Lattice QCD’s flavour physics results for phenomenologists - a mini review

Rencontres de Moriond

La Thuile, Italy

13.-20.03.2011

Andreas Jüttner
CERN Theory Division
Introduction

Why lattice QCD?

- perturbation theory works well for weak coupling
- bound state observables like the proton mass or the $B$-decay constant for example cannot be predicted by perturbation theory
Introduction

Why lattice QCD?

- perturbation theory works well for weak coupling
- bound state observables like the proton mass or the $B$-decay constant for example cannot be predicted by perturbation theory

but simulations of lattice QCD can do this:

![Graph](image-url)
**Introduction**

**What is ...?**

<table>
<thead>
<tr>
<th></th>
<th>QCD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_c$</td>
<td>3</td>
</tr>
<tr>
<td>$N_f$, fundamental</td>
<td>1+1+1+1+1+1</td>
</tr>
<tr>
<td>$SU(2)$ iso-spin brk.</td>
<td>✓</td>
</tr>
<tr>
<td>$m_{\pi}^{\pm}$</td>
<td>135MeV</td>
</tr>
<tr>
<td>$V$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$a$</td>
<td>0</td>
</tr>
</tbody>
</table>
## What is ...?

<table>
<thead>
<tr>
<th></th>
<th>QCD</th>
<th>Lattice QCD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_c$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$N_f$, fundamental</td>
<td>$1+1+1+1+1$</td>
<td>$0, 2, 2+1, 2+1+1$</td>
</tr>
<tr>
<td>$SU(2)$ iso-spin brk.</td>
<td>✓</td>
<td>$\times$</td>
</tr>
<tr>
<td>$m_\pi^\pm$</td>
<td>$135\text{MeV}$</td>
<td>$\lesssim m_\pi^{\text{sim}}$</td>
</tr>
<tr>
<td>$V$</td>
<td>$\infty$</td>
<td>$2-3\text{fm}$</td>
</tr>
<tr>
<td>$a$</td>
<td>0</td>
<td>$0.05-0.1\text{fm}$</td>
</tr>
</tbody>
</table>

*see Taku’s talk*
dynamical simulations are standard by now
first simulations now include dynamical charm quark
simulations with physical pion masses have become feasible
\(a^{-1} = 1/0.05 \text{fm} \approx 4 \text{GeV}\)

Plot kindly provided by G. Herdoiza
Simulations

- discretise space-time in euclidean QCD path integral

\[
\langle O[\bar{\psi}, \psi, A] \rangle_{\text{QCD}} = \frac{1}{Z} \int D\bar{\psi} D\psi DA O(\bar{\psi}, \psi, A) e^{-S_G(U) - S_q(\bar{\psi}, \psi, U)}
\]

and do numerical MC-integration
(discretisations: Wilson, staggered, twisted mass, domain wall, overlap, ...)
Simulations

- discretise space-time in euclidean QCD path integral

\[
\langle O[\bar{\psi}, \psi, A] \rangle_{\text{QCD}} = \frac{1}{Z} \int D\bar{\psi} D\psi DA O(\bar{\psi}, \psi, A) e^{-S_{G}(U) - S_{q}(\bar{\psi}, \psi, U)}
\]

and do numerical MC-integration
(discretisations: Wilson, staggered, twisted mass, domain wall, overlap, \ldots)

- extremely compute-intensive, so project time-scales of \(O(\text{years})\) even on fastest super-computers
Simulations

- discretise space-time in euclidean QCD path integral

\[ \langle O[\bar{\psi}, \psi, A] \rangle_{\text{QCD}} = \frac{1}{Z} \int D\bar{\psi} D\psi DA O(\bar{\psi}, \psi, A) e^{-S_G(U) - S_q(\bar{\psi}, \psi, U)} \]

and do numerical MC-integration
(discretisations: Wilson, staggered, twisted mass, domain wall, overlap, . . . )

- extremely compute-intensive, so project time-scales of \( O(\text{years}) \) even on fastest super-computers

- biggest machine is good but \( \neq \) best lattice QCD
even more important are fundamental understanding and usage of
  - field theory
  - algorithms

