Is $N = 8$ Supergravity an Ultraviolet Finite Quantum Field Theory?

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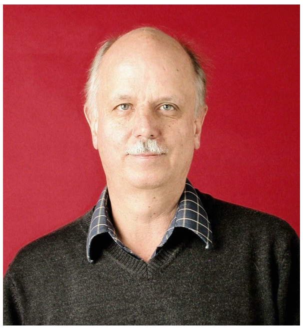

Advances in the computation of quantum amplitudes in supergravity theories raise the question whether maximal supergravity in $D = 4$ spacetime dimensions might actually be free of ultraviolet divergences. On the other hand, supersymmetric non-renormalization theorems give no indication of cancellations for anything beyond half-BPS counterterm operators. The jury is still out, and bets are being taken on the outcome.

Formulating an acceptable quantum theory of gravity remains the prime challenge to fundamental theoretical physics. A basic problem in formulating such a theory was already recognized in the earliest approaches to the problem in the 1930's: the dimensional character of Newton's constant gives rise to ultraviolet divergent quantum correction integrals. In the 1970's, this was confirmed explicitly in the first Feynman diagram calculations of the radiative corrections to systems containing gravity plus matter [1]. The time lag between the general perception of the UV divergence problem and its first concrete demonstration was due to the complexity of Feynman diagram calculations involving gravity. The necessary techniques were an outgrowth of the long struggle to Lorentz-covariantly control the quantization of non-abelian Yang-Mills theories in the Standard Model of weak and electromagnetic interactions and in quantum chromodynamics.

With the advent of supergravity[2] in the mid 1970's, hopes rose that the specific combinations of quantum fields in supergravity theories might possibly tame the gravitational UV divergence problem. Indeed, it turns out that all irreducible supergravity theories in four-dimensional spacetime, *i.e.* theories in which all fields are irreducibly linked to gravity by supersymmetry transformations, have remarkable cancellations in Feynman diagrams with one or two internal loops.

There is a sequence of such irreducible (or "pure") supergravity models, characterized by the number N of local (*i.e.* spacetime-dependent) spinor parameters. In four-dimensional spacetime, minimal, or $N = 1$, supergravity thus has 4 supersymmetries corresponding to the components of a single Majorana spinor transformation parameter. The maximal possible supergravity[3] in four dimensional spacetime has $N = 8$ spinor parameters, *i.e.* 32 independent supersymmetries.

The hopes for "miraculous" UV divergence cancellations in supergravity were subsequently dampened by the realization that the divergence-killing powers of supersymmetry most likely do not extend beyond the two-loop order for generic pure supergravity theories[4,5,6,7]. The three-loop anticipated invariant is quartic in curvatures, and has a purely gravitational part given by the square of the Bel-Robinson tensor[4].

The flowering of superstring theory in the 1980's and 1990's, in which the UV divergence problems of gravity are cured by a completely different mechanism replacing the basic field-theory point-particle states by extended relativistic object states, pushed the UV divergence properties of supergravity out of the limelight, leaving the supergravity UV problem in an unclear state.

Nonetheless, among some researchers a faint hope persisted that at least the maximal $N = 8$ supergravity might have special UV properties. This hope was bolstered by the fact that the fact that the maximal supersymmetric Yang-Mills theory, which has $N = 4$, *i.e.* 16-component supersymmetry, is completely free of ultraviolet divergences in four-dimensional spacetime[8]. This was the first interacting UV-finite theory in four spacetime dimensions.

It is this possibility of "miraculous" UV divergence cancellations in maximal supergravity that has now been confirmed in a remarkable 3-loop calculation by Z. Bern et al.[9]. Performing such calculations at high loop orders requires a departure from textbook Feynman-diagram methods, because the standard approaches can produce astronomical numbers of terms. Instead of following the standard propagator & vertex methods for the supergravity calculations, Bern et al. used another technique which goes back to Feynman: loop calculations can be performed using the unitarity properties of the quantum S-matrix. These involve cutting rules that reduce higher-loop diagrams to sums of products of leading-order "tree" diagrams without internal loops. This use of unitarity is an outgrowth of the optical theorem in quantum mechanics for the imaginary part of the S-matrix.

