $\rightarrow$ Do **not** lead to sampling theorems on the sphere!

- Driscoll & Healy (1994) sampling theorem:
  - Equiangular pixelisation of the sphere
  - Require $\sim 4L^2$ samples on the sphere
  - Semi-naive algorithm with complexity $\mathcal{O}(L^3)$
    (algorithms with lower scaling exist but they are not generally stable)
  - Require a precomputation or otherwise restricted use of Wigner recursions

- Gauss-Legendre sampling theorem:
  - Sample positions given by roots of Legendre functions
  - Require $\sim 2L^2$ samples on the sphere
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A novel sampling theorem on the sphere

- We have developed a new sampling theorem and corresponding fast algorithms by performing a factoring of rotations and then by associating the sphere with the torus through a periodic extension.
- Similar (in flavour but not detail!) to making a periodic extension in θ of a function ${}_sf$ on the sphere.

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(a) Function on sphere (b) Even function on torus (c) Odd function on torus

Figure: Associating functions on the sphere and torus

A novel sampling theorem on the sphere: inverse transform

- By a factoring of rotations, a reordering of summations and a separation of variables, the inverse transform of ${}_sf$ may be written:

Inverse spherical harmonic transform

$$ {}_sf(\theta, \varphi) = \sum_{m=-(L-1)}^{L-1} {}_sF_m(\theta)\, e^{im\varphi} $$

$$ {}_sF_m(\theta) = \sum_{m'=-(L-1)}^{L-1} {}_sF_{mm'}\, e^{im'\theta} $$

$$ {}_sF_{mm'} = (-1)^s\, i^{-(m+s)} \sum_{\ell=0}^{L-1} \sqrt{\frac{2\ell+1}{4\pi}}\, \Delta^\ell_{m'm}\, \Delta^\ell_{m',-s}\, {}_sf_{\ell m} $$

where $\Delta^\ell_{mn} \equiv d^\ell_{mn}(\pi/2)$ are the reduced Wigner functions evaluated at $\pi/2$.

A novel sampling theorem on the sphere: forward transform

- By a factoring of rotations, a reordering of summations and a separation of variables, the forward transform of ${}_sf$ may be written:

Forward spherical harmonic transform

$$ {}_sf_{\ell m} = (-1)^s \, \mathrm{i}^{m+s} \sqrt{\frac{2\ell+1}{4\pi}} \sum_{m'=-(L-1)}^{L-1} \Delta^{\ell}_{m' m} \, \Delta^{\ell}_{m', -s} \, {}_sG_{mm'} $$

$$ {}_sG_{mm'} = \int_0^{\pi} \mathrm{d}\theta \sin\theta \, {}_sG_m(\theta) \, \mathrm{e}^{-\mathrm{i}m'\theta} $$

$$ {}_sG_m(\theta) = \int_0^{2\pi} \mathrm{d}\varphi \, {}_sf(\theta, \varphi) \, \mathrm{e}^{-\mathrm{i}m\varphi} $$

- This formulation highlights similarities with Fourier series representation.
- The Fourier series expansion is only defined for periodic functions; thus, to recast these expressions in a form amenable to the application of Fourier transforms we must make a periodic extension in colatitude θ.

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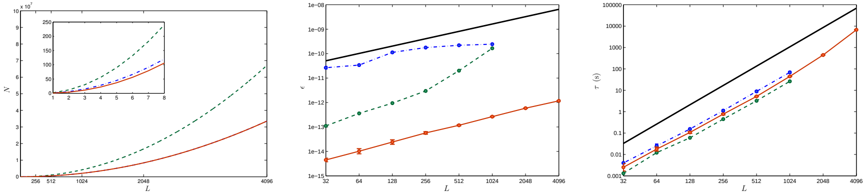
A novel sampling theorem on the sphere: properties