IBM BlueGene/Q
Systematics

- most current results systematics-dominated
- extrapolation of lattice data to the physical point very often tricky (it’s easy if you already know the experimental number ... but what if not?)
Systematics

- most current results systematics-dominated
- extrapolation of lattice data to the physical point very often tricky (it’s easy if you already know the experimental number ... but what if not?)

\[ a \rightarrow 0 \quad \text{Symanzik eff. th.} \]

\[ (a/r_0)^2 \]

Systematics

- most current results systematics-dominated
- extrapolation of lattice data to the physical point very often tricky (it’s easy if you already know the experimental number ... but what if not?)

\[ a \to 0 \quad \text{Symanzik eff. th.} \]

\[
\frac{r_0}{D_s} \propto (a/r_0)^2
\]

Systematics

- most current results systematics-dominated
- extrapolation of lattice data to the physical point very often tricky (it’s easy if you already know the experimental number ... but what if not?)

\[ a \to 0 \quad \text{Symanzik eff. th.} \]

\[ m_q \to m_q^{\text{phys}} \quad \text{chiral eff. th.} \]
Systematics continued

- renormalisation
  (Often NLO lattice PT - that’s quick and dirty. Lattice allows to do it non-perturbatively, so why not use it? In this way all reference to perturbation theory can be removed!!)
Systematics continued

- renormalisation
  (Often NLO lattice PT - that’s quick and dirty. Lattice allows to do it non-perturbatively, so why not use it? In this way all reference to perturbation theory can be removed!!!)

- hopefully not a show-stopper: reducing $a$ beyond $\approx 0.06\text{fm}$ turns out to be problematic

  critical slowing down of algorithms

  not clear which observables affected

  *I think that this needs to be studied more thoroughly than many collaborations wish to believe*
Systematics continued

- renormalisation
  (Often NLO lattice PT - that’s quick and dirty. Lattice allows to do it non-perturbatively, so why not use it? In this way all reference to perturbation theory can be removed!!!)

- hopefully not a show-stopper: reducing $a$ beyond $\approx 0.06\text{fm}$ turns out to be problematic

  critical slowing down of algorithms

  not clear which observables affected

  *I think that this needs to be studied more thoroughly than many collaborations wish to believe*

- scale setting
- finite size errors
- chosen discretisation
- ...
### Lattice phenomenology

| Experiment | $\Delta M_d$ | $=\begin{align*} SM \perp NPQCD & \times NPQCD \\ \text{const.} \times |V_{tb} V_{td}|^2 S\left(\frac{\bar{m}_t^2}{M_W^2}\right) & \times f_{B_d}^2 B_{B_d} \end{align*}$ | perturbation theory | lattice QCD, sum rules |
|------------|--------------|--------------------------------------------------|---------------------|-----------------------|
### Lattice phenomenology

<table>
<thead>
<tr>
<th>Experiment</th>
<th>SM $\perp$ NPQCD</th>
<th>NPQCD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta M_d$</td>
<td>$\text{const.} \times</td>
<td>V_{tb}V_{td}</td>
</tr>
<tr>
<td></td>
<td>perturbation theory</td>
<td>lattice QCD, sum rules</td>
</tr>
</tbody>
</table>

Other NP SM observables or parameters:

- meson and baryon spectra
- matrix elements relevant for phenomenology:
  - decay constants $(f_\pi, f_K, f_{D(s)}, f_{B(s)})$
  - form factors $(f_{\pi\pi}, f_{K\pi}, f_{D\to K}, f_{B\to \pi}, \ldots)$
  - mixing matrix elements $(B, B_{B(s)}, \ldots)$
  - hadr. $K$-decays $(A_0, A_2)$
Quality of lattice results

Very strong claims are made based on lattice QCD results:

- “We find a (2-3)$\sigma$ tension in the unitarity triangle”
  
  Laiho, Lunghi, Van de Water, PRD 81 (2010) 034503

- “. . . confirming CKM unitarity at the permille level”
  
  FLAG arXiv:1011.4408

- “. . . we find evidence of new physics in both $B_d$ and $B_s$ systems . . . ”
  
  CKMfitter Group PRD 83 (2011) 036004

- “Possible evidence for the breakdown of the CKM-paradigm of CP-violation”
  