In order to obtain information about the real part of the S-matrix, an additional necessary element in the unitarity-based technique is the use of dimensional regularization to render UV divergent diagrams finite. In dimensional regularization, the dimensionality of spacetime is changed from 4 to $4-\epsilon$, where ϵ is a small adjustable parameter. Traditional Feynman diagram calculations also often use dimensional regularization, but normally one just focuses on the leading $1/\epsilon$ poles in order to carry out a renormalization program. In the unitarity-based approach, all orders in ϵ need to be retained. This gives rise to logarithms in which real and imaginary contributions are related.

In the maximal $N = 8$ supergravity theory, the complexity of the quantum amplitudes factorizes, with details involving the various field types occurring on the external legs of an amplitude multiplying a much simpler set of scalar-field Feynman diagrams. It is to the latter that the unitarity-based methods may be applied. Earlier applications[10] of the cutting-rule unitarity methods based on iterations of two-particle cuts gave an expectation that one might have cancellations for $D < 10/L + 2$, where D is the spacetime dimension and L is the number of Feynman diagram loops (for $L > 1$). Already, this gave an expectation that $D = 4$ maximal supergravity would have cancellations of the UV divergences at the $L = 3$ and $L = 4$ loop orders. This would leave the next significant test at $L = 5$ loops. In the ordinary Feynman-diagram approach, a full calculation at this level would involve something like 10^{30} terms. Even using the unitarity-based methods, such a calculation would be a daunting, but perhaps not impossible, task.

The impressive new elements in the 3-loop calculation of Bern et alare the completeness of their calculation and the unexpected further patterns of cancellations found. This could suggest a possibility of unexpected UV cancellations at yet higher loop orders. Although the various 3-loop diagram classes were already individually expected to be finite on the basis of the earlier work by Bern et al., the new results show that the remaining finite amplitudes display additional cancellations, rendering them "superfinite". In particular, the earlier work employed iterated 2-particle cuts and did not consider all diagram types. The new complete calculation displays further cancellations between diagrams that can be analyzed using iterated 2-particle cuts and the additional diagrams that cannot be treated in this way. The set of three-loop diagrams is shown in Figure 1. The end result is that the sum of all diagram types is more convergent by two powers of external momentum than might otherwise have been anticipated.

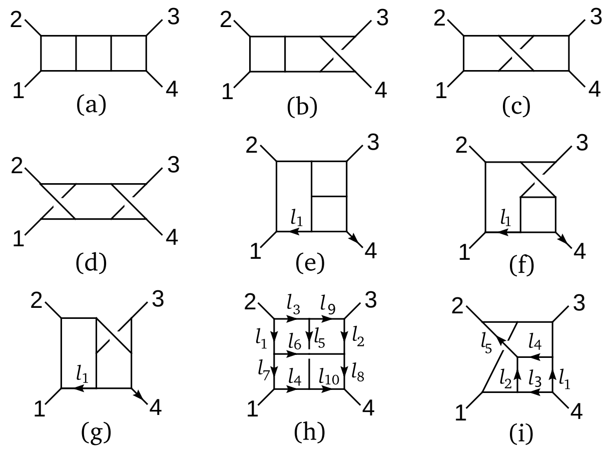

Figure 1: 3-loop Feynman diagram types leading to unanticipated 'superfiniteness' of maximal supergravity at this loop order. Diagrams (a)-(g) can be analyzed using iterated 2-particle cuts, leading to an expectation of ultraviolet divergence cancellation. Diagrams (h) and (i) cannot be treated this way, but the result of summing all diagrams (a-i) is a deeper cancellation of the leading UV behavior than anticipated.

Does such a mechanism cascade in higher-order diagrams, rendering the maximal N=8 theory completely free of ultraviolet divergences? No one knows at present. Such a scenario might pose puzzling questions for the superstring program, where it has been assumed that ordinary supergravity theories need string ultraviolet completions in order to form consistent quantum theories. On the other hand, there are hints [11] from superstring theory that precisely such an all-orders divergence cancellation might take place in the $N = 8$ theory. On the other hand, it is not clear exactly what one can learn from superstring theory about purely perturbative field-theory divergences.

