Unusual singular behaviour of the Entanglement Entropy in one dimension

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Collaboration with

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arXiv:1008.3892 *and* **work in progress...**

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Outline

- Introduction:
  - Entanglement in Quantum Mechanics
  - Von Neumann and Renyi entropies as a measure of Entanglement
- Entanglement entropy in 1D lattice spin chains: the Corner Transfer Matrix (CTM) method
- XYZ chain exact Entanglement Entropy
- Essential critical point for the entropy
- Conclusions

Why Entanglement?

- Classical computing $\longrightarrow$ Boolean Logic $\longrightarrow$ Bits
- Quantum Information $\longrightarrow$ Q-bits $\longrightarrow$ Entanglement
- How to define and measure it?
    - Von Neumann and Renyi entropies
- New challenges for our understanding of Nature
    - EPR paradox
    - Bell inequalities
    - Interpretation of Quantum Mechanics

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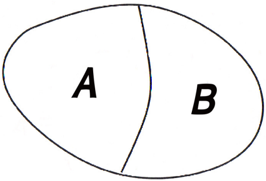
Quantum systems and sub-systems

- Consider a quantum system (e.g. a 1D quantum spin chain) in a pure state $|\psi\rangle$, whose density matrix is $\rho = |\psi\rangle\langle\psi|$.
- Divide the system into two subsystems, A and B.
  The Hilbert space then separates into two parts

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

- Suppose to do separated measures on each subsystem



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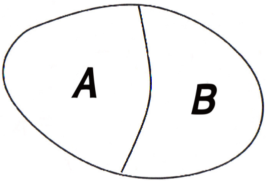

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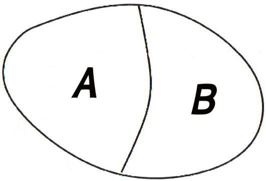

Separable and entangled states

- States that can be written as $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ are called separable
  In this case measurements on B do not affect A
- Not all states are separable

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- Generic state in $\mathcal{H}$

$$|\psi\rangle = \sum_{j=1}^{d} \lambda_j |j_A\rangle \otimes |j_B\rangle$$

with $d > 1$, $|j_A\rangle, |j_B\rangle$ linearly independent
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Observers and measures

- In subsystems A and B we have observers capable of doing measures on their subsystem only
- Consider two spins 1/2
  - $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ no entanglement
  - $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle)$ maximally entangled: measures in A affect those in B.
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Density Matrix of state $|\psi\rangle$ (Von Neumann 1927)

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Quantum entropy (Von Neumann) of Entanglement

$$S_A = -\mathrm{Tr}_A(\rho_A \log \rho_A) = S_B$$

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Von Neumann Entropy

- Quantum analog of Shannon Entropy

$$\rho_A = \sum_j \lambda_j |j_A\rangle\langle j_A| \qquad \Longrightarrow \qquad S_A = -\sum_j \lambda_j \log \lambda_j$$

Measures the amount of information in the given state

- Schumacher's theorem: information in a state seen by A can be compressed in a e^{S_A} set of Q-bits

- Bell states (maximally entangled) as unities of Entanglement

$$|\mathrm{Bell}\,1\rangle = \frac{|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle}{\sqrt{2}} \quad , \quad \mathrm{Bell}\,2\rangle = \frac{|\downarrow\downarrow\rangle - |\uparrow\uparrow\rangle}{\sqrt{2}}$$

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S measures how many Bell pairs are contained in a given state $|\psi\rangle$, i.e. closeness of the state to maximally entangled one.

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Other Entanglement estimators

- Renyi entropy

$$S_\alpha = \frac{1}{1-\alpha} \log \mathrm{Tr}_A \rho_A^\alpha$$

  - It reduces to Von Neumann for $\alpha \to 1$
  - Contains higher momenta and for $\alpha \to \infty$ the spectrum of the reduced density matrix ρ_A can be read
  - link with replica trick à la Calabrese Cardy
- Tsallis Entropy
- Concurrence
- ...

