MODIFICATION OF COULOMB LAW AND ENERGY LEVELS OF THE HYDROGEN ATOM IN A SUPERSTRONG MAGNETIC FIELD

B. Machet *(LPTHE, Paris)* & M.I. Vysotsky *(ITEP, Moscow)*

RESULTS

SCALES OF THE PROBLEM

ELECTRON IN THE "COULOMB FIELD OF A PROTON" + EXTREME B

- Resummation $\longrightarrow$ polarization operator
- the adiabatic approximation
- effective potential with screening

THE GROUND LEVEL OF THE HYDROGEN ATOM

- Results
- Principle of the method (Karnakov-Popov)

OUTLOOK AND PROSPECTS

RESULTS

e^- in the “Coulomb” field of a proton (H atom) in an $\simeq$ homogeneous, $\simeq$ constant ENORMOUS external magnetic field $B \geq 10^{17}\,G = 10^{13}\,T$.

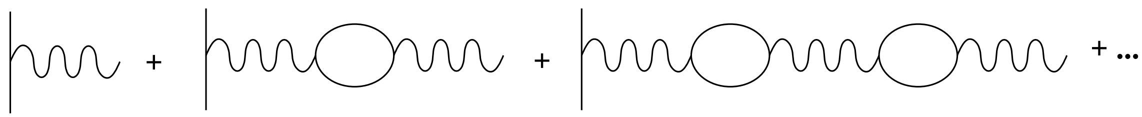


Fig. 1. *Modification of the Coulomb potential due to the dressing of the photon propagator.*

$$\longrightarrow \qquad \boxed{\Phi(z) \approx \frac{e}{z}\left[1 - e^{-m_e|z|\sqrt{6}} + e^{-m_e|z|\sqrt{6+\frac{2\alpha}{\pi}\frac{B}{B_{cr}}}}\right]}$$

The Coulomb potential becomes “screened” at short distances $\leq 1.5\frac{1}{m_e}$ when $B \geq \frac{3\pi}{\alpha}B_{cr}$.
$\longrightarrow$ energy levels of the ground state (Lowest Landau Level) in the Coulomb field of the proton

$$\boxed{\ln\left[\frac{B/B_a}{1+\frac{\alpha^3}{3\pi}B/B_a}\right] = \lambda + 2\log\lambda + 2\psi\left(1-\frac{1}{\lambda}\right) + \ln 2 + 4\gamma + \psi(1+|m|)}$$

$$\lambda = \sqrt{-E_0/E_I}, \qquad \boxed{E_I = \frac{\mu e^4}{2} = 13.6\ eV}$$

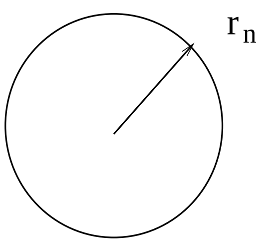
- no screening + shallow well approx: $E_0 = -E_I \ln B / B_a \stackrel{B\to\infty}{\longrightarrow} -\infty$

- screening + shallow well approx: $E_0 \stackrel{B\to\infty}{\longrightarrow} -4E_I \ln^2 \left(\sqrt{3\pi/\alpha^3}\right) \to -4\ keV$;

- screening and not shallow well: $E_0 \stackrel{B\to\infty}{=} -1.7\ keV$

SCALES OF THE PROBLEM

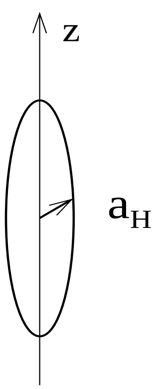
• **Bohr's atom** (no B)



$$E_n = -E_I/n^2, \quad r_n = n^2 a_B, \quad \boxed{a_B = Bohr\ radius = 1/m_e e^2 = 5.29\ 10^{-9} cm}$$

$$a_B = \lambda_c/2\pi\alpha = r_e/\alpha^2, \quad \lambda_c/2\pi = 1/m_e = 3.86\ 10^{-11} cm, \quad r_e = 2.82\ 10^{-13} cm$$

• **in the presence of** B



Landau radius $\boxed{a_H = \sqrt{1/eB}}$.

