Spread spectrum for compressive sampling & Application to MRI

Gilles Puy

Institute of Electrical Engineering, EPFL, CH-1015 Lausanne
Institute of the Physics of Biological Systems, EPFL, CH-1015 Lausanne, Switzerland

BASP Frontiers, Villars, September 2011

Research in collaboration with:

J. Marques, R. Gruetter, J.-P. Thiran, D. Van De Ville, R. Gribonval,

P. Vandergheynst, and Y. Wiaux

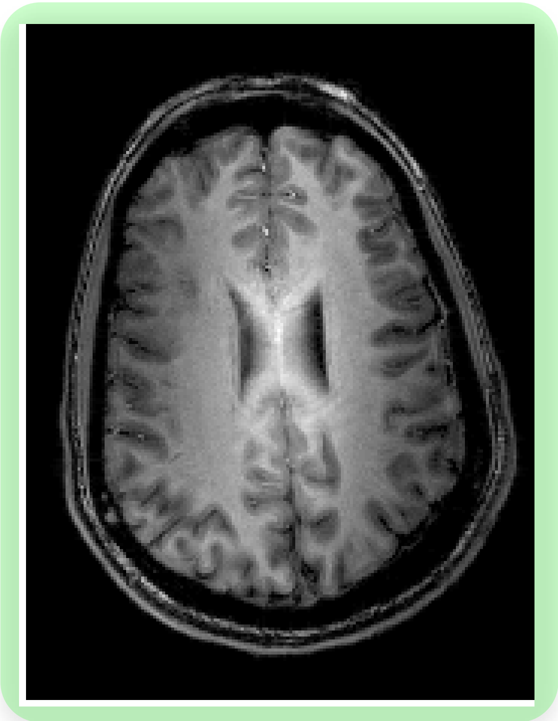
- I -

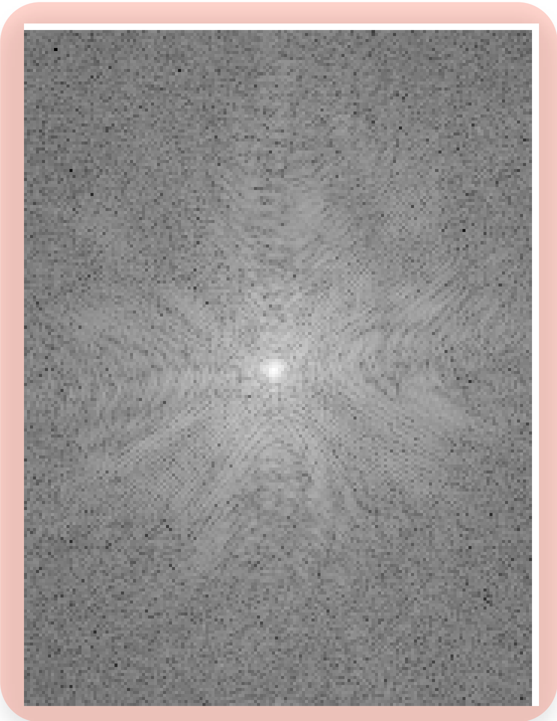
Motivation

Motivation

• Images in MRI are encoded in the Fourier domain by application of gradient magnetic fields:

$$y(\boldsymbol{k}_i) \equiv \int_{\mathrm{D}_\tau} x(\boldsymbol{\tau})\, \mathrm{e}^{-2\mathrm{i}\pi \boldsymbol{k}_i \cdot \boldsymbol{\tau}}\, \mathrm{d}\boldsymbol{\tau} \equiv \widehat{x}(\boldsymbol{k}_i).$$





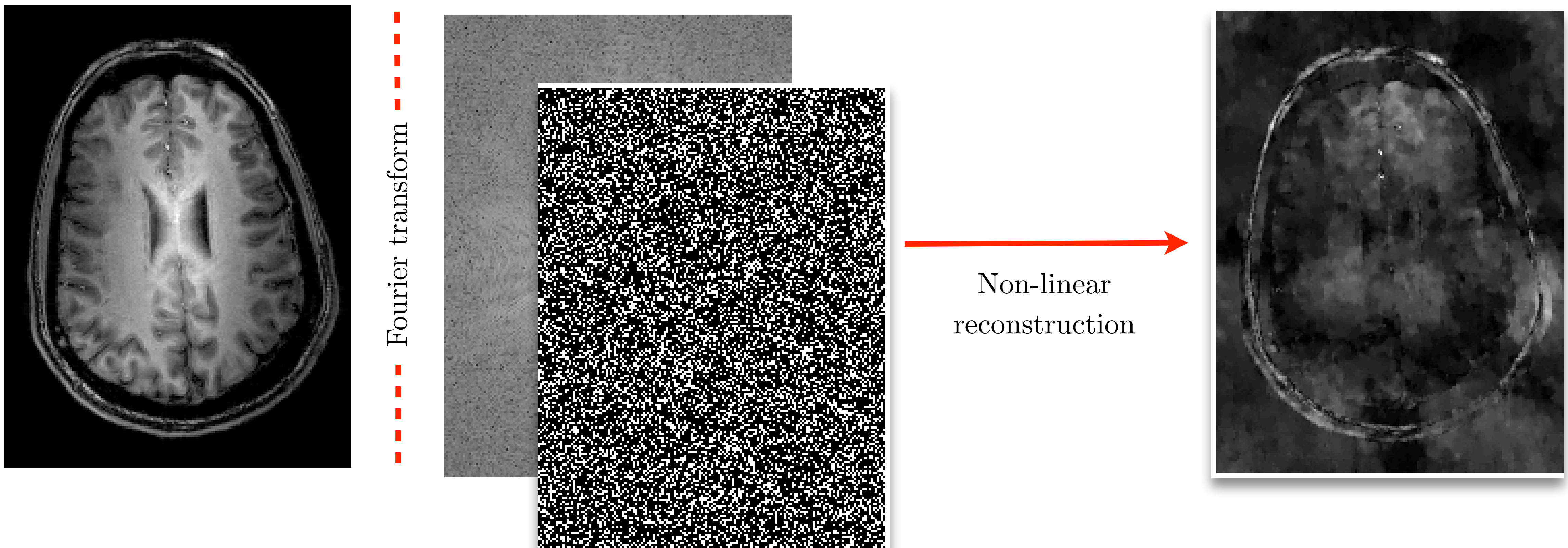
• Standard acquisition strategies probe all frequencies one after the other, providing complete information at the required resolution.

Motivation

- MRI images are sparse in well-chosen basis such as wavelet bases.
- Compressed sensing: sparse signals can be reconstructed from a few number of linear and non-adaptive measurements.
- Simple implementation: uniform random selection of Fourier coefficients.