  Lunghi, Soni, PLB 697, 323-328 (2011)
Quality of lattice results

Very strong claims are made based on lattice QCD results:

- “We find a (2-3)$\sigma$ tension in the unitarity triangle”
  
  Laiho, Lunghi, Van de Water, PRD 81 (2010) 034503

- “. . . confirming CKM unitarity at the permille level”
  
  FLAG arXiv:1011.4408

- “. . . we find evidence of new physics in both $B_d$ and $B_s$ systems . . . ”
  
  CKMfitter Group PRD 83 (2011) 036004

- “Possible evidence for the breakdown of the CKM-paradigm of CP-violation”
  
  Lunghi, Soni, PLB 697, 323-328 (2011)

How to deal with lattice results?
Are these statements water-proof?
FLAG

→ Flavia Net Lattice Averaging Group (FLAG) was founded to allow also to an outsider to judge the quality and 'state-of-the-art'-fulness of lattice results relevant to flavor physics

FLAG

→ Flavia Net Lattice Averaging Group (FLAG) was founded to allow also to an outsider to judge the quality and 'state-of-the-art'-fulness of lattice results relevant to flavor physics


→ criteria:  publication status
  chiral extrapolation
  continuum extrapolation
  finite volume errors
  renormalisation
  renormalisation scale running

→ quantities: $m_{u,d}$, $m_s$, $f_+^{K\pi}(0)$, $f_K/f_\pi$, $B_K$, NLO LEC's, potentially more in the future

FLAG

→ Flavia Net Lattice Averaging Group (FLAG) was founded to allow also to an outsider to judge the quality and 'state-of-the-art'-fulnes of lattice results relevant to flavor physics


→ criteria: publication status, chiral extrapolation, continuum extrapolation, finite volume errors, renormalisation, renormalisation scale running

→ quantities: $m_{u,d}, m_s, f^{K\pi}_+(0), f_K/f_\pi, B_K$, NLO LEC's, potentially more in the future


Other efforts by Laiho, Lunghi, Van de Water:

Lattice QCD inputs to the CKM unitarity triangle analysis,
A FLAG example - the kaon sector

\[ |V_{us}| \]

\[ A_\mu \rightarrow K(p_K) \]

\[ \Gamma(K \rightarrow \mu \bar{\nu}_\mu (\gamma)) = \frac{|V_{us}|^2}{|V_{ud}|^2} \left( \frac{f_K}{f_\pi} \right)^2 \frac{m_K (1 - m_\mu^2/m_K^2)}{m_\pi (1 - m_\mu^2/m_\pi^2)} \times 0.9930(35) \]

\[ \Gamma(K \rightarrow \pi \nu) = C_K^2 \frac{G_F m_K^5}{192\pi^2} \int S_{EW} [1 + \Delta_{SU(2)} + \Delta_{EM}] \times |V_{us}|^2 |f_{K^0(0)}|^2 \]