Remember:

Larmor frequency $\omega = eB/m_e$,

Larmor radius $= m_e v/eB = (L \equiv m\ \not{h})/eBr \Rightarrow r = m\ \not{h}/eBr \Rightarrow r^2 = m/eB = m a_H^2$

$$\frac{a_B^2}{a_H^2} = \frac{B}{B_a}; \quad B_a = \text{atomic B} = m_e^2 e^3 = 1/e a_B^2 = 2.35\ 10^9 G.$$

We deal with $B \approx 10^{17} G \Rightarrow \boxed{a_H \ll a_B}$. *Remember*: $B_{earth} \approx 1G, \quad B_{LHC} \approx a\ few\ 10^5 G$

$$B_{cr}\ or\ B_{Schwinger} = m_e^2/e = B_a/\alpha^2 = 4.35\ 10^{13} G.$$

For $B \gg B_{cr}$, electromagnetic fields start to interact with each other: non-linearity; e^- becomes relativistic.

For us $\quad B \gg B_{cr} \gg B_a$

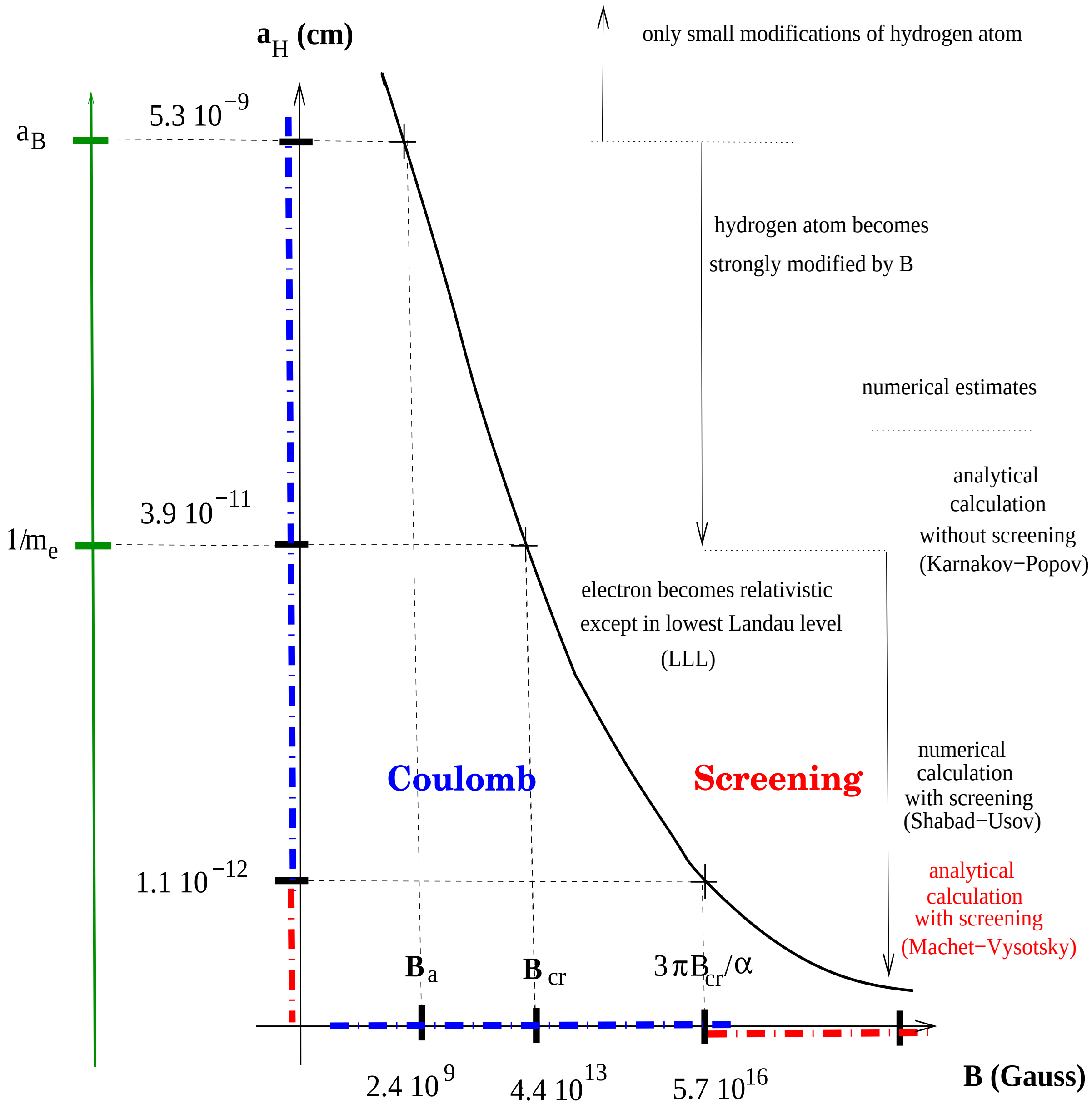


Fig. 2. *Landau radius a_H versus magnetic field B.*

ELECTRON IN THE "COULOMB FIELD OF A PROTON" + EXTREME B

Very strong $B \longrightarrow$ fast oscillations $\perp B$, slow oscillations $\parallel B \Rightarrow$ adiabatic approximation $\boxed{\psi^{LLL}_{n,n_\rho=0,m,\sigma_3=-1} = R_{0m}(\vec{\rho})\chi_n(z)}$

e^- relativistic except in the LLL $\longrightarrow$ solve non-relativistic Schrœdinger equation for $\chi(z)$ (1-dim) with effective potential U_{eff} incorporating the resummation of Fig. 1.