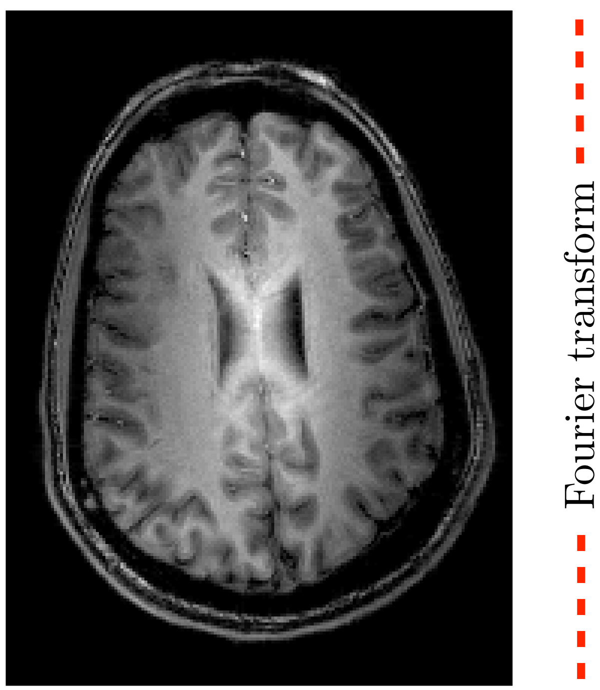
Motivation

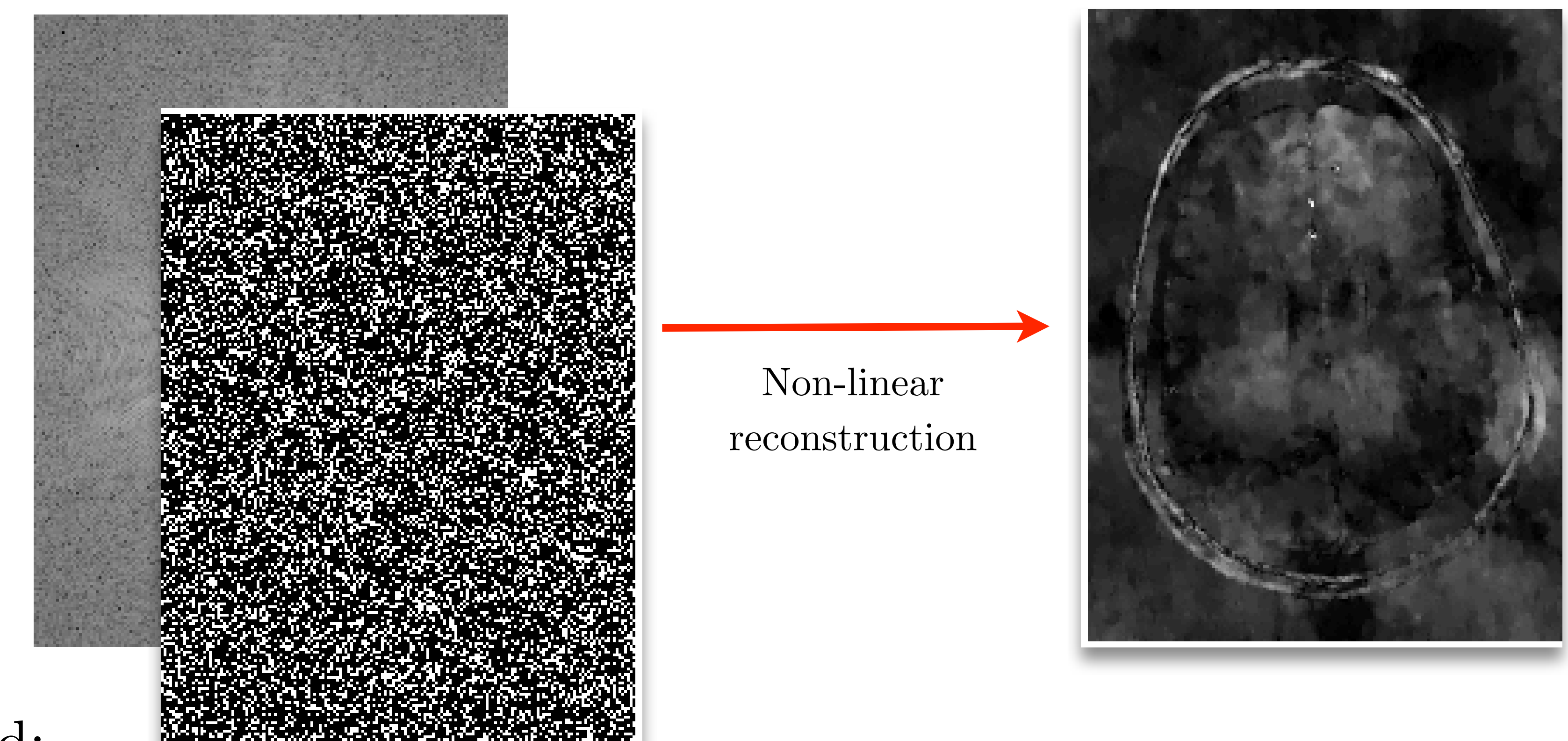
- MRI images are sparse in well-chosen basis such as wavelet bases.
- Compressed sensing: sparse signals can be reconstructed from a few number of linear and non-adaptive measurements.
- Simple implementation: uniform random selection of Fourier coefficients.



Motivation

- MRI images are sparse in well-chosen basis such as wavelet bases.
- Compressed sensing: sparse signals can be reconstructed from a few number of linear and non-adaptive measurements.
- Simple implementation: uniform random selection of Fourier coefficients.





- State of the art method:
  - Energy of MRI images are usually concentrated at low frequencies.
  - Concentrate most of the measurements at low frequencies: Variable density sampling.

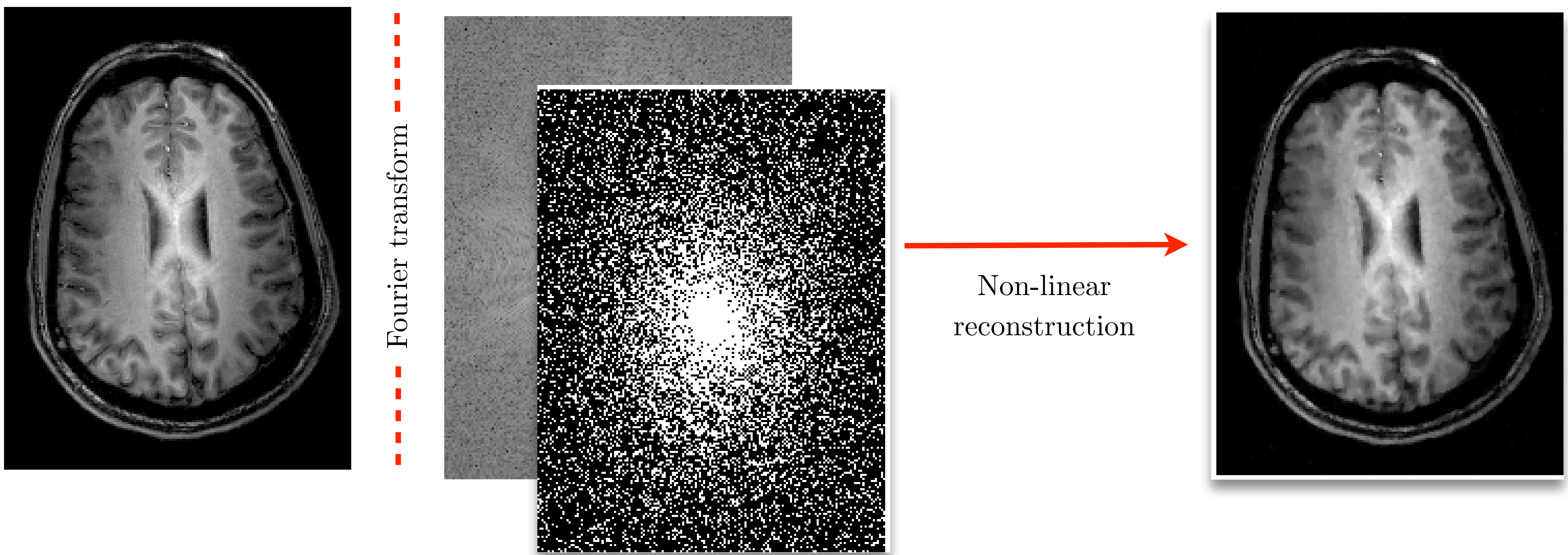
Motivation

- State of the art method:
  - Energy of MRI images are usually concentrated at low frequencies;
  - Concentrate most of the measurements at low frequencies.

[1] Lustig et al, “Sparse MRI: The Application of Compressed Sensing for Rapid MR Imaging” Magn. Reson. Med, 2007.

Motivation

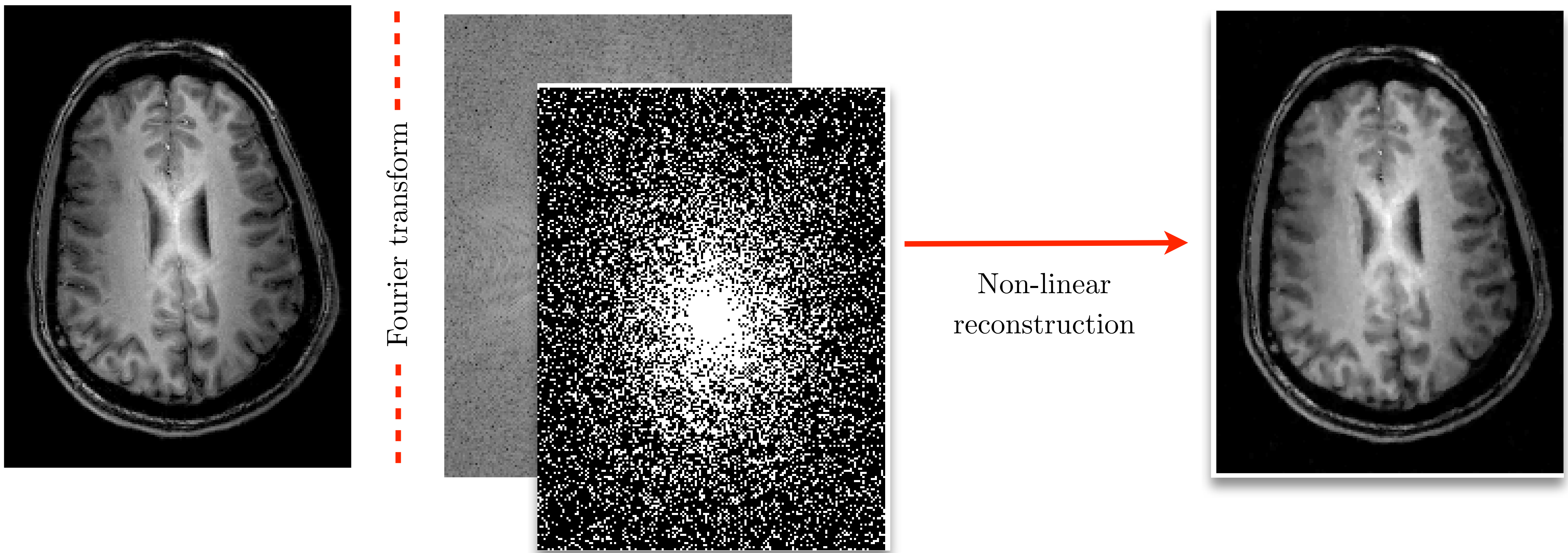
- State of the art method:
  - Energy of MRI images are usually concentrated at low frequencies;
  - Concentrate most of the measurements at low frequencies.



[1] Lustig et al, "Sparse MRI: The Application of Compressed Sensing for Rapid MR Imaging" Magn. Reson. Med, 2007.