A FLAG example - the kaon sector

Lattice QCD’s flavour physics results for phenomenologists

Andreas Jüttner
**A FLAG example - the kaon sector**

<table>
<thead>
<tr>
<th>Collaboration</th>
<th>$N_f$</th>
<th>chiral extrapolation</th>
<th>continuum extrapolation</th>
<th>finite volume errors</th>
<th>$f_K / f_\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ETM 10E</td>
<td>2+1+1</td>
<td>C</td>
<td>●</td>
<td>●</td>
<td>1.224(13)$_{\text{stat}}$</td>
</tr>
<tr>
<td>MILC 10</td>
<td>2+1</td>
<td>C</td>
<td>●</td>
<td>★</td>
<td>1.197(2)(+$^3_{-7}$)</td>
</tr>
<tr>
<td>RBC/UKQCD 10A</td>
<td>2+1</td>
<td>P</td>
<td>●</td>
<td>★</td>
<td>1.204(7)(25)</td>
</tr>
<tr>
<td>BMW 10</td>
<td>2+1</td>
<td>A</td>
<td>★★</td>
<td>★★</td>
<td>1.192(7)(6)</td>
</tr>
<tr>
<td>JLQCD/TWQCD 09A</td>
<td>2+1</td>
<td>C</td>
<td>●</td>
<td>■</td>
<td>1.210(12)$_{\text{stat}}$</td>
</tr>
<tr>
<td>MILC 09A</td>
<td>2+1</td>
<td>C</td>
<td>●</td>
<td>★★</td>
<td>1.198(2)(+$^6_{-8}$)</td>
</tr>
<tr>
<td>Aubin 08</td>
<td>2+1</td>
<td>A</td>
<td>●</td>
<td>★★</td>
<td>1.191(3)(+$^6_{-13}$)</td>
</tr>
<tr>
<td>PACS-CS 08, 08A</td>
<td>2+1</td>
<td>A</td>
<td>★★</td>
<td>■</td>
<td>1.189(20)</td>
</tr>
<tr>
<td>RBC/UKQCD 08</td>
<td>2+1</td>
<td>A</td>
<td>●</td>
<td>■</td>
<td>1.205(18)(62)</td>
</tr>
<tr>
<td>HPQCD/UKQCD 07</td>
<td>2+1</td>
<td>A</td>
<td>●</td>
<td>★★</td>
<td>1.189(2)(7)</td>
</tr>
<tr>
<td>NPLQCD 06</td>
<td>2+1</td>
<td>A</td>
<td>●</td>
<td>■</td>
<td>1.218(2)(+$^{11}_{-24}$)</td>
</tr>
<tr>
<td>ETM 10D</td>
<td>2</td>
<td>C</td>
<td>●</td>
<td>★★</td>
<td>1.190(8)$_{\text{stat}}$</td>
</tr>
<tr>
<td>ETM 09</td>
<td>2</td>
<td>A</td>
<td>●</td>
<td>★★</td>
<td>1.210(6)(15)(9)</td>
</tr>
<tr>
<td>QCDSF/UKQCD 07</td>
<td>2</td>
<td>C</td>
<td>●</td>
<td>★★</td>
<td>1.21(3)</td>
</tr>
</tbody>
</table>

*Lattice QCD’s flavour physics results for phenomenologists*  
*Andreas Jüttner*
A FLAG example - the kaon sector

For both $N_f = 2 + 1$ and $N_f = 2$ FLAG identified high quality lattice results and provides averages:

<table>
<thead>
<tr>
<th>$N_f$</th>
<th>$f_+^{K\pi}(0)$</th>
<th>$f_K/f_\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2+1</td>
<td>0.9560(57)(62)</td>
<td>1.193(5)</td>
</tr>
<tr>
<td>2</td>
<td>0.9599(34)(41)</td>
<td>1.210(6)(17)</td>
</tr>
</tbody>
</table>

Together with experimental input:

- $|V_{us}f_+^{K\pi}(0)| = 0.2163(5)$
- $|f_K/V_{us}| = 0.2758(5)$
- $|V_{ud}| = 0.97425(22)$

FLAG did two kind of analysis:

a) test the SM: $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 \overset{?}{=} 1$

b) using SM-correlations
FLAG tests the SM: $|V_{ul}|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$
FLAG tests the SM: $|V_{ul}|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$
FLAG tests the SM: $|V_{ul}|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$
FLAG tests the SM: $|V_{ul}|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$

$$
\begin{array}{cccc}
N_f & |V_u|^2|_{no|V_{ud}|} & |V_u|^2|_{V_{ud}|} \text{ from } f^{K\pi}(0) & |V_u|^2|_{V_{ud}|} \text{ from } f_K/f_\pi \\
2+1 & 1.002(15) & 1.0000(7) & 0.9999(6) \\
2 & 1.037(36) & 1.0004(10) & 0.9985(16)
\end{array}
$$

So indeed, CKM-unitarity confirmed at the per-mil level.
**FLAG tests the SM:**

\[ |V_{ul}|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \]

---

**Results:**

| \(N_f\) | \(|V_{ul}|^2\) | \(|V_{ud}|^2\) from \(f_+^{K\pi}(0)\) | \(|V_{ud}|^2\) from \(f_K/f_\pi\) |
|---|---|---|---|
| 2+1 | 1.002(15) | 1.0000(7) | 0.9999(6) |
| 2   | 1.037(36) | 1.0004(10) | 0.9985(16) |