$$\boxed{\left[-\frac{1}{2m_e}\frac{d^2}{dz^2} + U_{eff}\right]\chi_n(z) = E_n\chi_n(z)}$$

We have to determine what is $U_{eff}(z)$.

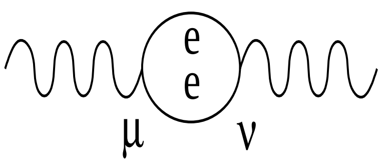
Remember; Coulomb: $\Phi_c(z) = e/z, \quad U_c(z) = -e^2/z.$

• **If no screening**, U_{eff} is given by Landau: $\quad U_{eff} = -e^2\int \frac{|R_{0m}(z)|^2}{\sqrt{z^2+\rho^2}}\, d^2\rho; \quad \stackrel{z\gg a_H}{\rightarrow} -e^2/z; \quad \stackrel{z\to 0}{\rightarrow} -e^2/a_H \; finite$

• **Including screening**

$\Phi(k) \stackrel{no\ resum}{=} -4\pi g/k^2 \stackrel{resum}{\longrightarrow} \dfrac{-4\pi g}{k^2+\Pi(k^2)} = \dfrac{-4\pi g}{k_\parallel^2 + k_\perp^2 + \Pi(k^2)}.$

Remember: in 4D, $\Pi_{\mu\nu} = -(g_{\mu\nu} - k_\mu k_\nu/k^2)\Pi(k^2)$.



$= \Pi_{\mu\nu}$. Static charge $\rightarrow$ we need only Π_{00}.

$$\boxed{\Phi(k) = \frac{4\pi g}{k_\parallel^2 + k_\perp^2 + \dfrac{2\alpha}{\pi}\dfrac{B}{B_{cr}} m_e^2\, e^{\left(\frac{-k_\perp^2}{2m_e^2}\frac{B_{cr}}{B}\right)} P\left(\dfrac{k_\parallel^2}{4m_e^2}\right)}}$$

$\Pi(k^2) = -4g^2 P(t), \quad t = \dfrac{-k^2}{4m_e^2}$. P is the same as in 2 D: $\boxed{P(t) = 1 - \dfrac{\ln(\sqrt{1+t}+\sqrt{t})}{\sqrt{t(1+t)}}}$.