Lattice models

- Consider a square lattice with IRF. To each site i assign a spin σ_i and to each plaquette delimited by sites i, j, k, l Boltzmann weights

$$w(\sigma_i, \sigma_j, \sigma_k, \sigma_l) = \exp\{-\epsilon(\sigma_i, \sigma_j, \sigma_k, \sigma_l)/kT\}$$

- Total energy of the system

$$\mathcal{E} = \sum_{\square} \epsilon(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$$

the sum is over all plaquettes (faces) of the lattice and i, j, k, l are the surrounding sites. The partition function is

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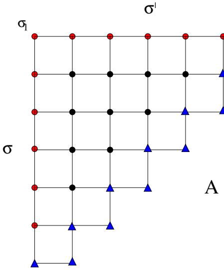

- Define the element of the Corner Transfer Matrix (CTM) as

$$A_{\bar{\sigma}\bar{\sigma}'} = \begin{cases} \sum_{\bullet}\prod_{\square} w(\sigma_i, \sigma_j, \sigma_k, \sigma_l) & \text{if} \quad \sigma_1 = \sigma_1' \\ = 0 & \text{if} \quad \sigma_1 \neq \sigma_1' \end{cases}$$

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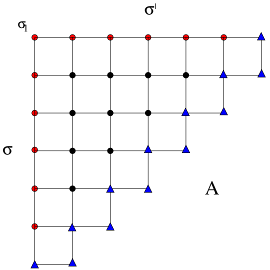

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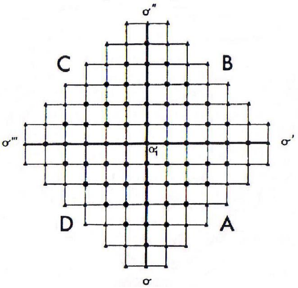
- Define $B_{\bar{\sigma}\bar{\sigma}'}$ in the same way as $A_{\bar{\sigma}\bar{\sigma}'}$ only with the last figure rotated anticlockwise by 90°. Similarly define $C_{\bar{\sigma}\bar{\sigma}'}$ and $D_{\bar{\sigma}\bar{\sigma}'}$ by rotating by 180° and 270°.
- Now we can build up the whole lattice by using the 4 CTM's



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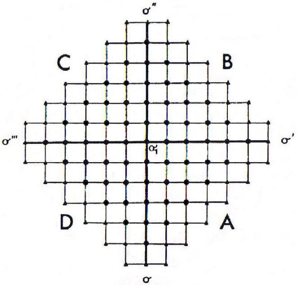

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Density matrix and corner transfer matrix I

- Quantum spin chain with L sites, Hamiltonian H and ground state $|0\rangle$. Vacuum wave function $\langle\bar{\sigma}|0\rangle = \psi_0(\bar{\sigma})$. Density matrix $\rho = |0\rangle|\langle 0|$.
- Matrix element (assume ψ_0 real)

$$\rho(\bar{\sigma}, \bar{\sigma}') = \langle\bar{\sigma}|0\rangle\langle 0|\bar{\sigma}'\rangle = \psi_0(\bar{\sigma})\, \psi_0(\bar{\sigma}')$$

- Suppose there is a relation between this quantum chain of hamiltonian H and a classical spin lattice model of row to row transfer matrix T in the sense that $[H, T] = 0$
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Density matrix and CTM

- Consider a vector $|\psi\rangle \in \mathcal{H}$ Hilbert space of H (or of T)

$$|\psi\rangle = |0\rangle + \sum_{k\neq 0} c_k |k\rangle$$

where $|k\rangle$ are the excited states of H with T eignevalues λ_k.

- Apply N times the operator T to such vector

$$T^N|\psi\rangle = \lambda_0^N \left(|0\rangle + \sum_k \left(\frac{\lambda_k}{\lambda_0}\right)^N c_k |k\rangle \right)$$

- In the limit $N \to \infty$

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i.e. $\psi_0(\bar\sigma)$ is the partition function evolving the model from an initial $|\bar\sigma\rangle$ to a final $|0\rangle$and $\rho(\bar\sigma, \bar\sigma')$ is a product of two semi-infinite partition functions evolving the system from $\bar\sigma$ to $+\infty$ and from $\bar\sigma'$ to $-\infty$.