Motivation

- State of the art method:
  - Energy of MRI images are usually concentrated at low frequencies;
  - Concentrate most of the measurements at low frequencies.



- What is the optimal shape of the variable density profile ?
- Can we modify the acquisition to obtain better reconstructions ?

[1] Lustig et al, "Sparse MRI: The Application of Compressed Sensing for Rapid MR Imaging"
Magn. Reson. Med, 2007.

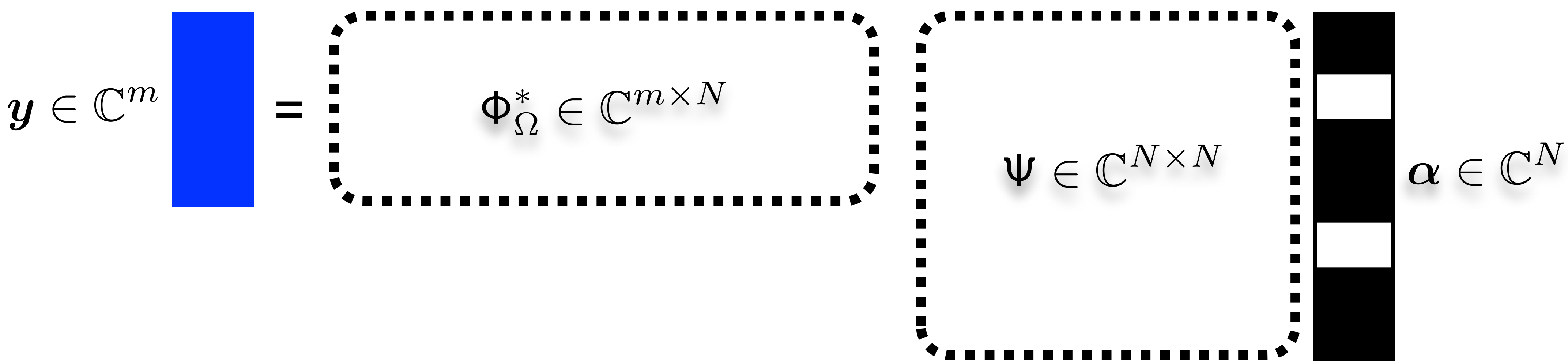
- II -

Role of the coherence

Role of the coherence

- $\boldsymbol{x} \in \mathbb{C}^N$ is a s-sparse signal in an orthonormal basis $\Psi \in \mathbb{C}^{N \times N}$, i.e. $\boldsymbol{\alpha} = \Psi^* \boldsymbol{x}$ contains s non-zero entries.
- $\boldsymbol{x}$ is probed by projection onto m randomly selected vectors of another orthonormal basis $\Phi = (\boldsymbol{\phi}_1, ..., \boldsymbol{\phi}_N) \in \mathbb{C}^{N \times N}$.
- $\Omega = \{l_1, \ldots, l_m\}$ denotes the indices of the selected basis vectors.
- The measurement vector $\boldsymbol{y} \in \mathbb{C}^m$ reads as:

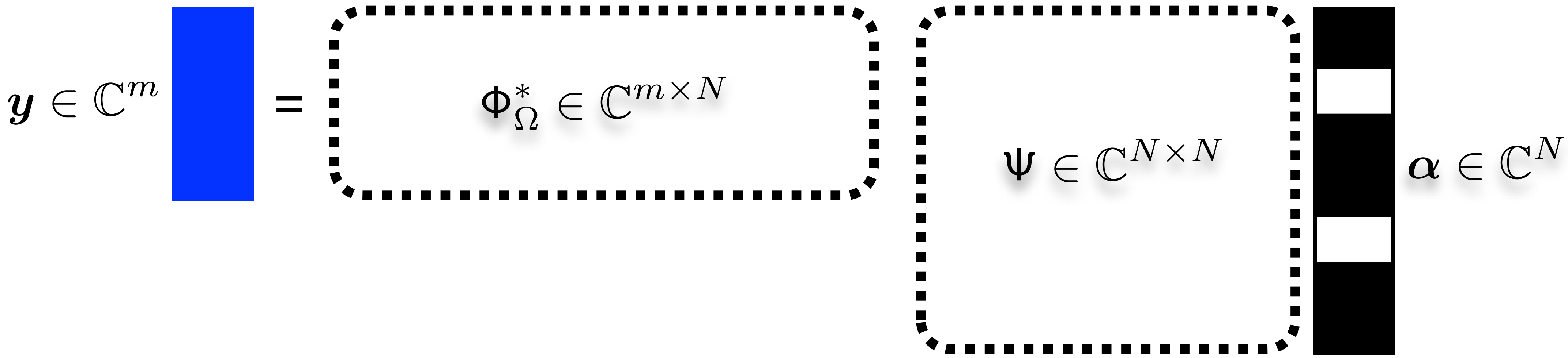
$$\boldsymbol{y} = \mathsf{A}_\Omega \, \boldsymbol{\alpha}, \text{ with } \mathsf{A}_\Omega = \Phi_\Omega^* \Psi \in \mathbb{C}^{m \times N}.$$



- $\boldsymbol{\alpha}$ is recovered by solving the ℓ_1-minimization problem

$$\boldsymbol{\alpha}^\star = \arg \min_{\bar{\boldsymbol{\alpha}} \in \mathbb{C}^N} \|\bar{\boldsymbol{\alpha}}\|_1 \text{ subject to } \boldsymbol{y} = \mathsf{A}_\Omega \bar{\boldsymbol{\alpha}}.$$

Role of the coherence



- For some universal constants $C > 0$ and $\gamma > 1$, if

$$m \geqslant C N \mu^2 s \log^4(N),$$

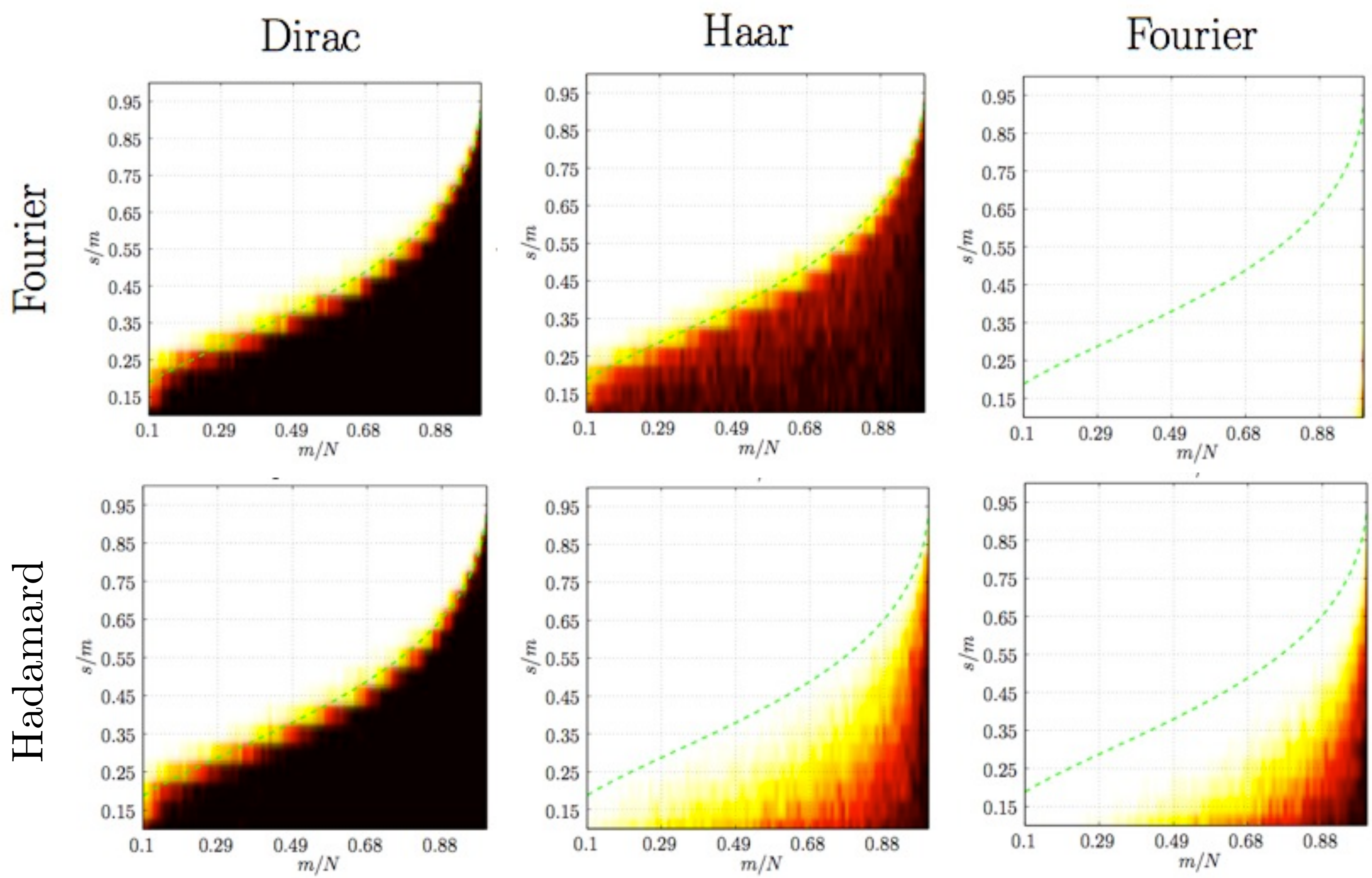
then $\boldsymbol{\alpha}$ is the unique minimizer of the ℓ_1-minimization problem with probability at least $1 - N^{-\gamma \log^3(N)}$.

