Enumeration of plane hypermaps with general boundary conditions

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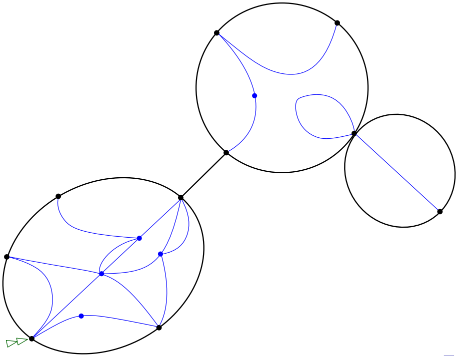
Definitions and Notations

Definition

A **plane map** is a connected graph drawn on the plane without edge crossings, and considered up to homeomorphism.
The finite connected component of its complementary is called *inner faces* and the edges incident to the infinite conn. comp. form the **boundary**.

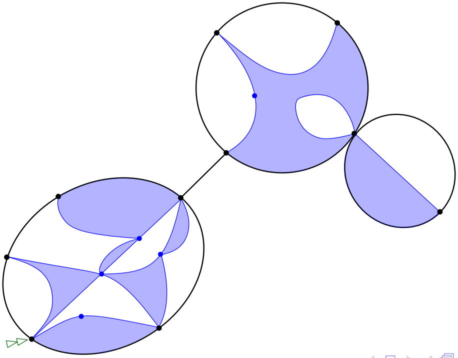


Definitions and Notations

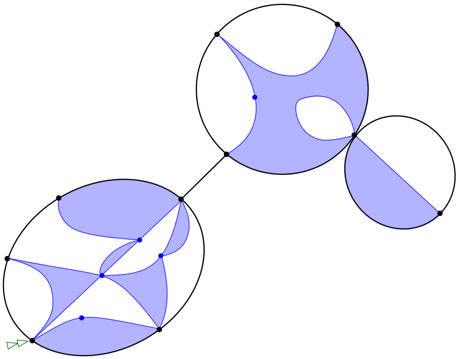
Definition

A **plane hypermap** is a plane map whose inner faces are properly bicoloured: each black face is incident to white faces only and reciprocally.

NB: We allow multiple edges (and loops), which makes it a generalization of maps.



$\longrightarrow$ A plane hypermap is naturally furnished with a (quasi) **canonical orientation** for its edges: edges incident to white (resp. black) faces will be oriented clockwise (resp. counter-clockwise).



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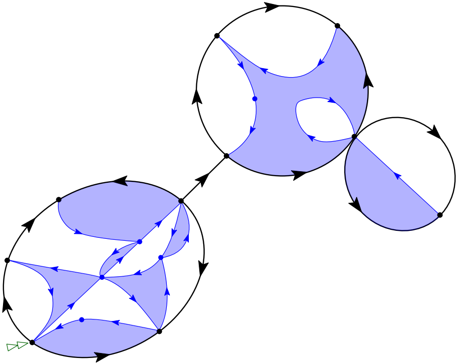


Goal

Determine the number of such hypermaps with fixed number of *vertices*, number of *black/white faces* with fixed degree and fixed configuration of arrows on the boundary (*boundary condition*).

NB: Of course not using Tutte decomposition !

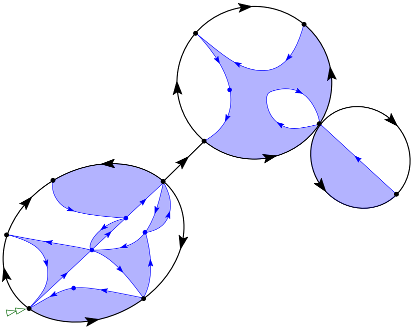
15 vertices

32 edges

$3, 4, 1$ black faces of degree $2, 3, 6$

$1, 3, 2, 3$ white faces of degree $1, 2, 3, 4$

boundary condition: $(1, 1, 3, 1, 1, 3, 1, 2)$



In order to compute these numbers, we define the generating function:

$$H_\pi(t, t_i^\bullet, t_j^\circ) := \sum_{n, n_i^\circ, n_j^\bullet \geqslant 0} \mathcal{M}^{[\pi]}_{n,(n_i^\circ),(n_j^\bullet)}\ t^n \prod_i (t_i^\circ)^{n_i^\circ} \prod_j (t_j^\bullet)^{n_j^\bullet}$$

where $\mathcal{M}^{[\pi]}_{n,(n_i^\circ),(n_j^\bullet)}$ is the number of all hypermaps with n vertices, n_i° white faces of degree i, $n_j^\bullet$ of degree j and fixed boundary conditions $\pi := (\ell_1^\bullet, \ell_1^\circ, \ldots, \ell_k^\bullet, \ell_k^\circ)$.