---

So indeed, CKM-unitarity confirmed at the per-mil level.
FLAG assumes first row unitarity

**Nice result:** Assuming first row unitarity lattice QCD makes prediction for $|V_{ud}|$ with same precision as super-allowed nuclear beta decays and fully compatible with it.
$\epsilon_K$, $f_{B(s)}$ and $B_{B(s)}$

Lattice QCD input to SM-tests (*cf.* Soni’s talk in this session)

<table>
<thead>
<tr>
<th>$K$-observables</th>
<th>$B$-observables</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon_K \propto B_K$</td>
<td>$\Delta M_d \propto f_{B_d}^2 B_{B_d}$</td>
</tr>
<tr>
<td>$\Delta M_s \propto f_{B_s}^2 B_{B_s}$</td>
<td>$\Delta \Gamma_s \propto (f_{B_s} B_{B(s)})^2$</td>
</tr>
</tbody>
</table>
\( \epsilon_K, f_{B(s)} \text{ and } B_{B(s)} \)

lattice QCD input to SM-tests \textit{(cf. Soni’s talk in this session)}

<table>
<thead>
<tr>
<th>( K )-observables</th>
<th>( B )-observables</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \epsilon_K \propto B_K )</td>
<td>( \Delta M_d \propto f_{B_d}^2 B_{B_d} )</td>
</tr>
<tr>
<td>( \Delta M_s \propto f_{B_s}^2 B_{B_s} )</td>
<td>( \Delta \Gamma_d \propto (f_{B_d} B_{B_d})^2 )</td>
</tr>
<tr>
<td>( \Delta \Gamma_s \propto (f_{B_s} B_{B(s)})^2 )</td>
<td></td>
</tr>
</tbody>
</table>

b-quarks on the lattice not straight forward! Reason: \( m_b \approx 4 \text{GeV} \approx a^{-1} \)

This causes head-aches and huge cut-off effects \( \rightarrow \) naively impossible

Techniques for \( b \)-quarks on the lattice (separation of scales):

- static approx (leading HQET) \textit{(INF-TOV, ALPHA, ETM)}
- HQET \textit{(ALPHA)}
- NRQCD \textit{(HPQCD)}
- relativistic heavy quark actions \textit{(Fermilab, RBC/UKQCD, PACS-CS)}
recent breakthrough thanks to chirally symmetric lattice actions (Domain Wall Fermions, see Taku’s talk (next))
well advanced lattice calculations, $\delta \approx 3 - 4\%$
FLAG average currently being updated
NON-FLAG: $f_{B(s)}$ and $B_{B(s)}$

Lattice QCD's flavour physics results for phenomenologists

Andreas Jüttner
NON-FLAG: $f_{B(s)}$ and $B_{B(s)}$

- given it’s
  - potentially huge impact
  - non-straight forward lattice computation,

$f_{B(s)}$ and $B_{B(s)}$ urgently need confirmation by other (non-staggered) groups using approaches with ideally uncorrelated systematics
NON-FLAG: $f_{B(s)}$ and $B_{B(s)}$

- given it’s
  - potentially huge impact
  - non-straight forward lattice computation,

$f_{B(s)}$ and $B_{B(s)}$ urgently need confirmation by other (non-staggered) groups using approaches with ideally uncorrelated systematics

- again, cleanest would be to renormalise the operators non-perturbatively and it is very well understood how to do it

- $N_f = 2$ QCD and $N_f = 2 + 1$ QCD are different theories - it is not meaningful to average results

- consecutive updates of results by one and the same collaboration are highly correlated
NON-FLAG: $K \rightarrow \pi\pi$

$$\langle \pi\pi(I)|H^{\text{eff}}|K \rangle$$

Maiani Testa: standard lattice approach will project on unphysical 2-pion state
\[ \langle \pi\pi(I) | H^{\text{eff}} | K \rangle \]

Maiani Testa: standard lattice approach will project on unphysical 2-pion state current non-standard approaches allow to compute it anyway:

- indirectly via $\chi$PT

  \[ RBC \text{ PRD68, (2003), CP-PACS PRD68, (2003)} \]

depends crucially on SU(3) chiral PT which turns out to converge badly repeat efforts with unphysically light strange quarks?
\[ \langle \pi\pi(l) | \mathcal{H}^{\text{eff}} | K \rangle \]

Maiani Testa: standard lattice approach will project on unphysical 2-pion state current non-standard approaches allow to compute it anyway:

- indirectly via $\chi$PT \[ RBC \text{ PRD68, (2003), CP-PACS PRD68, (2003)} \]
depends crucially on SU(3) chiral PT which turns out to converge badly repeat efforts with unphysically light strange quarks?