- Properties of our new sampling theorem:
  - Equiangular pixelisation of the sphere
  - Require $\sim 2L^2$ samples on the sphere (and still fewer than Gauss-Legendre sampling)
  - Exploit fast Fourier transforms to yield a fast algorithm with complexity $\mathcal{O}(L^3)$
  - No precomputation and very flexible regarding use of Wigner recursions
  - Extends to spin function on the sphere with no change in complexity or computation time



(a) Number of samples (b) Numerical accuracy (c) Computation time

Figure: Performance of our sampling theorem (MW=red; DH=green; Gauss-Legendre=blue)

A novel sampling theorem on the sphere: quadrature

- Sampling theorems effectively encode (often implicitly) an exact quadrature rule for evaluating the integral of a band-limited function on the sphere.
- The quadrature rule can be made explicit:

$$\int_{\mathbb{S}^2} \mathrm{d}\Omega(\theta, \varphi)\, {}_s f(\theta, \varphi) = \sum_{t=0}^{L-1} \sum_{p=0}^{2L-2} q_{\mathrm{MW}}(\theta_t)\, {}_s f(\theta_t, \varphi_p) \, .$$

- A similar quadrature rule can be given for the Driscoll & Healy sampling theorem. However, $2L$ samples in colatitude θ are required $\Rightarrow \sim 4L^2$ samples on the sphere.

Compressive sensing on the sphere

- A reduction in the number of samples required to represent a band-limited signal on the sphere has important implications for compressive sensing.
- Many natural signals are sparse in measures defined in the spatial domain, such as in the magnitude of their gradient.
- A more efficient sampling of a band-limited signal on the sphere improves both the dimensionality and sparsity of the signal in the spatial domain.
- For a given number of measurements, a more efficient sampling theorem improves the quality of compressive sampling reconstruction.
- Illustrate with a total variation (TV) inpainting problem on the sphere.

TV inpainting

- Consider inpainting problem $\boldsymbol{y} = \Phi \boldsymbol{x} + \boldsymbol{n}$ in the context of different sampling theorems, where:
  - the samples of f are denoted by the concatenated vector $\boldsymbol{x} \in \mathbb{R}^N$;
  - N is the number of samples on the sphere of the chosen sampling theorem;
  - M noisy measurements $\boldsymbol{y} \in \mathbb{R}^M$ are acquired;
  - the measurement operator $\Phi \in \mathbb{R}^{M \times N}$ represents a random masking of the signal;
  - the noise $\boldsymbol{n} \in \mathbb{R}^M$ is assumed to be iid Gaussian with zero mean.
- Define TV norm on the sphere:

$$\int_{\mathrm{S}^2} \mathrm{d}\Omega \, |\nabla f| \simeq \sum_{t=0}^{N_\theta - 1} \sum_{p=0}^{N_\varphi - 1} |\nabla f| \, q(\theta_t) \simeq \sum_{t=0}^{N_\theta - 1} \sum_{p=0}^{N_\varphi - 1} \sqrt{q^2(\theta_t)(\delta_\theta \boldsymbol{x})^2 + \frac{q^2(\theta_t)}{\sin^2\theta_t}(\delta_\varphi \boldsymbol{x})^2} \equiv \|\boldsymbol{x}\|_{\mathrm{TV}} \, .$$

- TV inpainting problem solved directly on the sphere:

$$\boldsymbol{x}^\star = \arg\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_{\mathrm{TV}} \text{ such that } \|\boldsymbol{y} - \Phi \boldsymbol{x}\|_2 \leq \epsilon \, .$$

- TV inpainting problem solved in harmonic space:

$$\hat{\boldsymbol{x}}^\star = \arg\min_{\hat{\boldsymbol{x}}} \|\Lambda \hat{\boldsymbol{x}}\|_{\mathrm{TV}} \text{ such that } \|\boldsymbol{y} - \Phi \Lambda \hat{\boldsymbol{x}}\|_2 \leq \epsilon \, ,$$

where Λ represents the inverse spherical harmonic transform and harmonic coefficients are represented by the concatenated vector $\hat{\boldsymbol{x}} \in \mathbb{C}^{L^2}$.