And then we can sum over all boundary conditions **of length** $2k$, $k \geqslant 1$, to get the generating function

$$H_k(x_1, \ldots, x_k; y_1, \ldots, y_k; t, t_i^\bullet, t_j^\circ) := \sum_{\ell_1^\bullet, \ell_1^\circ, \ldots, \ell_k^\bullet, \ell_k^\circ \in \mathbb{N}} \frac{H_{\pi=(\ell_1^\bullet, \ldots, \ell_k^\circ)}}{x_1^{\ell_1^\bullet+1} y_1^{\ell_1^\circ+1} \cdots x_k^{\ell_k^\bullet+1} y_k^{\ell_k^\circ+1}}$$

of what we call k-alternating hypermaps.

$$H_k = \sum_{\ell_1^\bullet,\ell_1^\circ,\dots,\ell_k^\bullet,\ell_k^\circ\in\mathbb{N}} \frac{\sum t^{\#vertices} \prod_{b \text{ black face}} t^\bullet_{\deg(b)} \prod_{w \text{ white face}} t^\circ_{\deg(w)}}{x_1^{\ell_1^\bullet+1} y_1^{\ell_1^\circ+1}\cdots x_k^{\ell_k^\bullet+1} y_k^{\ell_k^\circ+1}}$$

15 vertices

32 edges

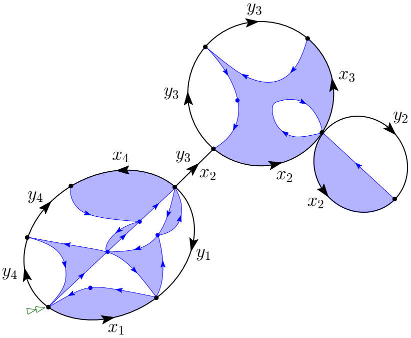
$3, 4, 1$ black faces of degree $2, 3, 6$

$1, 3, 2, 3$ white faces of degree $1, 2, 3, 4$

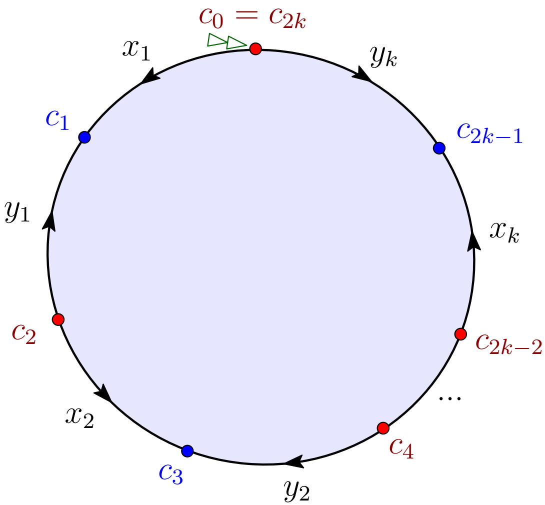
boundary condition: $(1, 1, 3, 1, 1, 3, 1, 2)$

Contribution to $H_{k=4}$:

$$\frac{t^{15}\,(t_2^\bullet)^3\,(t_3^\bullet)^4\,(t_6^\bullet)^1\,(t_1^\circ)^1\,(t_2^\circ)^3\,(t_3^\circ)^2\,(t_4^\circ)^3}{x_1^2\,y_1^2\,x_2^4\,y_2^2\,x_3^2\,y_3^4\,x_4^2\,y_4^3}$$



Accessibly pointed hypermap: Decomposition into slices



Sketch of an hypermap in our future proofs: inner faces are blurred, some corners are coloured and bridges are omitted.