Fourier transform $\longrightarrow \Phi(z)$. But exact P too cumbersome to integrate $\longrightarrow$ use approximate $\boxed{\overline{P}(t) = \frac{2t}{3+2t}}$.

$$\boxed{\Phi(z) \approx \frac{e}{z}\left[1 - e^{-m_e|z|\sqrt{6}} + e^{-m_e|z|\sqrt{6+\frac{2\alpha}{\pi}\frac{B}{B_{cr}}}}\right]}$$

* for $B \ll 3\pi B_{cr}/\alpha$, $\quad \Phi(z) = \frac{e}{|z|}\left(1 + \mathcal{O}(\alpha B/B_{cr})\right)$
* for $B \gg 3\pi B_{cr}/\alpha$

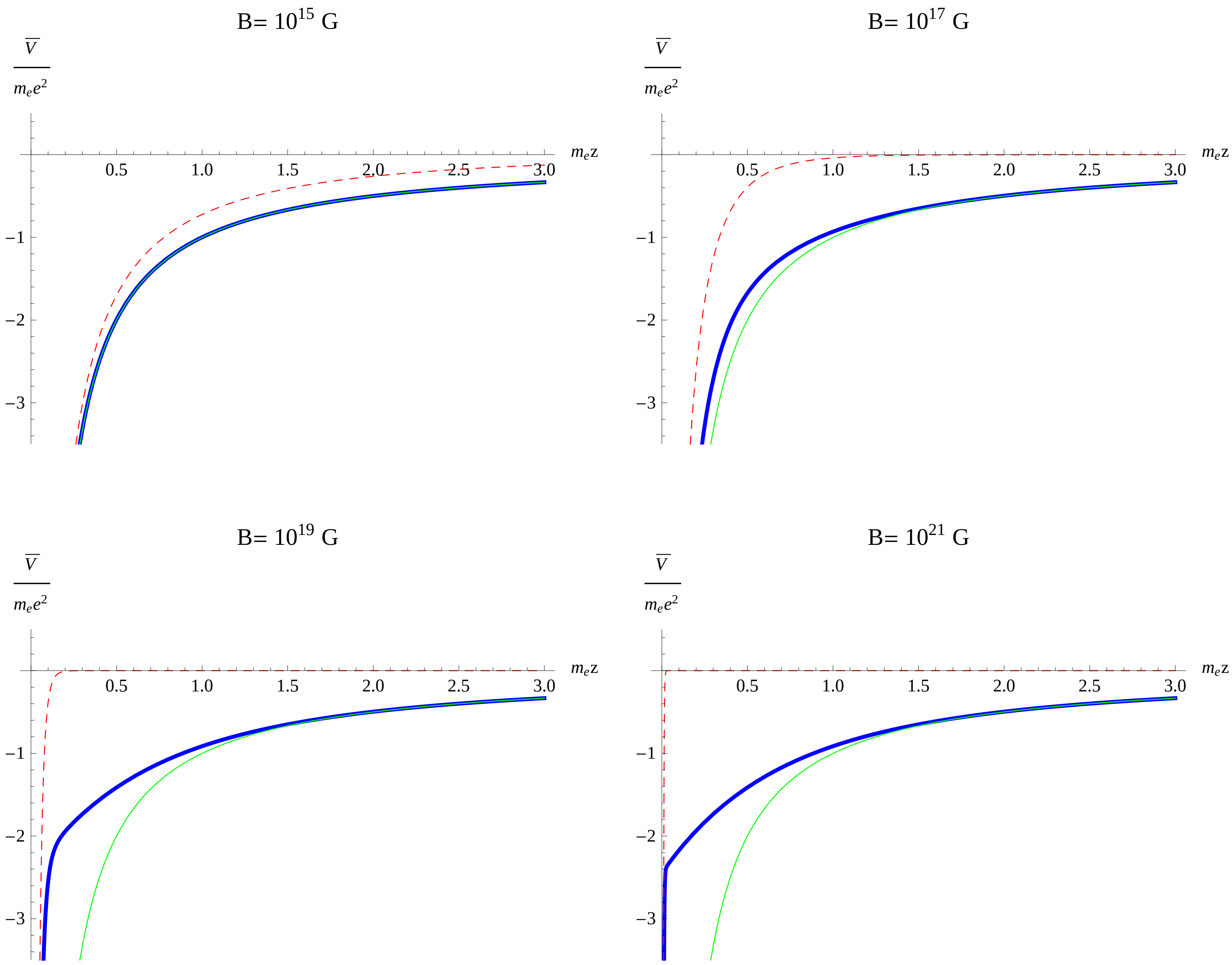
$$\Phi(z) = \begin{cases} \frac{e}{|z|}\, e^{-m_e|z|\sqrt{\frac{2\alpha}{\pi}\frac{B}{B_{cr}}}} & for \quad |z| < \frac{1}{\sqrt{\frac{2\alpha}{\pi}\frac{B}{B_{cr}}}}\frac{1}{m_e}\ln(\sqrt{\frac{\alpha}{3\pi}\frac{B}{B_{cr}}}), \quad screening \\ \frac{e}{|z|}\left(1 - e^{-m_e|z|\sqrt{6}}\right) & for \quad \frac{1}{m_e} > |z| > \frac{1}{\sqrt{\frac{2\alpha}{\pi}\frac{B}{B_{cr}}}}\frac{1}{m_e}\ln(\sqrt{\frac{\alpha}{3\pi}\frac{B}{B_{cr}}}) \\ \frac{e}{|z|} & for \quad |z| > \frac{1}{m_e}, \quad Coulomb \end{cases}$$

