Hypermaps and Ising maps with an alternating boundary

Ariane Carrance - University of Vienna

joint works with [J. Bouttier], [V. Baillard and B. Eynard]
+ WIP with B. Eynard and T. Lejeune

Journée Cartes
Lyon
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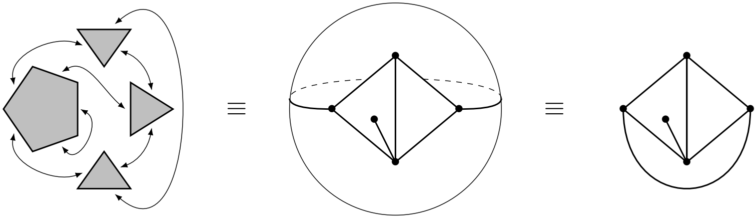
(Planar) maps

Map: gluing of polygons along their edges $\iff$ graph embedded in a surface



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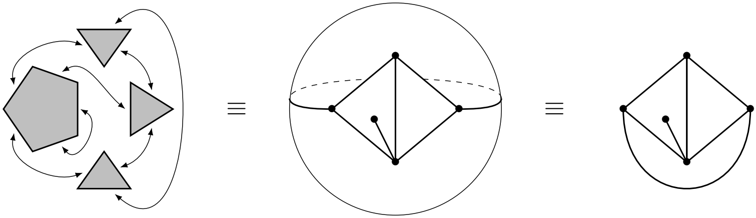
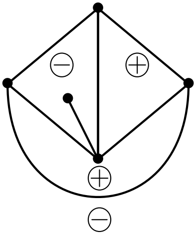


- **Faces**: the original polygons
- **Vertices, edges**: the ones of the polygons, with identifications
- **Planar** map: embedded into the sphere

The Ising model on random maps: definitions

- Endow map $\mathfrak{m}$ with **spin configuration** $\sigma \in \{\oplus, \ominus\}^{\mathcal{F}(\mathfrak{m})}$
- On a given map, pick σ proportionally to its weight $\nu^{E_{\textsf{mono}}(\sigma)}$ $(\nu \geq 0)$



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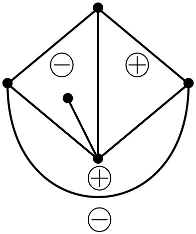
$$Z\left(t, \{t_i\}_{3 \leq i \leq d}, \nu\right) := \sum_{(\mathfrak{m},\sigma)} t^{V(\mathfrak{m})} \prod_i t_i^{F_i(\mathfrak{m})} \nu^{E_{\textbf{mono}}(\sigma)}$$



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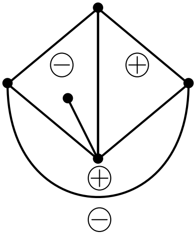

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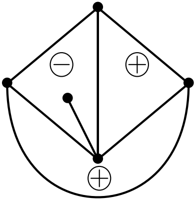
$\implies$ new behaviors, influence of **boundary conditions**



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$$A\left(w; t, \{t_i\}_{3\leq i\leq d}, \nu\right) := \sum_{r\geq 0} \frac{1}{w^{r+1}} \sum_{(\mathfrak{m},\sigma)\in\mathcal{I}_r} t^{V(\mathfrak{m})} \prod_i t_i^{F_i(\mathfrak{m})} \nu^{E_{\textsf{mono}}(\sigma)}$$

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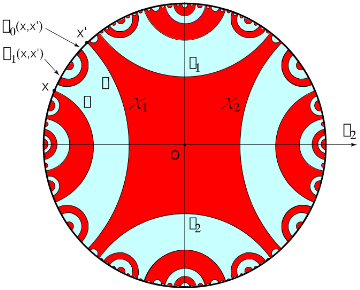
$\rightarrow$ natural for *antiferromagnetic* ($\nu < 1$) regime (*and hyperbolic lattices*...)



[Gandolfo–Ruiz–Shlosman 2015]

Background on the monochromatic boundary

- $B(x) := B(x; t, \{t_i\}, \nu)$ generating function of Ising maps with a monochromatic boundary
- $Y(x) := B(x) - \sum_{3\leq i\leq d} \frac{t_i}{i} x^i$

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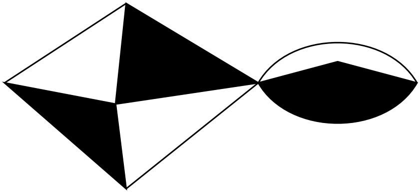
Theorem (Eynard 2003)

The function Y is **algebraic**: it satisfies $E(x, Y(x)) = 0$, where E is an explicit non-trivial polynomial with coefficients algebraic in $t, \{t_i\}, \nu$.
Moreover, it admits an explicit **rational parametrization**: $\exists\, x(z), y(z)$ explicit rational functions of a formal variable z with coefficients algebraic in $t, \{t_i\}, \nu$, such that $Y(x(z)) = y(z)$.