One thing that seems clear is that ordinary Feynman diagram techniques coupled with the "non-renormalization" theorems of supersymmetry are unlikely to be able to explain finiteness properties of $N = 8$ supergravity at arbitrary loop order. Earlier expectations[4,5,6,7] were that the first loop order at which divergences that cannot be removed by field redefinitions would be three loops in all pure $D = 4$ supergravities. A key element in this anticipation was the expectation that the maximal amount of supersymmetry that can be *linearly* realized in Feynman diagram calculations (aka "off-shell supersymmetry") is half the full supersymmetry of the theory, or 16 out of 32 supercharges for the maximal $N = 8$ theory.

Similarly to the way in which chiral integrals of $N = 1$, $D = 4$ supersymmetry achieve invariance from integrals over less than the theory's full superspace, provided the integrand satisfies a corresponding BPS type constraint, there are analogous invariants involving integration over varying portions of an extended supersymmetric theory's full superspace [6]. "Half-BPS" operators require integration over just half the full set of fermionic variables. And if half the full supersymmetry were the maximal amount that can be linearly realized (so giving strong results from the corresponding Ward identities), such operators would be the first to be allowed as UV counterterms.

The results of Ref. [9] show that the half-BPS expectation for the first allowed counterterms is too conservative in the case the maximal theory. But more recent advances in the understanding of supersymmetric non-renormalization theorems push the divergence onset boundary out slightly for the maximal theory, so that half-BPS counterterms that require superspace integrals over half the 32 component superspace are now expected to be the last *disallowed* counterterms instead of the first *allowed* ones. The resulting current expectations for first divergences from a traditional Feynman diagram plus non-renormalization viewpoint are shown for various spacetime dimensions in Table 1.

Dimension D	11	10	8	7	6	5	4
Loop order L	2	2	1	2	3	4	5
Gen. form	$\partial^{12}R^4$	$\partial^{10}R^4$	R^4	$\partial^4 R^4$	$\partial^6 R^4$	$\partial^6 R^4$	$\partial^4 R^4$

Table 1: Current maximal supergravity divergence expectations from Feynman rules and non-renormalization theorems.

The behavior of maximal $N = 4$ supersymmetric Yang-Mills theory in dimensions $D > 4$ may be a model for what is happening. Contrary to earlier expectations of UV divergences at the 4 loop order in $D = 5$ spacetime, the unitarity-based methods indicate that this SYM onset should be postponed to the 6-loop order. But here, the standard Feynman diagram methods have a comeback through the realization that the 4-loop finiteness could be explained using more sophisticated "harmonic superspace" methods.[12]

There are two new recent elements to the non-renormalization theorem perspective. One is the realization that maximal SYM can be formulated in a "1/2 SUSY + 1" formalism which is not however Lorentz covariant [13]. Such a SYM formulation dimensionally reduces to (8,1) supersymmetry in $D = 2$. Although considerations of gauge invariance implications in various dimensions are still ongoing, this formulation should be just the minimum needed to rule out the half-BPS operators. Moreover, there is an analogous "1/2 SUSY + 1" formulation for maximal supergravity dimensionally reduced to $D = 2$, having (16,1) supersymmetry. Providing this can be successfully lifted to a viable quantization formalism in $D = 4$, it should be just enough to rule out the $D = 4$ 3-loop candidate counterterm, now known from Ref. [10] not to occur [14].

The second new approach to the derivation of non-renormalization theorems is via "algebraic renormalization", which uses BRST cohomological techniques and has been used to give yet another demonstration of the finiteness of $D = 4$, $N = 4$ SYM [15]. Similar techniques for maximal supergravity are anticipated also to kill the eligibility of the 1/2 BPS $D = 4$ 3-loop candidate counterterm.

The overall picture that emerges from the non-renormalization theorems and the currently known divergence results from calculation is that the half-BPS operators are ruled out as UV counterterms, but that operators with less than half BPS character (thus requiring superspace integrals with more than half of the theory's full supersymmetry) are not. The most accessible test of this proposition will occur at 4 loops in $D = 5$. As is not uncommon in this subject, bets are being taken on the outcome, the payoff to be made in bottles of wine.

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