$$\boxed{U_{eff}(z) = -e^2 \int d^2\rho\, \frac{|R_{0m}(\rho)|^2}{\sqrt{z^2+\rho^2}}\left[1 - e^{-m_e|z|\sqrt{6}} + e^{-m_e|z|\sqrt{6+\frac{2\alpha}{\pi}\frac{B}{B_{cr}}}}\right]}$$

$R_{0m}(\rho) = \frac{1}{\sqrt{\pi(2a_H^2)^{(1+|m|)}|m|!}}\rho^{|m|}e^{im\rho}e^{-\rho^2/4a_H^2}$ = wave function of e^- in magnetic field (Landau Q.M.)

At short distance, the photon gets an effective mass $m_\gamma^2 = \alpha\frac{B}{B_{cr}}m_e^2 = e^3 B$ *(2-dim massive QED, Schwinger model).*

$$\Phi(z) \longrightarrow \textit{potential energy } V(z) = -e\Phi(z)$$



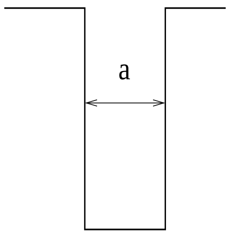
Modified Coulomb potential energy at $B = 10^{17}G$ (blue) and its long distance (green-pale) and short distance (red-dashed) asymptotics.

THE GROUND LEVEL OF THE HYDROGEN ATOM

- **no screening + shallow well approximation**

$$E_0 = -E_I \ln \frac{B}{B_a} \overset{B\to\infty}{\longrightarrow} -\infty \; (\text{“}fall\ to\ the\ center''\text{)}$$

- **shallow well approximation (Landau) + screening**



$$E_0 = -2m_e \left[\int_{a_H}^{a_B} U(z)\, dz \right]^2 \overset{B\to\infty}{\longrightarrow} -2m_e\alpha^2 \ln^2 \left(\sqrt{\frac{3\pi}{\alpha^3}} \right) = -4E_I \times 72.3 = -4\ keV$$

shallow well approximation valid for $m_e|U|a^2 \ll 1$; for 1-D Coulomb potential: $a \ll a_B$, not realized because Coulomb extends to ∞.