Earlier results on *pure* m*-angulations* with an alternating boundary

- "Frozen antiferromagnetic limit": $\nu \to 0$
- $t_i = 0$ for $i \neq m$



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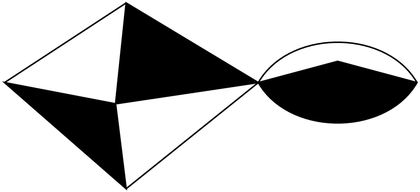

Theorem (Bouttier–C. 2021)

The function $A(w; t, t_m, \nu = 0)$ is algebraic over all its parameters. Moreover, it admits an explicit rational parametrization $(w(z), a(z))$ that can be expressed **as rational functions of the monochromatic case**:

$$w(z) = \frac{y(z)^2}{x(z)^{m-2}}, \quad a(z) = 1 - \frac{x(z)^{m-1}}{y(z)}.$$

These expressions satisfy the relation: $w(A-1) + xY = 0$.

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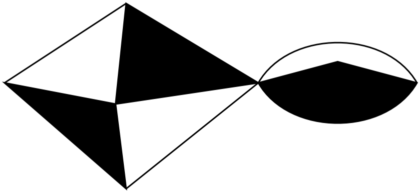

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$\to$ What happens for $\nu > 0$?

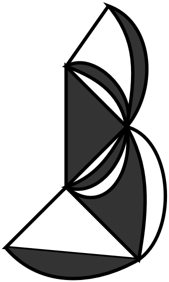
Main results

Theorem (Baillard–C.–Eynard 2026+)

- For general face degree parameters $\{t_i\}$, $A(w; t, \{t_i\}, \nu)$ is **algebraic** over all its parameters. Moreover, we have an explicit strategy to obtain such a polynomial.
- For Ising quadrangulations ($t_i = 0$ for $i \neq 4$), $A(w)$ admits an explicit **rational parametrization**: $\exists\, a(h), w(h)$ explicit rational functions of a formal variable h with coefficients algebraic in $t, \{t_i\}, \nu$, such that $A(w(h)) = a(h)$.
- In that case, $(w, A(w))$ can no longer be expressed as rational functions of $(x, Y(x))$ satisfying $w(1 - A) + xY = 0$.

Proof idea 1: Correspondence with hypermaps

- Hypermap: map with properly bicolored faces, allowing *digons*
- Can also impose boundary condition such as alternating (like for pure m-angulations)
- $H\left(w;t,t_2,\{t_i\}_{3\le i\le d}\right) := \sum_{r\ge 0}\frac{1}{w^{r+1}}\sum_{\mathfrak{m}\in\mathcal{H}_r} t^V t_2^{F_2}\prod_{3\le i\le d} t_i^{F_i}$



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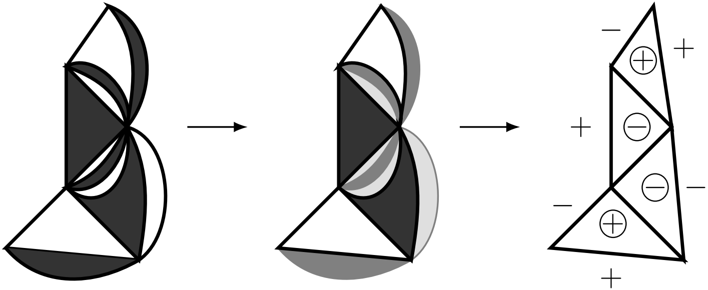

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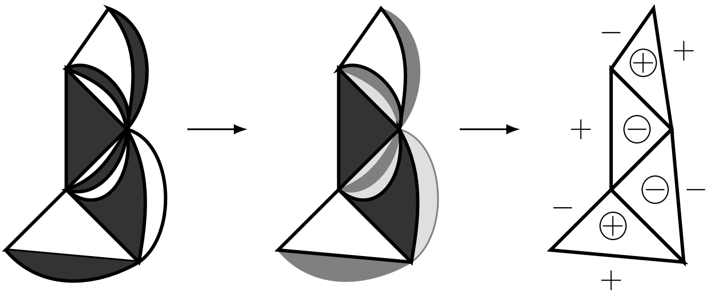

- $\implies H\left(w; t(1-\nu^2), t_2=\nu, \{t_i(1-\nu^2)\}_{3\leq i\leq d}\right) = \frac{1}{1-\nu^2} A(w; t, \{t_i\}, \nu)$

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N.B. Elimination of **2** catalytic variables (x and y) without any use of kernel method/generalization thereof

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- $\implies$ Explicit rational parametrization of $w, A(w)$

Perspectives

WIP with Bertrand Eynard and Thomas Lejeune:

Exact enumeration:

- **Bijective** proofs through **slice decomposition** (see Thomas's talk this afternoon!)
- $\rightsquigarrow$ "explicit" equation in general case, involving relation $\lambda + xy = 0$
- $\rightsquigarrow$ Other rational parametrizations?

Asymptotics for Ising quadrangulations:

- On the critical line of **Ising quadrangulations without a boundary** $(\nu, t_c(\nu))$: *not quite typical* behavior for Ising maps with a boundary

Thank you for your attention!