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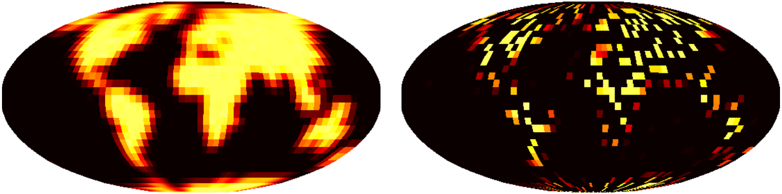

(a) Ground truth (b) Measurements

Figure: Earth topographic data reconstructed in the harmonic domain for $M/L^2 = 1/2$

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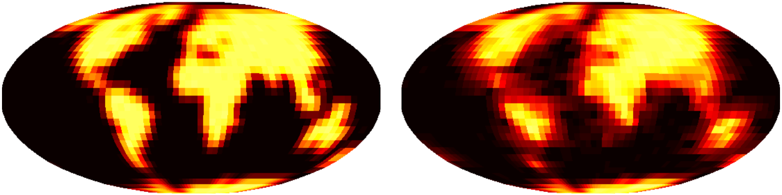

(a) Ground truth (b) DH reconstruction

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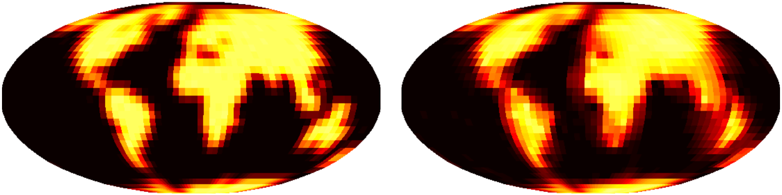

(a) Ground truth (b) MW reconstruction

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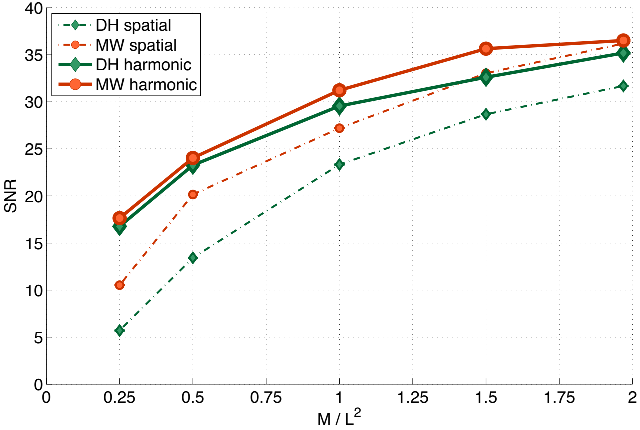

Figure: Reconstruction performance for the DH and MW sampling theorems

Summary

- We have developed a new sampling theorem on the sphere requiring fewer than half the number of samples of the canonical Driscoll & Healy sampling theorem.
- A reduction in the number of samples required to represent a band-limited signal on the sphere has important implications for compressive sensing, both in terms of the dimensionality and sparsity of signals.
- We have demonstrated improved reconstruction quality when solving an inpainting problem in the context of different sampling theorems.

Upcoming publications

- McEwen, J. D. and Wiaux, Y., *A novel sampling theorem on the sphere*, IEEE Trans. Sig. Proc., in press, 2011.
- McEwen, J. D., Puy, G., Thiran, J.-P., Vandergheynst, P., Ville, D. V. D., and Wiaux, Y., *Efficient and compressive sampling on the sphere*, IEEE Trans. Sig. Proc., submitted, 2011.

SSHT code

- Code to compute exact spin spherical harmonic transforms (SSHT) in the context of our new sampling theorem will be available very soon from:
  `http://www.jasonmcewen.org/`