Radio interferometric imaging with compressed sensing

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Outline

Radio interferometry

- The complex visibility measured by an interferometer is given by

$$y(\boldsymbol{u}, w) = \int_{D^2} A(\boldsymbol{l})\, x_{\mathrm{p}}(\boldsymbol{l})\, \mathrm{e}^{-\mathrm{i}2\pi[\boldsymbol{u}\cdot\boldsymbol{l} + w\,(n(\boldsymbol{l})-1)]}\, \frac{\mathrm{d}^2\boldsymbol{l}}{n(\boldsymbol{l})}$$
$$= \int_{D^2} A(\boldsymbol{l})\, x_{\mathrm{p}}(\boldsymbol{l})\, C^{(w)}(\|\boldsymbol{l}\|)\, \mathrm{e}^{-\mathrm{i}2\pi\boldsymbol{u}\cdot\boldsymbol{l}}\, \frac{\mathrm{d}^2\boldsymbol{l}}{n(\boldsymbol{l})},$$

  where $\boldsymbol{l} = (l, m)$, $\|\boldsymbol{l}\|^2 + n^2(\boldsymbol{l}) = 1$ and the w-component $C^{(w)}(\|\boldsymbol{l}\|)$ is given by

$$C^{(w)}(\|\boldsymbol{l}\|) \equiv \mathrm{e}^{\mathrm{i}2\pi w\left(1-\sqrt{1-\|\boldsymbol{l}\|^2}\right)}.$$

- Various assumptions are often made regarding the size of the field-of-view (FoV):
  - Small-field with $\|\boldsymbol{l}\|^2\, w \ll 1 \quad \Rightarrow \quad C^{(w)}(\|\boldsymbol{l}\|) \simeq 1$
  - Small-field with $\|\boldsymbol{l}\|^4\, w \ll 1 \quad \Rightarrow \quad C^{(w)}(\|\boldsymbol{l}\|) \simeq \mathrm{e}^{\mathrm{i}\pi w\|\boldsymbol{l}\|^2}$
  - Wide-field $\quad \Rightarrow \quad C^{(w)}(\|\boldsymbol{l}\|) = \mathrm{e}^{\mathrm{i}2\pi w\left(1-\sqrt{1-\|\boldsymbol{l}\|^2}\right)}$

- Interferometric imaging: **recover an image from noisy and incomplete Fourier measurements**.

Radio interferometric inverse problem

- Consider the resulting **ill-posed inverse problem** posed in the discrete setting:

$$\boldsymbol{y} = \Phi \boldsymbol{x} + \boldsymbol{n} \, ,$$

  with:
  - incomplete Fourier measurements taken by the interferometer $\boldsymbol{y}$;
  - linear measurement operator Φ;
  - underlying image $\boldsymbol{x}$;
  - noise $\boldsymbol{n}$.

- **Measurement operator** $\Phi = \mathbf{M}\,\mathbf{F}\,\mathbf{C}\,\mathbf{A}$ incorporates:
  - primary beam $\mathbf{A}$ of the telescope;
  - w-component modulation $\mathbf{C}$ (responsible for the spread spectrum phenomenon);
  - Fourier transform $\mathbf{F}$;
  - masking $\mathbf{M}$ which encodes the incomplete measurements taken by the interferometer.

Interferometric imaging with compressed sensing

- Solve by applying a **prior on sparsity** of the signal in a sparsifying basis Ψ or in the magnitude of its gradient.
- Image is recovered by solving:
  - Basis Pursuit denoising problem

$$\boldsymbol{\alpha}^\star = \arg\min_{\boldsymbol{\alpha}} \|\alpha\|_1 \ \text{ such that } \ \|\boldsymbol{y} - \Phi\Psi\boldsymbol{\alpha}\|_2 \leq \epsilon \ ,$$

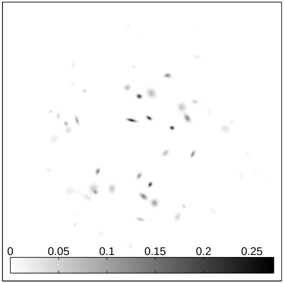
    where the image is synthesising by $\boldsymbol{x}^\star = \Psi\boldsymbol{\alpha}^\star$;
  - Total Variation (TV) denoising problem

$$\boldsymbol{x}^\star = \arg\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_{\text{TV}} \ \text{ such that } \ \|\boldsymbol{y} - \Phi\boldsymbol{x}\|_2 \leq \epsilon \ .$$

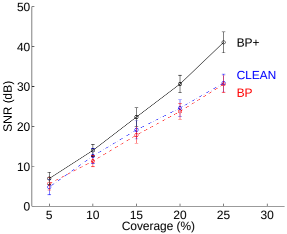
- ℓ_1-norm $\|\cdot\|_1$ is given by the sum of the absolute values of the signal.
- TV norm $\|\cdot\|_{\text{TV}}$ is given by the ℓ_1-norm of the gradient of the signal.
- Tolerance ϵ is related to an estimate of the noise variance.