- **general case with screening (Karnakov-Popov)**

$$\boxed{E_0 \overset{B\to\infty}{\longrightarrow} -1.7\ keV}$$

- $*$ even/odd degeneracy is broken
- $*$ corrections to odd levels very small
- $*$ corrections to even levels substantial for $B \geq 3\pi \frac{B_{cr}}{\alpha} = 6\ 10^{16} G$

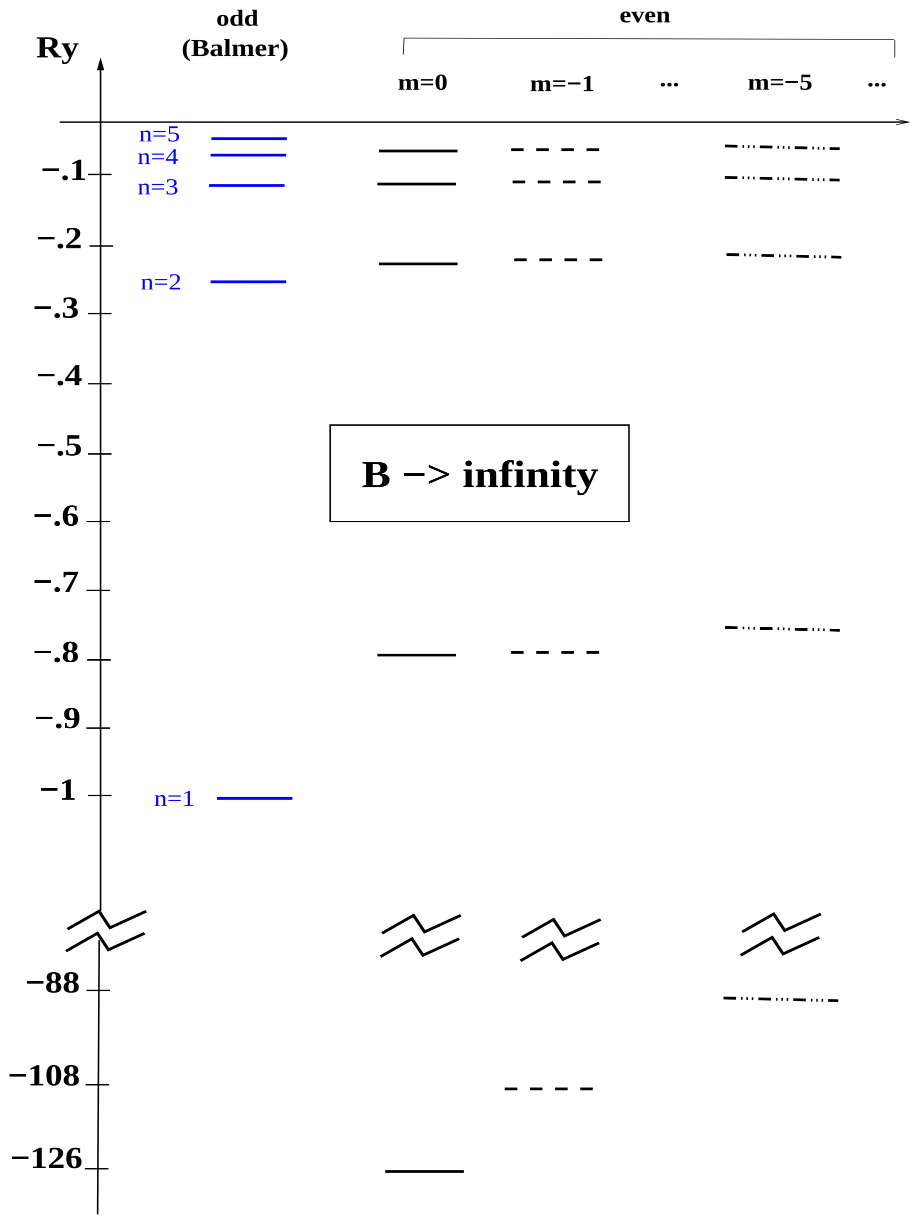


Fig. 5. *Spectrum of hydrogen levels in the limit of infinite magnetic field. Energies are given in rydberg units,* $Ry \equiv 13.6\ eV$.