$\mu = \max_{1 \leqslant i,j \leqslant N} |\langle \boldsymbol{\phi}_i, \boldsymbol{\psi}_j \rangle|$ is the mutual coherence between Φ and Ψ.

- For a fixed sensing basis, the reconstruction quality depends on sparsity basis.

[3] Candès et al, "Sparsity and Incoherence in Compressive Sampling," Inverse Problems, 2007.
[4] Rauhut, "Compressive Sensing and Structured Random Matrices," Theoret. Found. and Num. Methods for Sparse Recovery, 2010.

Role of the coherence

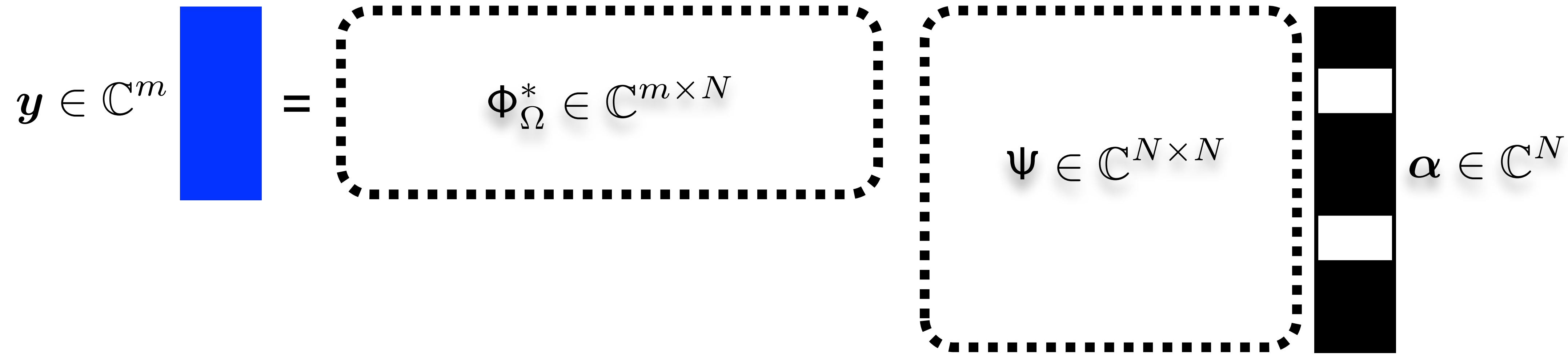


ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

- II -

Sampling profile optimization

Sampling profile optimization



- Sampling profile: A vector $\boldsymbol{p} = (p_j)_{1 \leqslant j \leqslant N}$ with $p_j \in (0,1], 1 \leqslant j \leqslant N$, and $\|\boldsymbol{p}\|_1 = \sum_{1 \leqslant j \leqslant N} p_j = m\,.$
- p_j represents the probability that index ϕ_j is selected.
- For some universal constants $C > 0$ and $\gamma > 1$, if

$$m \geqslant CN\mu^2(\boldsymbol{p})s\log^2(6N/\epsilon),$$

then $\boldsymbol{\alpha}$ is the unique minimizer of the ℓ_1-minimization problem with probability at least ε.

$\mu(\boldsymbol{p}) = \left(\frac{m}{N}\right)^{1/2} \max_{1 \leqslant i,j \leqslant N} \frac{|\langle \boldsymbol{\phi}_i, \boldsymbol{\psi}_j \rangle|}{p_i^{1/2}}$ is the weighted mutual coherence between Φ and Ψ .

- To reduce the number of measurement, we should find $\boldsymbol{p}$ that minimizes $\mu(\boldsymbol{p})$.

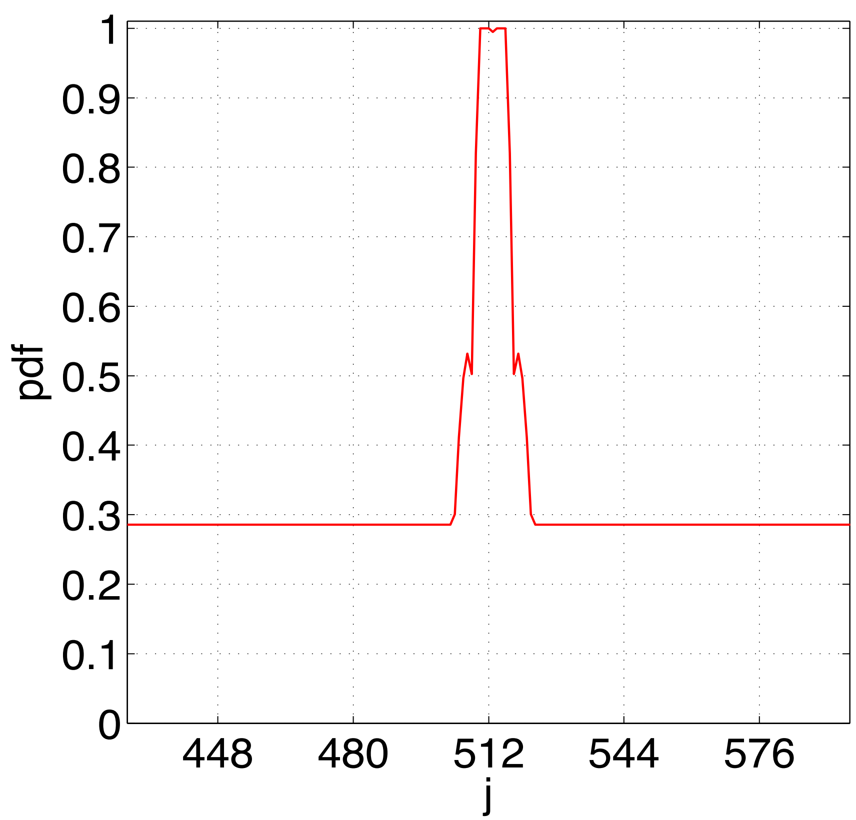
[4] Rauhut, "Compressive Sensing and Structured Random Matrices," Theoret. Found. and Num. Methods for Sparse Recovery, 2010.
[5] Puy et al., "On variable density compressive sampling", IEEE Signal Process. Lett., 2011.

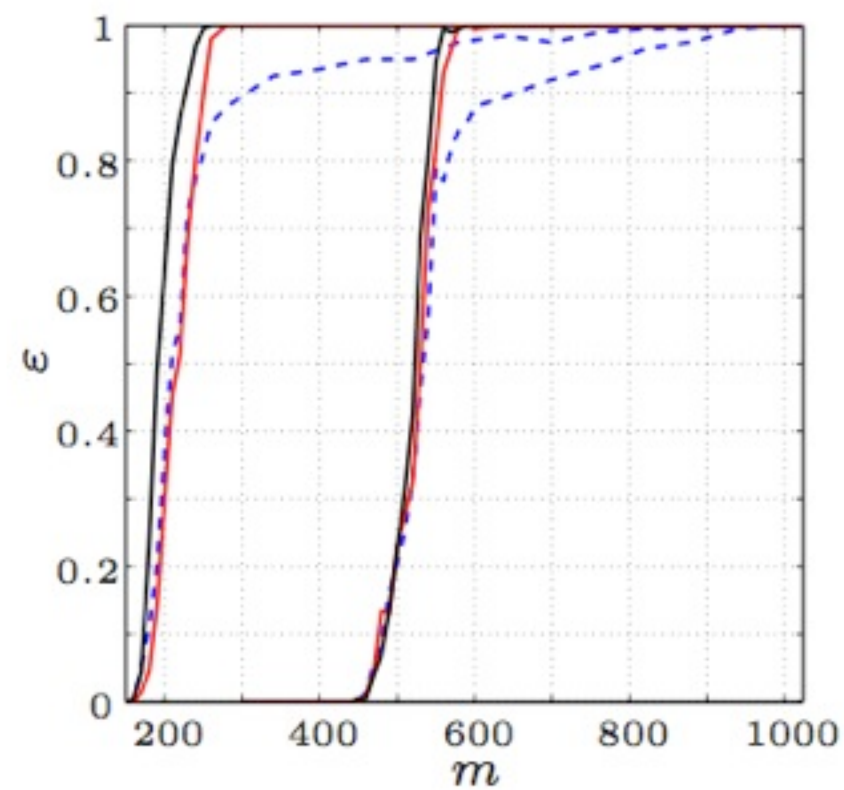
Sampling profile optimization

- Solve the following optimization problem

$$\min_{(\boldsymbol{p},\boldsymbol{q})\in\mathbb{R}^{N\times 2}} \|\mathsf{B}\,\boldsymbol{q}\|_\infty + \lambda\|\boldsymbol{p}\cdot\boldsymbol{q} - \mathbf{1}\|_2^2 \text{ s.t. } \boldsymbol{p}\in\mathcal{K}_\tau \qquad \mathcal{K}_\tau = \{\boldsymbol{p}\in[\tau,1]^N : \|\boldsymbol{p}\|_1 \leqslant m\}$$

$$\mathsf{B} = \max_{1\leqslant j\leqslant N} |\langle \boldsymbol{\phi}_i, \boldsymbol{\psi}_j\rangle|$$