Motivations of hypermaps

- Related to 2-matrix models and associated biorthogonal polynomial theory.
- Underlying fractal geometry and scaling limit that, we hope, differs from brownian sphere.
- Local limit and peeling algorithm for percolation theory.
- Study of phase transitions and underlying CFT theory.
- Discretization to 2D gravity and Liouville Quantum Theory.

Motivations of hypermaps with boundary

- *Related to 2-matrix models and associated biorthogonal polynomials theory.*

Consider the following partition function with potentials $V^\circ, V^\bullet$:

$$Z_N = \int_{\mathcal{H}_N \times \mathcal{H}_N} \mathrm{d}A\,\mathrm{d}B e^{-\frac{N}{t}\mathrm{Tr}(A^2+B^2-cAB+V^\circ(A)+V^\bullet(B))}$$

Some interesting expectation values of operators are of the form:

$$\left\langle \mathrm{Tr}\left(A^{\ell_1^\bullet}B^{\ell_1^\circ}\cdots A^{\ell_k^\bullet}B^{\ell_k^\circ}\right)\right\rangle$$

which can be gathered into the resultant:

$$H_k(x_1,\ldots,x_k;y_1,\ldots,y_k) := \left\langle \mathrm{Tr}\left(\frac{1}{x_1-A}\frac{1}{y_1-B}\cdots\frac{1}{x_k-A}\frac{1}{y_k-B}\right)\right\rangle$$

This function can be interpreted as the generating function of k-alternating hypermaps H_k as defined earlier.

Motivations of hypermaps with boundary

In order to construct random surfaces, one needs to fix the topology to perform asymptotic analysis.

- Scaling limit.
  Fixing the area and perimeter yields a compact surface from which the hypermap is a discretization. This induces the presence of a boundary and allows the study of scaling limits and fractal dimensions.
- Local limit and peeling.
  Exploring the neighbourhood of a vertex produces a hypermap with boundary. The peeling algorithm realizes this limit through a random walk on the boundary and connects to percolation theory.

Motivations of hypermaps with boundary

- Two-matrix models $\to$ Resultant to compute traces.
- Scaling limit $\to$ Balance area/perimeter and large number of faces.
- Local limit and peeling algorithm $\to$ Fixed neighbourhood of a mark.
- Physics: Study of boundary operators, holographic principle and relation with Liouville Quantum Field theory.

Accessibly pointed hypermap: Definition

Main idea: introducing a specific family of hypermaps.

Definition

A k-alternating hypermap is said **accessibly pointed** if it is pointed such that there exists a directed path from any vertex to the marked one.

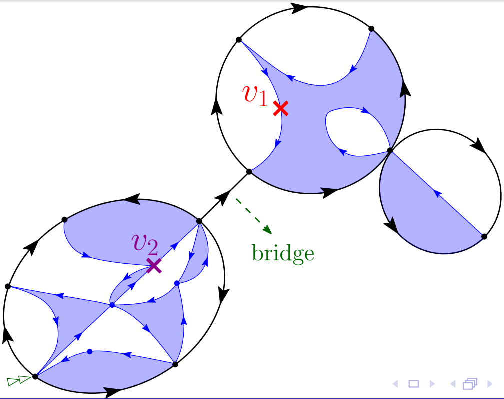
Summing and weighting as before on this specific family, we denote by A_k the associated generating function.

NB: It is equivalent to ask that a directed path exists from any vertex on the boundary of the hypermap to the marked one.