- directly, using (partially twisted, G-parity) boundary condition tricks to project on the desired state \[ \text{arXiv:0912.2917} \]
  - very feasible in $\Delta l = 3/2$ channel
  - very cost-intensive in $\Delta l = 1/2$ channel

N. Christ in his Kaon 2009 write-up \[ \text{arXiv:0912.2917}: \text{“$\Delta l = 1/2, 3/2$ amplitudes $A_0$ and $A_2$ with 10-20\% precision within the next 2-3 years” (RBC+UKQCD)} \]
Outlook

Kaons:
- well advanced calculations
- of $f_K / f_\pi$ and $f_+^{K\pi}(0)$ the latter is most likely to improve considerably in the near future
- tremendous progress in $B_K$ over the last years $\rightarrow \approx 4\%$ uncertainty
- dominant uncertainties in renormalisation procedure and running

B-physics:
- calculations are advanced but one could do much better (clean heavy-quark discretisation, continuum limit and non-perturbative renormalisation)
- more dedicated efforts based on different approaches are under way
- in the near future individual results will not be produced with much improved precision
Outlook

- **Kaons:**
  - well advanced calculations
  - of $f_K/f_\pi$ and $f_{K\pi}(0)$ the latter is most likely to improve considerably in the near future
  - tremendous progress in $B_K$ over the last years $\rightarrow \approx 4\%$ uncertainty
  - dominant uncertainties in renormalisation procedure and running

- **B-physics:**
  - calculations are advanced but one could do much better (clean heavy-quark discretisation, continuum limit and non-perturbative renormalisation)
  - more dedicated efforts based on different approaches are under way
  - in the near future individual results will not be produced with much improved precision

- **FLAG** and **ALVdW** started efforts to provide you with easily accessible summaries of lattice phenomenology - please make use of this service and get in touch with us if interested
Outlook

Kaons:
- well advanced calculations
- of $f_K/f_\pi$ and $f_{K\pi}^+(0)$ the latter is most likely to improve considerably in the near future
- tremendous progress in $B_K$ over the last years $\rightarrow \approx 4\%$ uncertainty
- dominant uncertainties in renormalisation procedure and running

B-physics:
- calculations are advanced but one could do much better (clean heavy-quark discretisation, continuum limit and non-perturbative renormalisation)
- more dedicated efforts based on different approaches are under way
- in the near future individual results will not be produced with much improved precision

FLAG and ALVdW started efforts to provide you with easily accessible summaries of lattice phenomenology - please make use of this service and get in touch with us if interested

Lattice QCD is continuously evolving and so does the quality of the results - so there is more to come

Thank you!
additional material
\[ \Delta S = 1, 2 \ - \ CP\text{-}violation \ in \ the \ K\text{-}system \]

\[ K_L \propto K_2 + \epsilon K_1 \]

\[ \begin{aligned}
\epsilon' \text{: direct} & \quad \epsilon \text{: indirect} \\
\text{KTeV, NA48 1999} & \quad \text{Cronin&Fitch 1964}
\end{aligned} \]

- **Indirect CP-violation** (\( \Delta S = 2 \) Kaon mixing):

\[
\epsilon_K = \frac{A(K_L \rightarrow (\pi\pi)_{I=0})}{A(K_S \rightarrow (\pi\pi)_{I=0})}
\]

\[ |\epsilon_K|_{\text{exp.}} = 2.28(2) \times 10^{-3} \quad \text{(PDG 2006)} \]