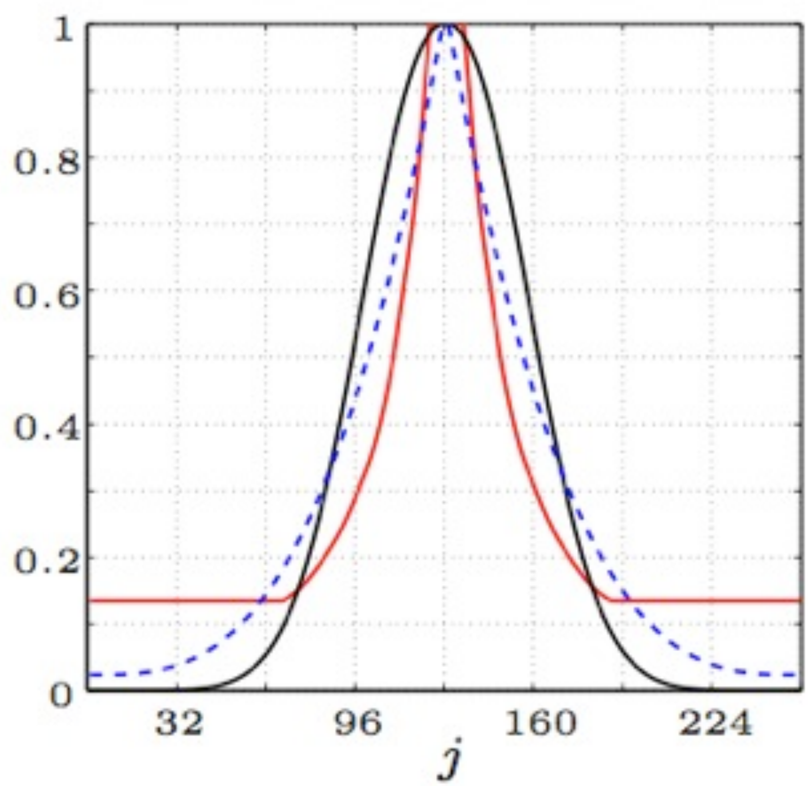
[5] Puy et al., “On variable density compressive sampling”, IEEE Signal Process. Lett., 2011.

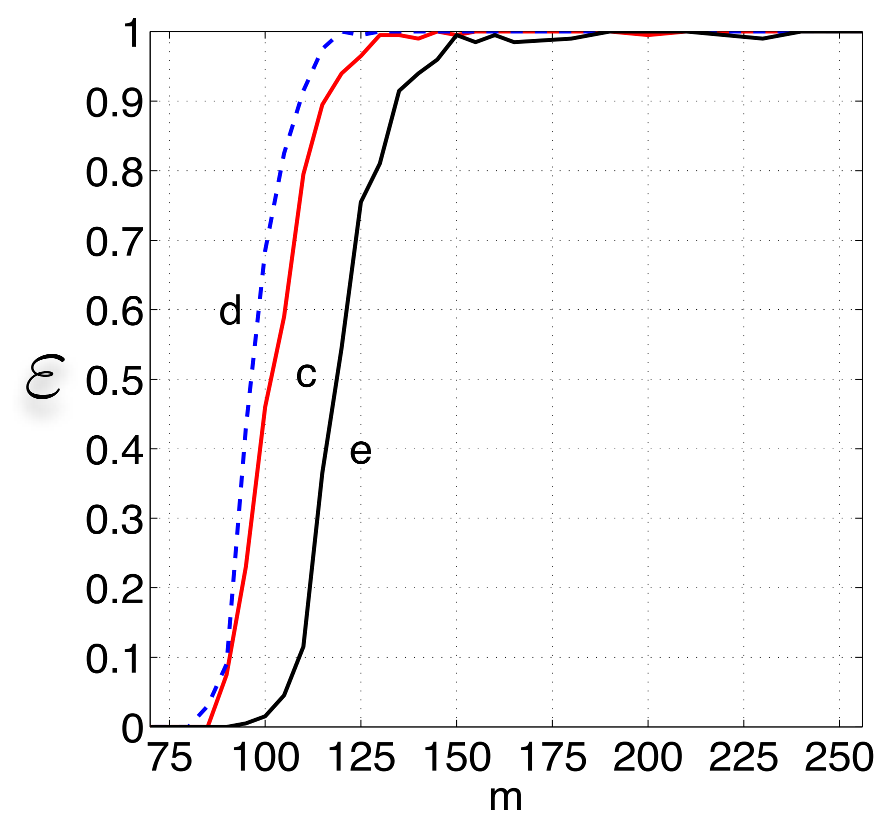
Sampling profile optimization

- Solve the following optimization problem

$$\min_{(\boldsymbol{p},\boldsymbol{q})\in\mathbb{R}^{N\times 2}} \|\mathsf{B}\,\boldsymbol{q}\|_\infty + \lambda\|\boldsymbol{p}\cdot\boldsymbol{q}-\mathbf{1}\|_2^2 \text{ s.t. } \boldsymbol{p}\in\mathcal{K}_\tau \qquad \mathcal{K}_\tau = \{\boldsymbol{p}\in[\tau,1]^N : \|\boldsymbol{p}\|_1 \leqslant m\}$$

prior information on the signal support in the sparsity basis





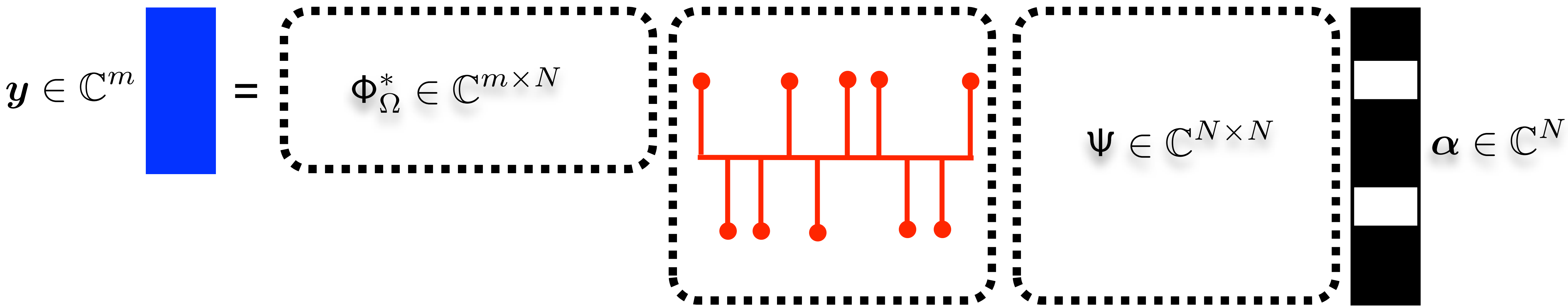
- III -

Spread spectrum technique

Spread spectrum technique

- Modulate the signal by a random sequence $\boldsymbol{c} = (c_l)_{1 \leqslant l \leqslant N} \in \mathbb{R}^N$ of +/- 1 entries with $\mathbb{P}(c_l = -1) = \mathbb{P}(c_l = +1) = 1/2$.
- The measurement vector becomes:

$$\boldsymbol{y} = \mathsf{A}_\Omega \, \boldsymbol{\alpha}, \text{ with } \mathsf{A}_\Omega = \Phi_\Omega^* \mathsf{C} \Psi \in \mathbb{C}^{m \times N}.$$



- We define the *modulus-coherence* as:

$$\beta(\Phi, \Psi) = \max_{1 \leqslant i,j \leqslant N} \sqrt{\sum_{k=1}^{N} |\phi_{ki}^* \psi_{kj}|^2},$$

Spread spectrum technique

- The mutual coherence $\mu = \max_{1 \leqslant i,j \leqslant N} |\langle \phi_i, \mathsf{C}\,\psi_j \rangle|$ of this new sensing system satisfies

$$\mu \leqslant \beta\,(\Phi, \Psi)\,\sqrt{2\log\left(2N^2/\epsilon\right)},$$

  with probability at least $1-\epsilon$.

- For some universal constants $\rho < \log^3(N)$ and $C_\rho > 0$, if

$$m \geqslant C_\rho\, N\beta^2\,(\Phi, \Psi)\, s\log^5(N),$$

  then $\boldsymbol{\alpha}$ is the unique minimizer of the ℓ_1-minimization problem with probability at least $1-\mathcal{O}\left(N^{-\rho}\right)$.

- *Universal sensing basis*: sensing basis $\Phi \in \mathbb{C}^{N\times N}$ with entries of equal complex magnitudes entries, e.g., the Fourier basis, the Hadamard basis.