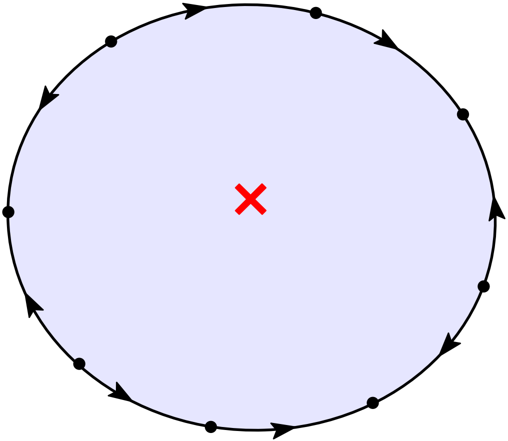
Accessibly pointed hypermap: Definition

Definition

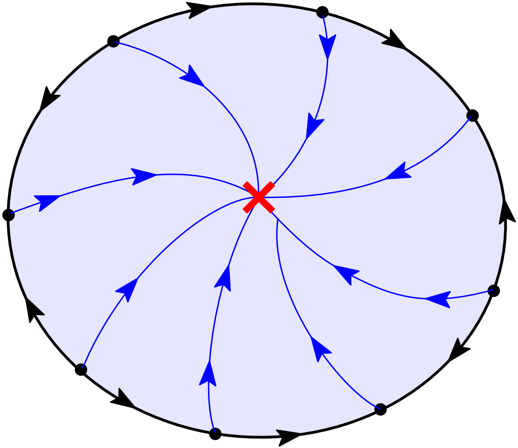
A k-alternating hypermap is said **accessibly pointed** if it is pointed such that there exists a directed path from any boundary vertex to the marked one.
We denote by A_k the associated generating function.



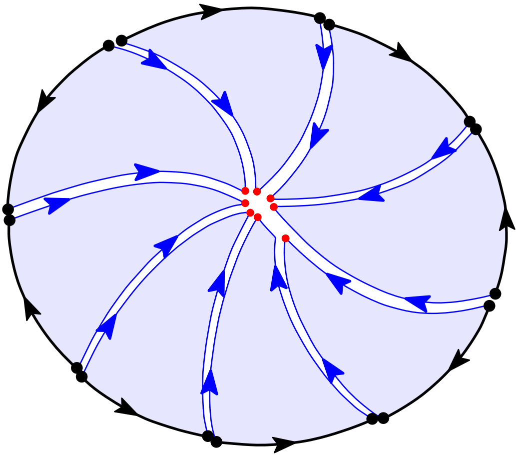
Start with an accessibly pointed hypermap...



...so there exists a path from all boundary vertices to the marked vertex...

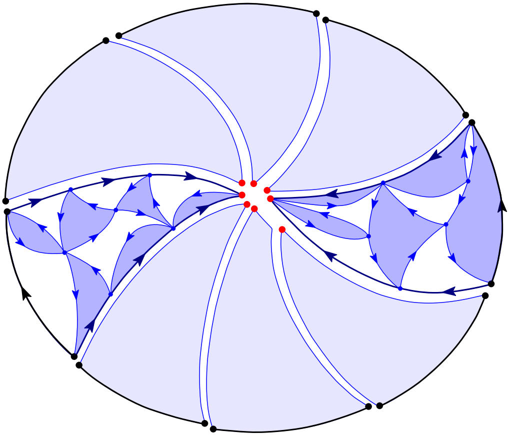


...then cut along these paths...



Accessibly pointed hypermap: Decomposition into slices

...and get an elementary piece of hypermap: a slice. Which has already been studied by M. Albenque, J. Bouttier and E. Guitter !



Definition of slices

Definition (Elementary slice)

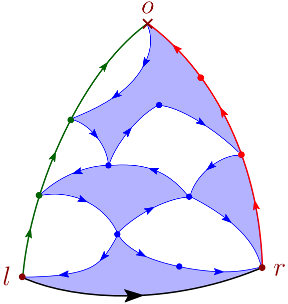
An **elementary slice** is a plane hypermap with three distinguished corners, denoted l, r, and o appearing in clockwise order on its outer face, and satisfying:

- the left boundary $[l, o]$ is a geodesic;
- the right boundary $[r, o]$ is the **unique** geodesic between these two corners;
- the unique edge $[l, r]$ is called the *base*.

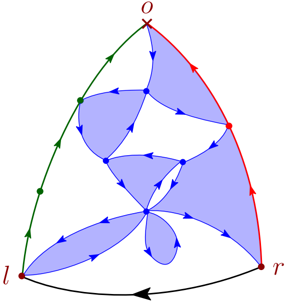
We also define the **increment** as the length difference between the right and the left boundary.

Definition (Elementary slice)

Slices of type $\mathcal{A}$ (resp. $\mathcal{B}$) are such that the base is oriented counter-clockwise (resp. clockwise).
We denote by $x(z) = x(z, t, t_i^\circ, t_j^\bullet)$ (resp. $y(z) = y(z, t, t_i^\circ, t_j^\bullet)$) their generating functions, where z marks the increment ($i =$ right - left).