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Reduced density matrix and CTM

- Now suppose to divide the spins in two subsystems A: $\bar{\sigma}_A = (\sigma_1, ..., \sigma_p)$ and B: $\bar{\sigma}_B = (\sigma_{p+1}, ..., \sigma_L)$, i.e. $\bar{\sigma} = (\bar{\sigma}_A, \bar{\sigma}_B)$
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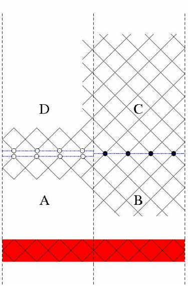

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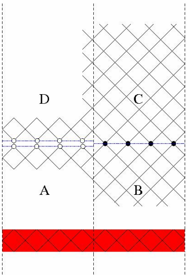

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Normalization by dividing by the trace

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$$S_A = -\mathrm{Tr}\rho_A \log \rho_A = -\mathrm{Tr}\frac{\hat{\rho}_A \log \hat{\rho}_A}{\mathrm{Tr}_A \hat{\rho}_A} + \mathrm{Tr}_A \hat{\rho}_A$$

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XYZ model

- Hamiltonian

$$H_{XYZ} = -J \sum_k (\sigma_k^x \sigma_{k+1}^x + \Gamma \sigma_k^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z)$$

  commutes with transfer matrix of 8-vertex model

  - for $\Gamma = 1$ it gives XXZ model
  - for $\Gamma = 1$, $\Delta = 1$ ferromagnetic XXX
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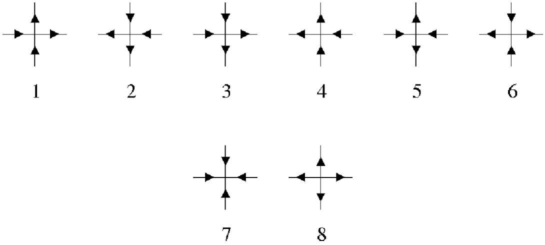
XYZ model

- XYZ is the hamiltonian limit of 8-vertex model, with partition function

$$Z = \sum \prod_{i=1}^{8} w_i^{n_i}$$

where the 8 Boltzmann weights $w_i = e^{-\beta \epsilon_i}$ appear n_i times each on the lattice.



$$w_1 = w_2 = a,\ w_3 = w_4 = b,\ w_5 = w_6 = c,\ w_7 = w_8 = d$$

Transfer matrix of 8-vertex

- Square lattice with M rows and N columns with periodic b.c. The vertical 8-vertex variables $t_i = \uparrow, \downarrow$ and the horizontal ones $s_j = \rightarrow, \leftarrow$ live on the links.
- Denote a row of arrows $\phi_r = (t_1, t_2, ..., t_N)$ $(r = 1...M)$. Row-to-row transfer matrix

$$T(\phi, \phi') = \prod_{n=1}^{N} w \begin{pmatrix} & t'_n & \\ s_n & & s_{n+1} \\ & t_n & \end{pmatrix}$$

can be diagonalized by Bethe ansatz (Baxter)
- The partition function is

$$Z = \prod_{r=1}^{M} T(\phi_r, \phi_{r+1})$$

- This can be generalized to nontrivial b.c. by the introduction of suitable double row transfer matrix

Transfer matrix of 8-vertex

- Square lattice with M rows and N columns with periodic b.c. The vertical 8-vertex variables $t_i = \uparrow, \downarrow$ and the horizontal ones $s_j = \rightarrow, \leftarrow$ live on the links.
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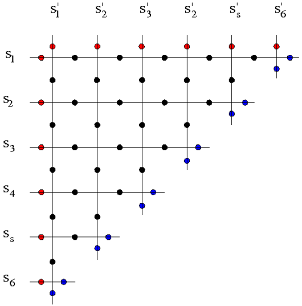
- CTM is defined with a slight modification w.r.t. the IRF models. There is no common spin on the two edges



$$A_{\bar{s},\bar{s}'} = \sum_{\bullet} \prod w_i$$

- and analogously B, C, D with 90° rotations. One can prove that $A = C$ and $B = D$.

Elliptic parametrization

- A convenient parametrization of the Boltzmann weights

$$\begin{aligned} a &= \rho\,\mathrm{snh}(\lambda - u) \\ b &= \rho\,\mathrm{snh}\,u \\ c &= \rho\,\mathrm{snh}\,\lambda \\ d &= \rho\,k\,\mathrm{snh}\lambda\,\mathrm{snh}u\,\mathrm{snh}(\lambda - u) \end{aligned}$$

- In this parametrization ($\mathrm{snh}x = -i\,\mathrm{sn}\,ix$, etc...)