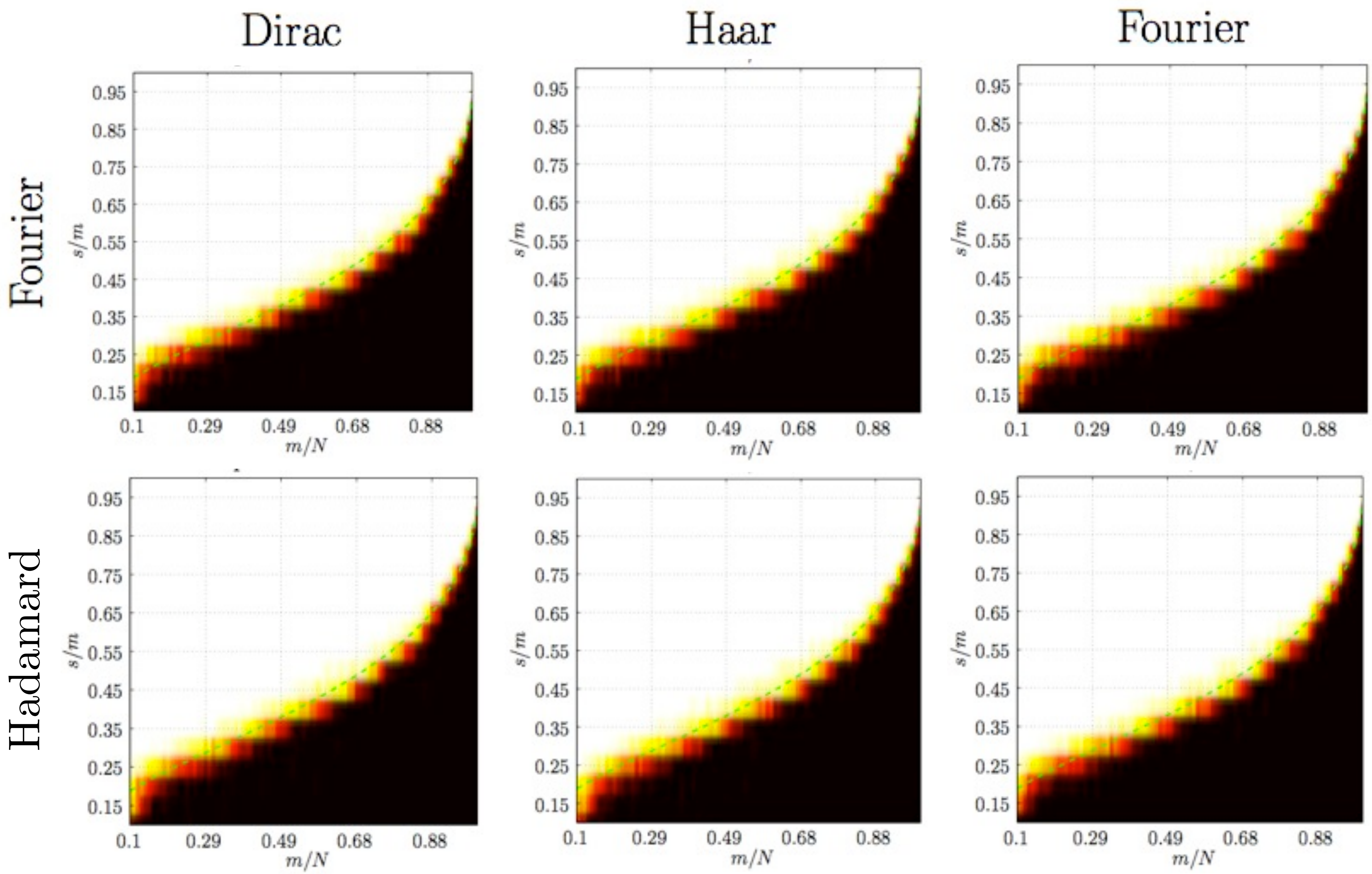
$$\beta\,(\Phi, \Psi) = \frac{1}{\sqrt{N}}$$

- And the recovery condition becomes

$$m \geqslant C_\rho\, s\log^5(N).$$

Universality **Optimality**

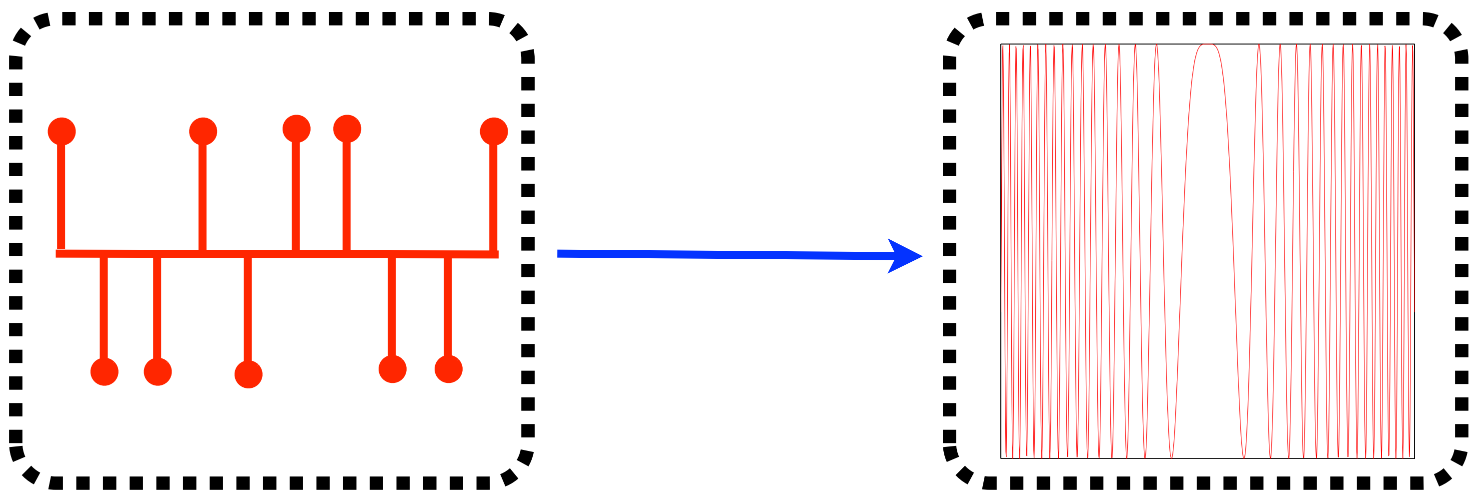
Spread spectrum technique



[6] Do T et al. "Fast compressive sampling with structurally random matrices," ICASSP, 2008.
[7] Romberg, "Compressive sensing by random convolution," SIAM J. Imaging Sciences, 2009.
[8] Tropp et al., "Beyond Nyquist: Efficient Sampling of Sparse Bandlimited Signals," IEEE Trans. Inf. Theory, 2010.
[9] Krahmer et al., "New and Improved Johnson-Lindenstrauss Embeddings via the Restricted Isometry Property," SIAM J. on Math. Analysis, 2011.
[10] Tropp, "Improved analysis of the subsampled randomized Hadamard transform," Adv. Adapt. Data Anal, 2011.

Spread spectrum technique

- Example of radio-interferometry and MRI.
- Replace the random pre-modulation by a linear chirp $e^{i\pi w \tau^2}$.



- As the random modulation, it is a wideband signal that does not change the norm of the original signal.
- But the modulation is analog... One should change the measurement model:

$$A_\Omega = F^*_\Omega C U \Psi \in \mathbb{C}^{m\times N}$$

Spread spectrum technique

- Numerical simulations: radio interferometric acquisition.

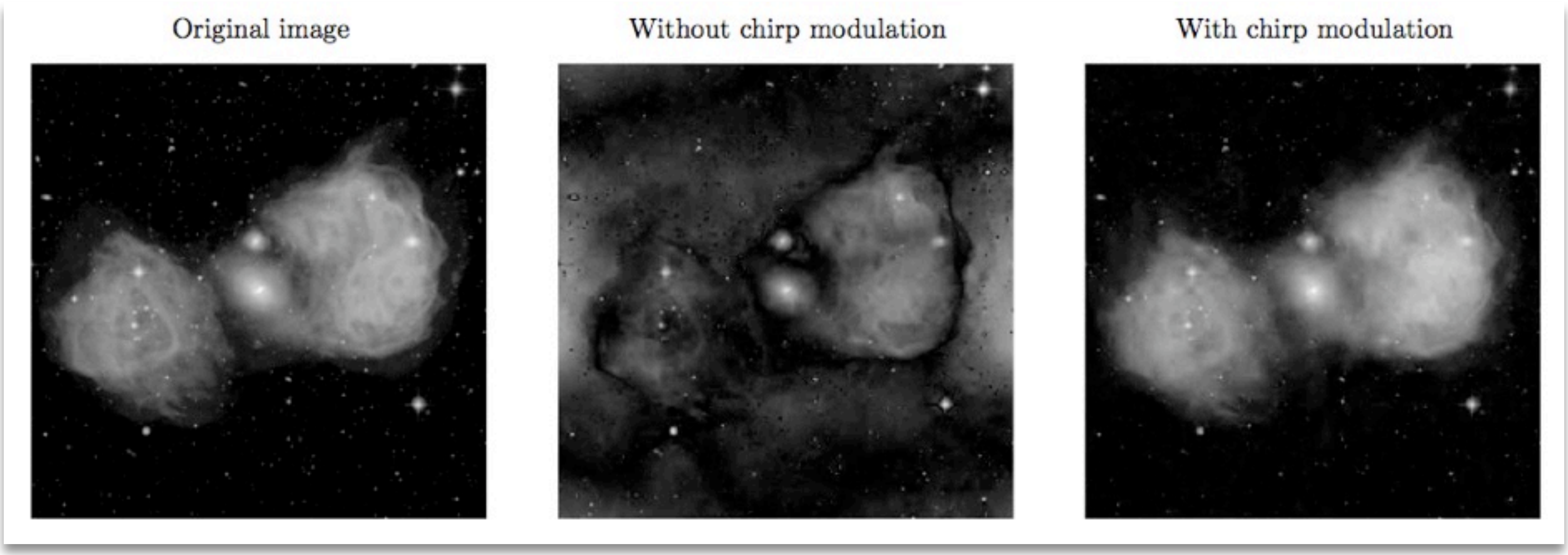


Image courtesy of NRAO/AUI and J. M. Uson

[10] Wiaux et al., "Spread spectrum for imaging techniques in radio interferometry," Mon. Not. R. Astron. Soc. 2009