Slice of type $\mathcal{A}$, $i = 0$



Slice of type $\mathcal{B}$, $i = -1$

Definition (Elementary slice)

Slices of type $\mathcal{A}$ (resp. $\mathcal{B}$) are such that the base is oriented counter-clockwise (resp. clockwise).
We denote by $x(z)$ (resp. $y(z)$) their generating functions, where z marks the increment (i = right - left).

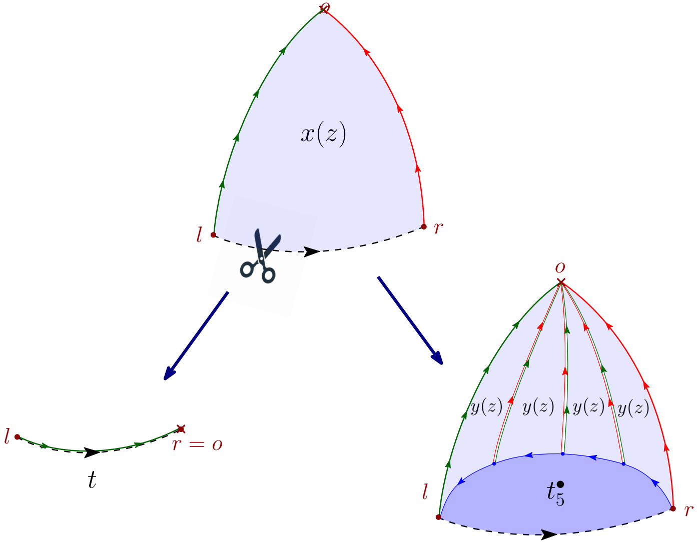
Slices are already well understood:

They are the rational parametrization of the spectral curve associated to H_1 and they verify a closed system of algebraic equations:

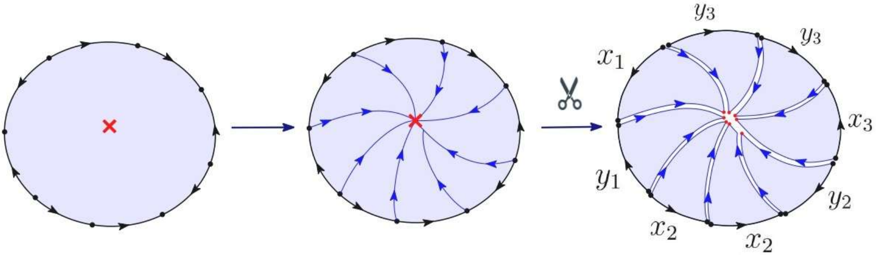
$$[z^{-k}]\Big(x(z) - tz - \sum_{d\geq 1} t_d^{\bullet}\, y(z)^{d-1}\Big) = 0 \qquad \text{for all } k \geq -1,$$

$$[z^{k}]\Big(y(z) - \sum_{d\geq 1} t_d^{\circ}\, x(z)^{d-1}\Big) = 0 \qquad \text{for all } k \geq 0.$$

Definition of slices



Accessibly pointed hypermap: back to decomposition into slices



Theorem 1 [J. Bouttier, B. Eynard, TL, 2026]

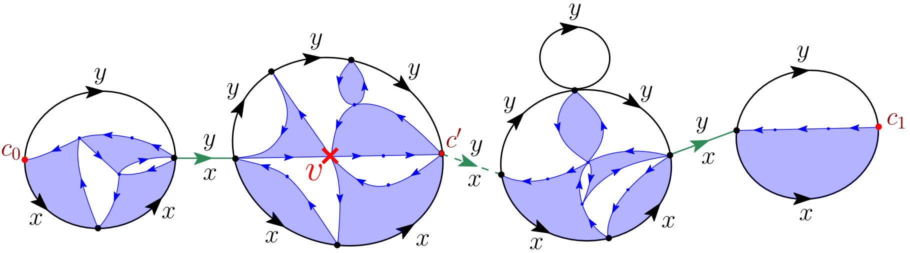
For any $k \geq 1$, the generating function of accessibly pointed k-alternating hypermaps reads

$$A_k(x_1, \ldots, y_k; t, t_i^\circ, t_j^\bullet) = [z^0] \frac{1}{(x_1 - x(z))(y_1 - y(z)) \cdots (y_k - y(z)))}.$$

Theorem 2 [J. Bouttier, B. Eynard, TL, 2026]