Interferometric imaging with Dirac sparsity

- BP denoising problem solved by Wiaux *et al.* (2009a) for the Dirac basis.
- Reconstruction performance is similar to CLEAN (which is a matching pursuit based approach).
- However, versatility of the framework allows easy addition of other priors, such as a positivity prior, and alternative sparsity basis.
- Implications for coherence.



(a) Typical simulation

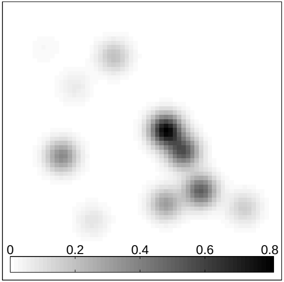


(b) Reconstruction performance

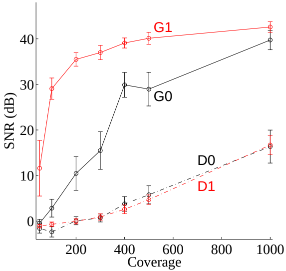
Figure: BP, BP+ and CLEAN reconstruction performance.

Spread spectrum phenomenon

- Spread spectrum phenomenon highlighted and studied in the context of radio interferometry by Wiaux *et al.* (2009b) (also see previous talk by Gilles Puy).
- Modulation by the w-component corresponds to a norm-preserving convolution in the Fourier plane $\rightarrow$ spreads the spectrum of the signal.
- Recall that for Fourier measurements the coherence is the maximum modulus of the Fourier transform of the sparsity basis vectors: $\mu = \max_{i,j} |\boldsymbol{f}_i \cdot \psi_j|$.
- Consequently, spreading the spectrum increases the incoherence between the sensing and sparsity bases, thus improving the fidelity of reconstruction.



(a) Typical simulation



(b) Reconstruction performance

Figure: BP reconstruction performance for Dirac (D) and Gaussian (G) sparsity bases, in the absence (0) and presence (1) of the spread spectrum phenomenon.

Spherical interferometric imaging

- Extend the standard compressed sensing imaging framework to wide fields by considering **interferometric images directly on the sphere**, rather than the equatorial plane (JDM & Wiaux 2010).
- Augment the usual interferometric measurement operator with an initial projection $\mathbf{P}$ from the sphere to the plane, *i.e.*

$$\boldsymbol{y} = \Phi_{\rm s}\, \boldsymbol{x}_{\rm s} + \boldsymbol{n}, \quad \text{where} \quad \Phi_{\rm s} = \Phi\, \mathbf{P}.$$

- Projection incorporates convolutional gridding on the sphere to afford use of FFTs
  (*cf.* gridding of continuous to discrete visibilities).
- Careful attention given to sampling densities to ensure accurate representation of band-limited signals:
  - Small FoV $\Rightarrow$ $L \simeq 2\pi B$
  - Wide FoV $\Rightarrow$ $L_{\rm FoV} \simeq 2\pi \cos(\theta_{\rm FoV}/2) B_{\rm FoV}$
- Spherical interferometric images recovered by solving the BP or TV denoising problems, replacing measurement operator Φ with its spherical equivalent $\Phi_{\rm s}$.

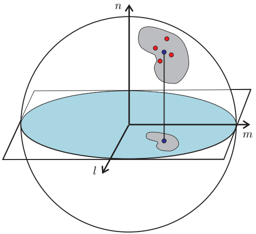
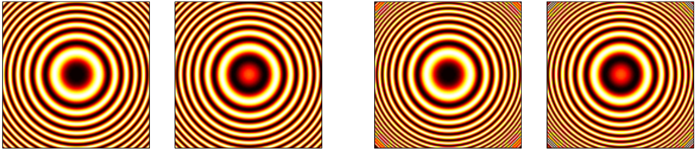


Figure: Projection operator.