Spread spectrum technique

- Assume that the phase of the non-zero coefficients of α are randomly and uniformly between 0 and 2π . For a universal constant $C > 0$, if

$$m \geqslant C N_{\bar{w}}\, \mu_{\bar{w}}^2 s \log^2(6N/\epsilon),$$

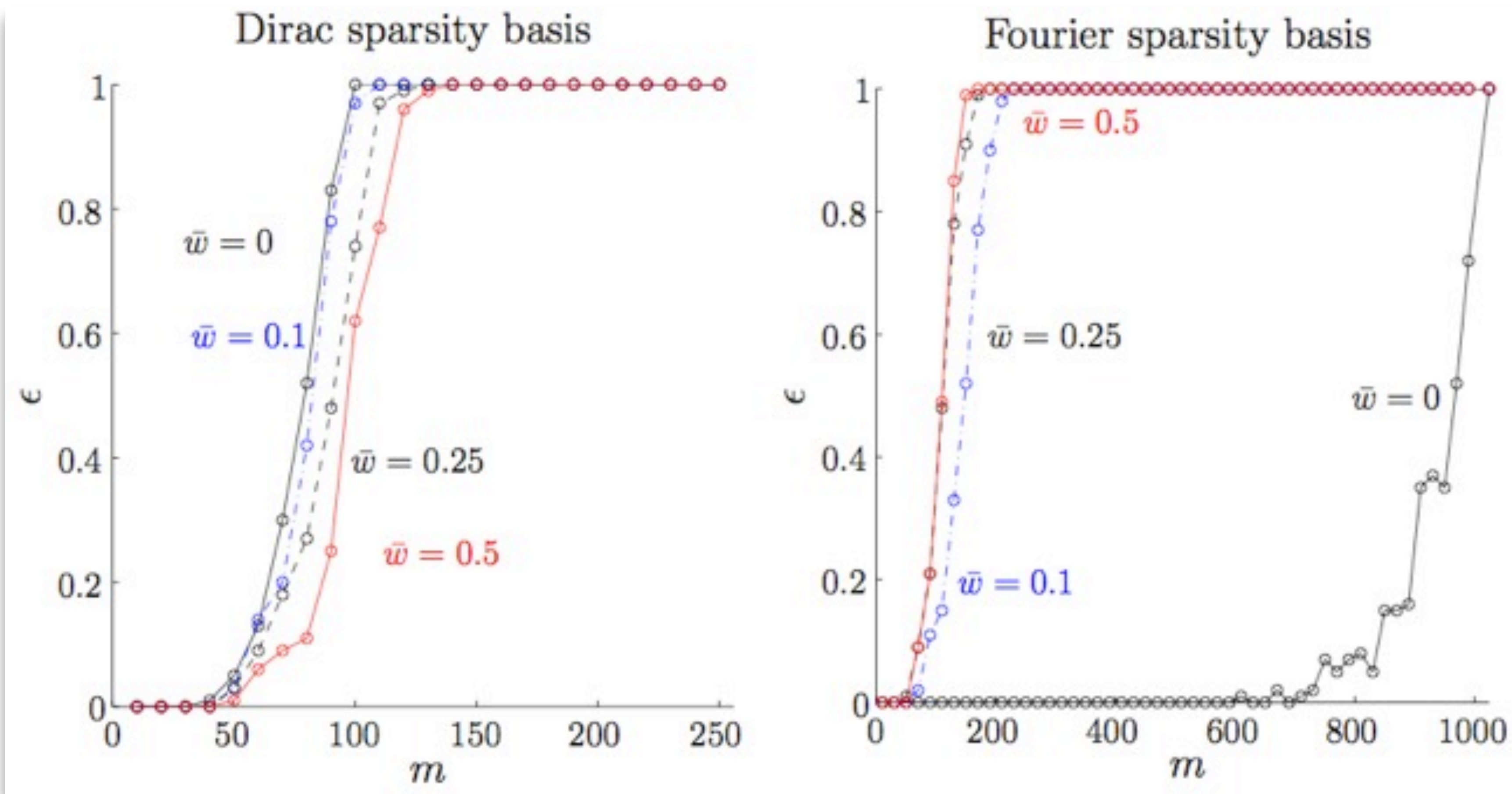
then $\boldsymbol{\alpha}$ is the unique minimizer of the ℓ_1-minimization problem with probability at least ϵ.

$$\mu_{\bar{w}} = \max_{1 \leqslant i,j \leqslant N} |\langle \boldsymbol{f}_i, \mathsf{CU}\boldsymbol{\psi}_j \rangle|$$

- Two effects are competing with each other:
  - $\mu_{\bar{w}}$ is decreasing with the spread spectrum phenomenon
  - $N_{\bar{w}}$ is increasing linearly with the chirp rate
- We should find the optimal trade-off between the two effects.

Spread spectrum technique

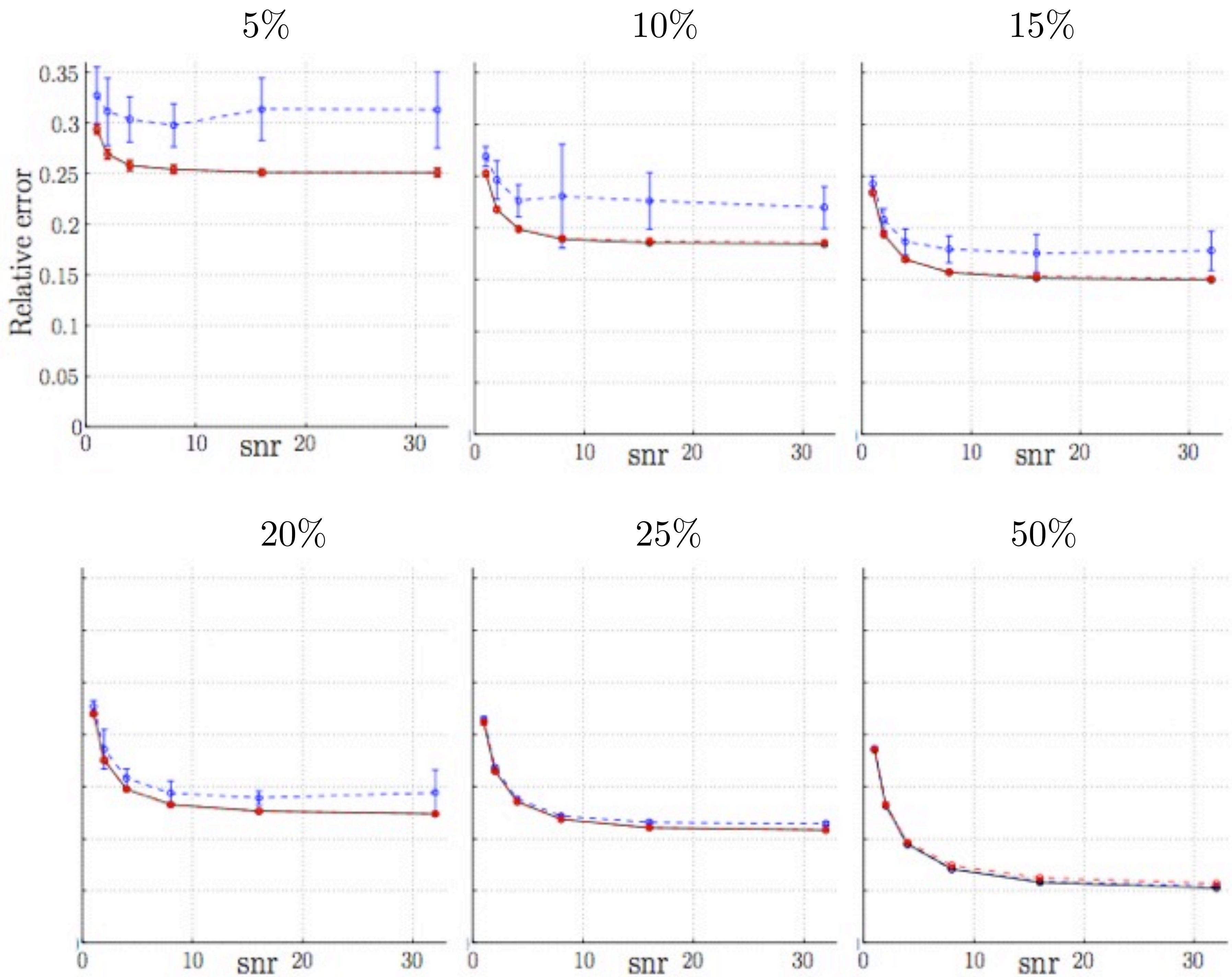
Sparsity basis	Dirac	Fourier
$N_{\bar{w}}\, \mu^2_{\bar{w}}$ at $\bar{w} = 0$	1.00	$1.02 \cdot 10^3$
$N_{\bar{w}}\, \mu^2_{\bar{w}}$ at $\bar{w} = 0.10$	2.58	$1.54 \cdot 10^1$
$N_{\bar{w}}\, \mu^2_{\bar{w}}$ at $\bar{w} = 0.25$	3.15	6.95
$N_{\bar{w}}\, \mu^2_{\bar{w}}$ at $\bar{w} = 0.50$	3.46	4.13