$$\Gamma = \frac{1 - k^2\mathrm{snh}^2\lambda}{1 + k^2\mathrm{snh}^2\lambda} \quad , \quad \Delta = -\frac{\mathrm{cnh}\lambda\,\mathrm{dnh}\lambda}{1 + k^2\mathrm{snh}^2\lambda}$$

- Phases:
  - ferroelettric order for $a > b + c + d$, $\Delta > 1$
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  - disorder for $a, b, c, d < \frac{1}{2}(a + b + c + d)$, $-1 < \Delta < 1$

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- In the thermodynamic limit Baxter (1977) proved the following formula for the diagonalized CTM

$$A_d(u) = C_d(u) = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & s^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & s^3 \end{pmatrix} \otimes \dots$$
$$B_d(u) = D_d(u) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & t^3 \end{pmatrix} \otimes \dots$$

where

$$s = \exp\left(-\frac{\pi u}{2I(k)}\right) \quad , \quad t = \exp\left(-\frac{\pi(\lambda - u)}{2I(k)}\right)$$

and $I(k)$ is the elliptic integral of I kind of modulus k

Reduced density matrix

- Define $x = (st)^2 = \exp\left(-\frac{\pi\lambda}{I(k)}\right)$ and use the CTM density matrix formula

$$\rho_A = ABCD = (AB)^2 = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & x^3 \end{pmatrix} \otimes \ldots$$

- $\rho = e^{\epsilon \mathcal{O}}$ where $\mathcal{O}$ is a operator with integer spectrum

$$\mathcal{O} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \otimes \ldots$$

$\epsilon = -\frac{\pi\lambda}{I(k)}$ depends on the XYZ parameters through elliptic functions

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Entanglement entropy of XYZ model

The trace of the reduced density matrix

$$\mathcal{Z} = \mathrm{Tr}\rho_A = \prod_{j=1}^{\infty}(1 + x^j) \qquad \text{and} \qquad S_A = -\epsilon \frac{\log \mathcal{Z}}{\partial \epsilon} + \log \mathcal{Z}$$

leads to the final formula for Von Neumann

$$S_A = \epsilon \sum_{j=1}^{\infty} \frac{j}{(1 + e^{j\epsilon})} + \sum_{j=1}^{\infty} \log(1 + e^{-j\epsilon})$$

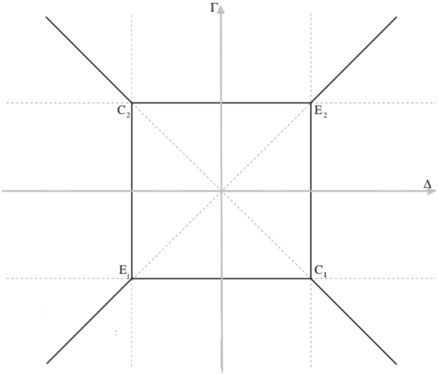
and for Rényi entropy

$$S_\alpha = \frac{\alpha}{\alpha - 1} \sum_{j=1}^{\infty} \log(1 + q^{2j}) + \frac{1}{1 - \alpha} \sum_{j=1}^{\infty} \log(1 + q^{2j\alpha})$$

that can also be written in theta function terms

$$S_\alpha = \frac{1}{6(1 - \alpha)} \left[\alpha \log \frac{\theta_4(0, q)\theta_3(0, q)}{\theta_2^2(0, q)} + \log \frac{\theta_2^2(0, q^\alpha)}{\theta_3(0, q^\alpha)\theta_4(0, q^\alpha)} \right]$$