$$\boxed{\ln\left[\frac{B/B_a}{1+\frac{\alpha^3}{3\pi}B/B_a}\right] = \lambda + 2\log\lambda + 2\psi\left(1-\frac{1}{\lambda}\right) + \ln 2 + 4\gamma + \psi(1+|m|)}$$

$*$ **if no screening**: $\Rightarrow \lambda \equiv \sqrt{-E_0/E_I} \xrightarrow{B\to\infty} \infty$.
When $B\to\infty$, l.h.s $\to\infty \Rightarrow \psi\left(1-1/\lambda\right)$ has poles at $1-1/\lambda = 0,-1,-2,\ldots-n+1 \Rightarrow \lambda = 1/n$. One recovers the Balmer series

$*$ **with screening**: when $B\to\infty$, $\ln\frac{3\pi}{\alpha^3} = r.h.s.$. No pole $\Rightarrow$ departure from Balmer series.

For $m=0, B\to\infty$, $\boxed{\lambda = \frac{1}{1} - \frac{2}{\ln\left[1+\frac{B/B_a}{1+\frac{\alpha^3}{3\pi}\frac{B}{B_a}}\right] - \ln 2 - \gamma + 1}}$

For arbitrary n and $m=0$, $\boxed{\lambda = \frac{1}{n} - \frac{2/n^2}{\ln\left[1+\frac{B/B_a}{1+\frac{\alpha^3}{3\pi}\frac{B}{B_a}}\right] - \ln 2 - \gamma + 1/n + 2\ln n - 2H_{n-1}}}$

$$H_{n-1} = \text{Harmonic number} = \Sigma_{k=1}^{n-1}\frac{1}{k}$$

- **Principle of the method (Karnakov-Popov)**

If $\chi(z) \approx cst$, integrate Shrœdinger $\longrightarrow$

$$\chi'(z) \approx 2m_e \int_0^z dt\, U(t)\chi(t)$$

(valid for $z \ll a_B$; (self energy negligible).

At the limit of the screening domain, χ and Whittaker (Coulomb) should match $\Rightarrow \frac{\chi'}{\chi} \overset{z>few\ 1/m_e}{\approx} \frac{W'}{W}$

- **Validity** of K.P. method : $\chi(z) \approx cst$. It is achieved, in particular thanks to the screening.

OUTLOOK AND PROSPECTS

- accuracy of adiabatic approximation
- 3D potential $V(z, \vec{\rho} \neq 0)$
- beyond LLL; relativistic e^- (not very physical)
- effective mass of $e^- \longrightarrow E_0 = m_e \left[1 - \frac{e^4}{2}\lambda^2 + \frac{e^2}{4\pi} \ln^2\left(\frac{eB}{m_e^2}\right)\right]$; corrections can be large but are universal, they cancel in transitions
- trace levels when B goes from 0 to ∞ (Zeldovich effect)
- string-like behavior when $B \to \infty$ (Kondo, Shabad-Usov)
- exciton physics in semi-conductors. $B_a \to 2000\ G$ for InSb $\Rightarrow B_{cr} \sim \frac{B_a}{\alpha^2} \approx 10^7 G$
- graphene : 2D QED for massless fermions. $c \to v_f \ll c \Rightarrow \alpha \to \frac{e^2}{v_f} \gg 1/137$.

Same physics at much smaller B ?