Spherical interferometric imaging: advantages

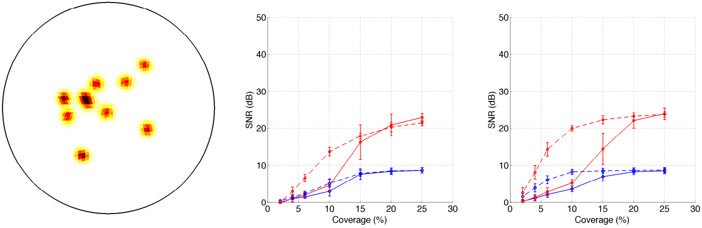
- **Enhance both sparsity and incoherence** in the wide-field spherical imaging framework.
- By recovering interferometric images on the sphere, distorting projections are eliminated and the number of samples required to represent signal is reduced $\rightarrow$ sparsity enhanced.
- Wider FoV $\rightarrow$ high frequency content in w-component modulation $\rightarrow$ more effective SS phenomenon $\rightarrow$ incoherence enhanced.
- Reconstruction fidelity improved.



(a) Assuming $\|\boldsymbol{l}\|^4\, w \ll 1$ (b) No small-field assumption

Figure: Real part and imaginary part of SS modulation for FoV $\theta_{\text{FoV}} = 90^\circ$.

Spherical interferometric imaging: reconstruction

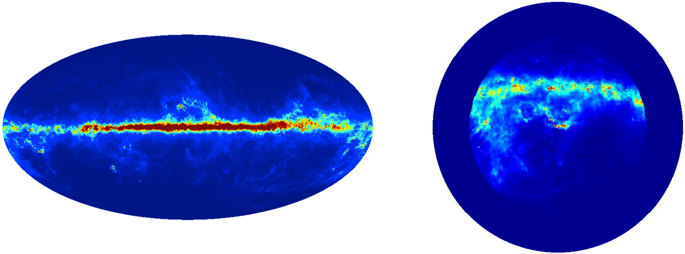


(a) Spherical image (b) $\mathrm{SNR_s}$ for BP (c) $\mathrm{SNR_s}$ for TV

Figure: Spherical interferometric imaging reconstruction performance (blue = plane; red = sphere; solid = no SS; dashed = SS).

Reconstruction of Galactic dust map

- Consider more realistic, higher resolution simulation of 94GHz FDS map of predicted submillimeter and microwave emission of diffuse interstellar Galactic dust (Finkbeiner *et al.* 1999) (available form LAMBDA website: `http://lambda.gsfc.nasa.gov`).
- Reconstruct FoV $\theta_{\mathrm{FoV}} = 90^\circ$ from 25% of visibilities.

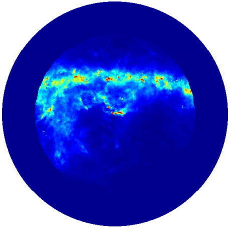


(a) Mollweide projection of full-sky

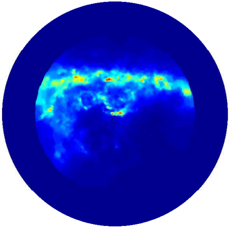
(b) Orthographic projection of FoV

Figure: FDS map of predicted emission of diffuse interstellar Galactic dust.

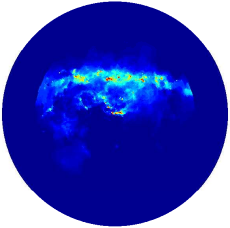
Reconstruction of Galactic dust map



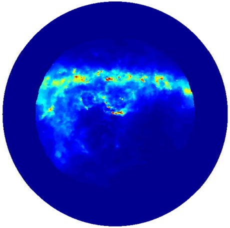
(a) Ground truth



(b) Planar reconstruction with SS (14dB)



(c) Spherical reconstruction without SS (7dB)



(d) Spherical reconstruction with SS (19dB)

Figure: Simulated TV reconstructions of diffuse FDS map.

Summary & future

- Previous works:
  - Y. Wiaux, L. Jacques, G. Puy, A. M. M. Scaife, P. Vandergheynst (2009a): Compressed sensing imaging techniques for radio interferometry
  - Y. Wiaux, G. Puy, Y. Boursier, P. Vandergheynst (2009b): Spread spectrum for imaging techniques in radio interferometry
  - JDM and Y. Wiaux (2010): Compressed sensing for wide-field radio interferometric imaging
- Current techniques idealised in order to remain as close as possible to the theoretical compressed sensing setting.
- Now that the effectiveness of these techniques has been demonstrated, it is of paramount importance to adapt them to **realistic interferometric configurations**.

Summary & future

- Visibility coverage due to real interferometric observing strategies.
- Continuous visibility coverage → incorporate a gridding operator in the measurement operator.
- Reconstruction can then be incorporated in the iterative self-calibration of radio interferometric telescopes.
- Study the spread spectrum phenomenon in the presence of varying w (using the w-projection algorithm).

(a) Realistic visibility coverage (b) Uniformly random and discrete visibility coverage

Figure: Visibility coverage.