- **Direct CP-violation** (\( \Delta S = 1 \) non-leptonic Kaon decay)

\[
A(K^0 \rightarrow \pi^+ \pi^-) = \sqrt{\frac{2}{3}} A_0 e^{i\delta_0} + \sqrt{\frac{1}{3}} A_2 e^{i\delta_2}
\]

\[
A(K^0 \rightarrow \pi^0 \pi^0) = \sqrt{\frac{2}{3}} A_0 e^{i\delta_0} - \sqrt{\frac{1}{3}} A_2 e^{i\delta_2}
\]

\[
\frac{\omega}{\sqrt{2}} e^{i\phi} \left( \frac{\epsilon'}{\text{Re}A_2 - \text{Im}A_0} \right) = \epsilon'/\epsilon |_{\text{exp.}} = 1.72(18) \times 10^{-3} \quad \text{(PDG 2006)}
\]

\[ \Delta l = \frac{1}{2}: \quad \omega = \frac{\text{Re}A_2}{\text{Re}A_0} \]

\[ \omega^{-1} |_{\text{exp.}} \approx 22 \]
## Results for $B_K$


<table>
<thead>
<tr>
<th>Source uncertainty/error</th>
<th>Uncertainty/error on $B_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>statistics</td>
<td>1.2%</td>
</tr>
<tr>
<td>chiral &amp; continuum extrapolation</td>
<td>1.9%</td>
</tr>
<tr>
<td>scale and quark-mass uncertainties</td>
<td>0.8%</td>
</tr>
<tr>
<td>finite volume errors</td>
<td>0.6%</td>
</tr>
<tr>
<td>renormalization factor</td>
<td>3.4%</td>
</tr>
<tr>
<td>total systematic</td>
<td>4.0%</td>
</tr>
<tr>
<td>total</td>
<td>4.2%</td>
</tr>
</tbody>
</table>
# Results for $B_K$


<table>
<thead>
<tr>
<th>Source uncertainty/error</th>
<th>uncertainty/error on $B_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>statistics</td>
<td>1.2%</td>
</tr>
<tr>
<td>chiral &amp; continuum extrapolation</td>
<td>1.9%</td>
</tr>
<tr>
<td>scale and quark-mass uncertainties</td>
<td>0.8%</td>
</tr>
<tr>
<td>finite volume errors</td>
<td>0.6%</td>
</tr>
<tr>
<td>renormalization factor</td>
<td>3.4%</td>
</tr>
<tr>
<td></td>
<td>$a \gtrsim 0.09\text{fm}$</td>
</tr>
<tr>
<td></td>
<td>$m_\pi \gtrsim 240\text{MeV}$</td>
</tr>
<tr>
<td></td>
<td>← mainly NLO running</td>
</tr>
<tr>
<td>total systematic</td>
<td>4.0%</td>
</tr>
<tr>
<td>total</td>
<td>4.2%</td>
</tr>
</tbody>
</table>
## Results for $f_{+}^{K\pi}(0)$


<table>
<thead>
<tr>
<th>Source uncertainty/error</th>
<th>uncertainty/error on $f_{+}^{K\pi}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>statistical</td>
<td>0.3%</td>
</tr>
<tr>
<td>chiral extrapolation</td>
<td>0.4%</td>
</tr>
<tr>
<td>continuum extrapolation</td>
<td>0.1%</td>
</tr>
<tr>
<td>total systematic</td>
<td>0.4%</td>
</tr>
<tr>
<td>total</td>
<td>0.5%</td>
</tr>
</tbody>
</table>

- dominant uncertainty from chiral extrapolation
## Results for $f_{+}^{K\pi}(0)$


<table>
<thead>
<tr>
<th>Source uncertainty/error</th>
<th>uncertainty/error on $f_{+}^{K\pi}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>statistical</td>
<td>0.3%</td>
</tr>
<tr>
<td>chiral extrapolation</td>
<td>0.4%</td>
</tr>
<tr>
<td>continuum extrapolation</td>
<td>0.1%</td>
</tr>
<tr>
<td>total systematic</td>
<td>0.4%</td>
</tr>
<tr>
<td>total</td>
<td>0.5%</td>
</tr>
</tbody>
</table>