Phase diagram of XYZ model

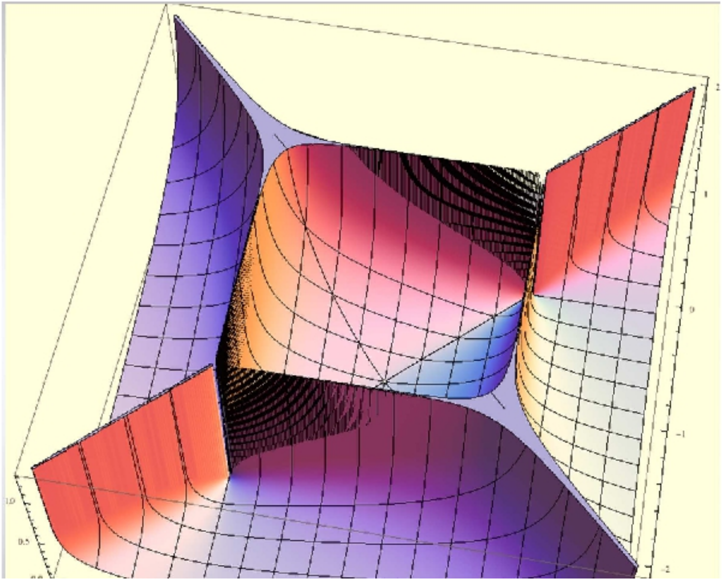


Approaching criticality the Calabese - Cardy (2004) formula holds

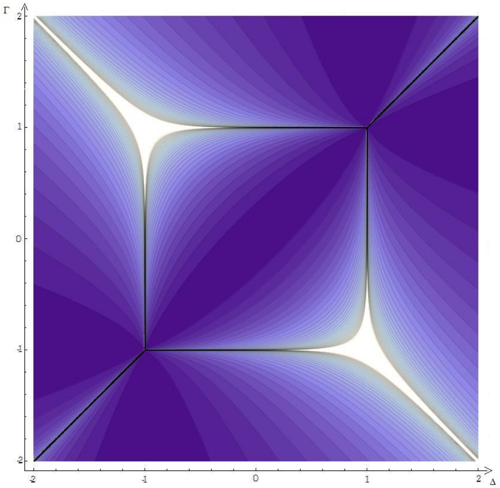
$$S_A = \frac{c}{6} \log \frac{\xi}{a} + \mathrm{cost.}$$

everywhere but at the $E_{1,2}$ points

Entanglement Entropy 3D plot



Isoentropic lines



Tricritical points

- $C_{1,2}$: conformal points - entropy diverges close to them - linear spectrum
- $E_{1,2}$: Non-conformal points - entropy goes from 0 to ∞ arbitrarily close to them, depending on direction.
  They corrspond to Isotropic ferromagnetic Heisenberg $\longrightarrow$ quadratic spectrum
- Points similar to $E_{1,2}$ previously observed in XY model in magnetic field (Franchini, Its, Korepin)

Conformal points

Expansion close to conformal points $C_{1,2}$ agree with expectations

$$
\begin{aligned}
S_\alpha &= \frac{1}{12}\left(1+\frac{1}{\alpha}\right)\log\xi - \frac{1}{6}\left(2-\frac{1}{\alpha}\right)\log 2 \\
&+ \frac{\alpha}{1-\alpha}\left[\frac{\xi^{-2}}{16}+\frac{\xi^{-4}}{512}+O(\xi^{-6})\right] \\
&- \frac{1}{1-\alpha}\left[(4\xi)^{-2/\alpha}+\frac{1}{2}(4\xi)^{-4/\alpha}+O(\xi^{-6/\alpha})\right]
\end{aligned}
$$

Leading correction $\xi^{-\delta/\alpha}$ with $\delta = 2$. Operator responsible of this correction (Calabrese, Cardy, Peschel - 2010) has conformal dimensions $(\Delta, \bar{\Delta}) = (1, 1)$

Non-conformal points

Expanding around E_1:

$$\Gamma = -1 + \delta \cos\phi \qquad , \qquad \Delta = -1 - \delta \sin\phi \qquad \left(0 \leq \phi \leq \frac{\pi}{2}\right)$$

one finds

$$\lambda \sim I(k') \qquad \text{and} \qquad \varepsilon = \frac{I(k')}{I(k)}$$

So ε varies from 0 at $\phi = 0$ to ∞ at $\phi = \frac{\pi}{2}$. Consequently the entropy explores all values from 0 to ∞ approaching E_1 from various directions $\Longrightarrow$ essential singularity.

- Highly symmetric point, higly degenerate ground state $\Longrightarrow$ level crossing, entanglement can change discontinously
- EE can be used as a marker to detect such essential phase transition points
- Cardy-Calabrese formula is non longer valid: what substitues it?

Conclusions

- We have got Von Neumann and Rényi EE from integrability in the XYZ spin chain, valid everywhere
- It can be written in nice modular form (theta functions) and its modular properties should be investigated further
- Inspecting this formula near critical points, we have discovered essential singularities with unusual critical behaviour
- EE can be used as a marker to discriminate behaviours of phase transition points.
- An approach taking into account finite size effects would help to clarify these issues