- IV -

s_2MRI

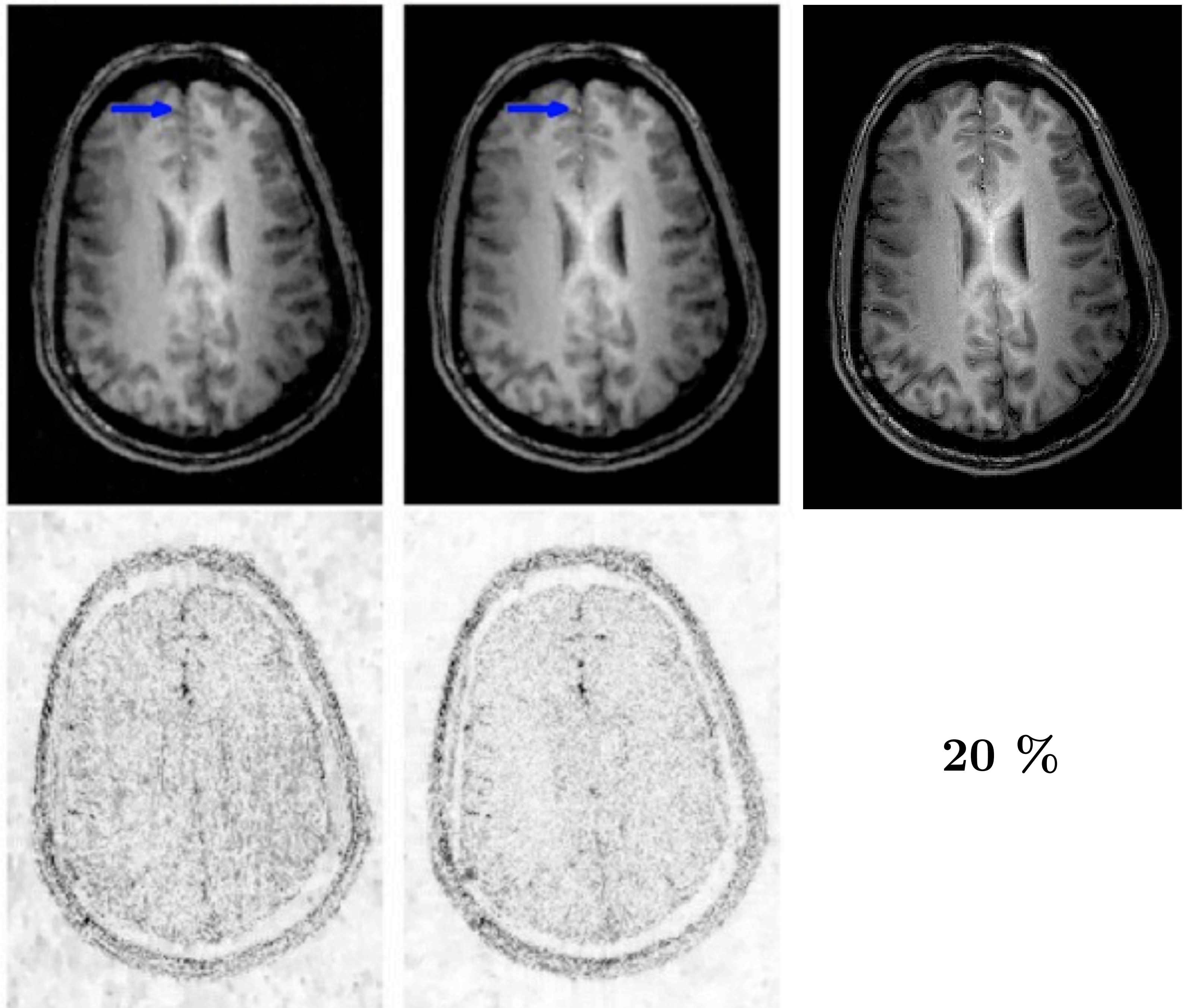
s_2MRI



[11] Puy et al., “Spread spectrum magnetic resonance imaging,” IEEE Tran. Med. Imag., submitted. 2011.

s_2MRI

Variable density sampling | s_2MRI | Original image

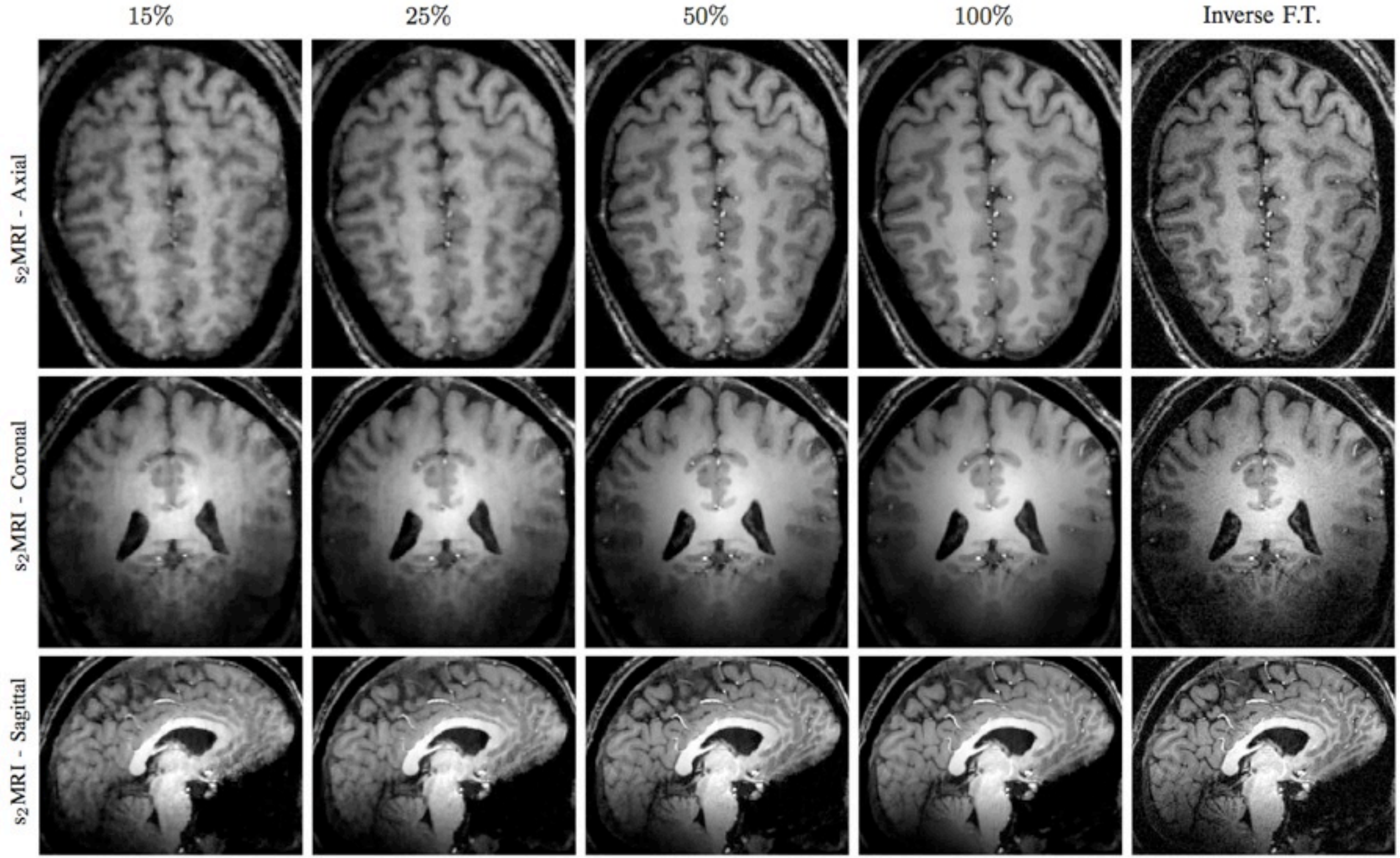


20 %

s_2MRI

- Proof of concept:
  - Real 3D *in vivo* acquisition on a 7T scanner (Siemens, Erlangen, Germany).
  - Chirp modulation implemented with the use of a 2^{nd} order shim coil.

s_2MRI



ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

- V -

Conclusion

Conclusion

- Incoherence between sparsity basis and sensing basis is an essential concept for compressive sampling.
- We proposed:
  - a coherence-driven optimization procedure for variable density sampling.
  - a spread spectrum technique to optimize the acquisition in MRI or radio-interferometry.
- In a discrete setting, the spread spectrum technique is universal and optimal for sensing matrices such as the Fourier or Hadamard basis.
- s_2MRI performs better than VDS in all simulations.

Conclusion

Thank you for your attention.