- dominant uncertainty from chiral extrapolation

\[ \geq 330\text{MeV} \]
## Results for $f_K/f_\pi$

<table>
<thead>
<tr>
<th>Source uncertainty/error</th>
<th>Uncertainty/error on $f_K/f_\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Statistics</td>
<td>0.6%</td>
</tr>
<tr>
<td>Chiral extrapolation</td>
<td></td>
</tr>
<tr>
<td>- Functional form</td>
<td>0.3%</td>
</tr>
<tr>
<td>- Pion mass range</td>
<td>0.3%</td>
</tr>
<tr>
<td>Continuum extrapolation</td>
<td>0.3%</td>
</tr>
<tr>
<td>Exited states</td>
<td>0.2%</td>
</tr>
<tr>
<td>Scale setting</td>
<td>0.1%</td>
</tr>
<tr>
<td>Finite volume</td>
<td>0.1%</td>
</tr>
<tr>
<td>Total syst</td>
<td>0.5%</td>
</tr>
<tr>
<td>Total</td>
<td>0.8%</td>
</tr>
</tbody>
</table>

- **“dominant” uncertainties**: chiral and continuum extrapolation (other collabs reach much smaller stat. error than BMW)
Results for $f_K/f_\pi$

<table>
<thead>
<tr>
<th>Source uncertainty/error</th>
<th>Uncertainty/error on $f_K/f_\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>statistics</td>
<td>0.6%</td>
</tr>
<tr>
<td>chiral extrapolation</td>
<td></td>
</tr>
<tr>
<td>- functional form</td>
<td>0.3%</td>
</tr>
<tr>
<td>- pion mass range</td>
<td>0.3% $\gtrsim 190\text{MeV}$</td>
</tr>
<tr>
<td>continuum extrapolation</td>
<td>0.3% $\gtrsim 0.064\text{fm}$</td>
</tr>
<tr>
<td>exited states</td>
<td>0.2%</td>
</tr>
<tr>
<td>scale setting</td>
<td>0.1%</td>
</tr>
<tr>
<td>finite volume</td>
<td>0.1%</td>
</tr>
<tr>
<td>total syst</td>
<td>0.5%</td>
</tr>
<tr>
<td>total</td>
<td>0.8%</td>
</tr>
</tbody>
</table>

“dominant” uncertainties: chiral and continuum extrapolation (other collabs reach much smaller stat. error than BMW)
Results for $f_K/f_\pi$


table III. Errors in % for $f_{B_s} \sqrt{\hat{B}_{B_s}}$, $f_{B_d} \sqrt{\hat{B}_{B_d}}$ and $\xi$.

<table>
<thead>
<tr>
<th>Source of error</th>
<th>$f_{B_s} \sqrt{\hat{B}_{B_s}}$</th>
<th>$f_{B_d} \sqrt{\hat{B}_{B_d}}$</th>
<th>$\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stat + chiral extrap.</td>
<td>2.3</td>
<td>4.1</td>
<td>2.0</td>
</tr>
<tr>
<td>Residual $a^2$ extrap. uncertainty</td>
<td>3.0</td>
<td>2.0</td>
<td>0.3</td>
</tr>
<tr>
<td>$r_1^{3/2}$ uncertainty</td>
<td>2.3</td>
<td>2.3</td>
<td>-</td>
</tr>
<tr>
<td>$g_{B^*B\pi}$ uncertainty</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$m_s$ and $m_b$ tuning</td>
<td>1.5</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Operator matching</td>
<td>4.0</td>
<td>4.0</td>
<td>0.7</td>
</tr>
<tr>
<td>Relativistic corr.</td>
<td>2.5</td>
<td>2.5</td>
<td>0.4</td>
</tr>
<tr>
<td>Total</td>
<td>6.7</td>
<td>7.1</td>
<td>2.6</td>
</tr>
</tbody>
</table>

**TABLE V.** Errors in % for $f_{B_s}$, $f_{B_d}$, and $f_{B_s}/f_{B_d}$.

<table>
<thead>
<tr>
<th>Source of error</th>
<th>$f_{B_s}$</th>
<th>$f_{B_d}$</th>
<th>$f_{B_s}/f_{B_d}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stat + chiral extrap.</td>
<td>2.2</td>
<td>3.5</td>
<td>1.6</td>
</tr>
<tr>
<td>HPQCD, PRD 80 014503 (2009)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Critical slowing down

$f_{D_s}$

Lattice QCD’s flavour physics results for phenomenologists

Andreas Jüttner