The generating functions H_1 and A_1 are related by the following formula:

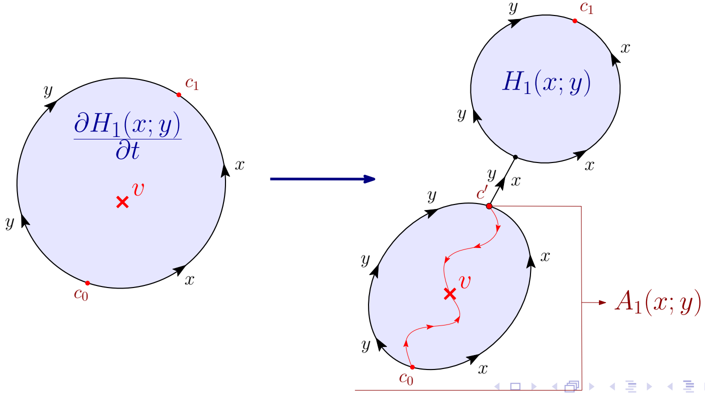
$$\frac{\partial H_1}{\partial t}(x, y) =?$$



Theorem 2 [J. Bouttier, B. Eynard, TL, 2026]

The generating functions H_1 and A_1 are related by the following formula:

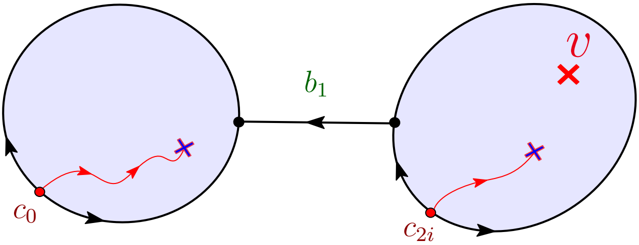
$$\frac{\partial H_1}{\partial t}(x, y, t) = A_1(x, y)\, H_1(x, y)$$



k-alternating hypermap: Relation with $A_i, i \leqslant k$

Lemma

A marked vertex is accessible from at least one of the outward corners of any k-alternating hypermap.



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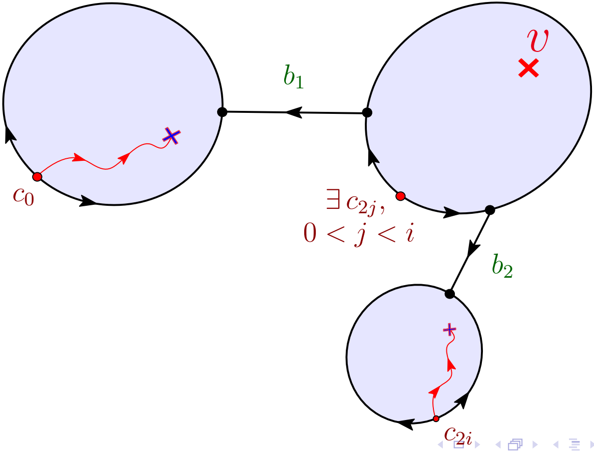
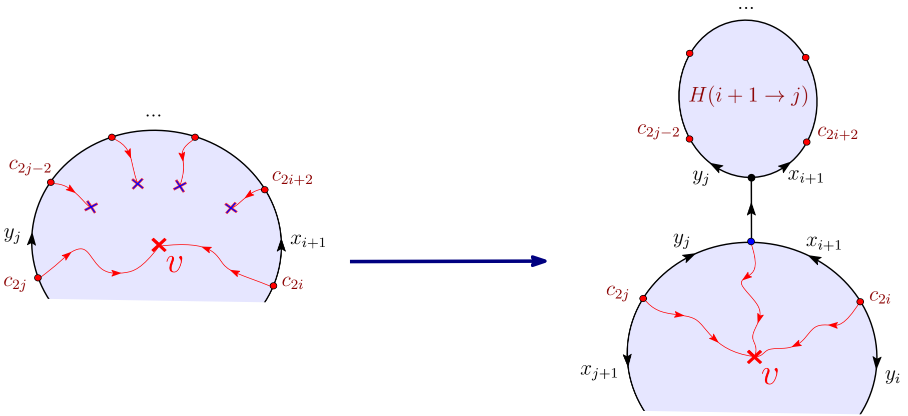
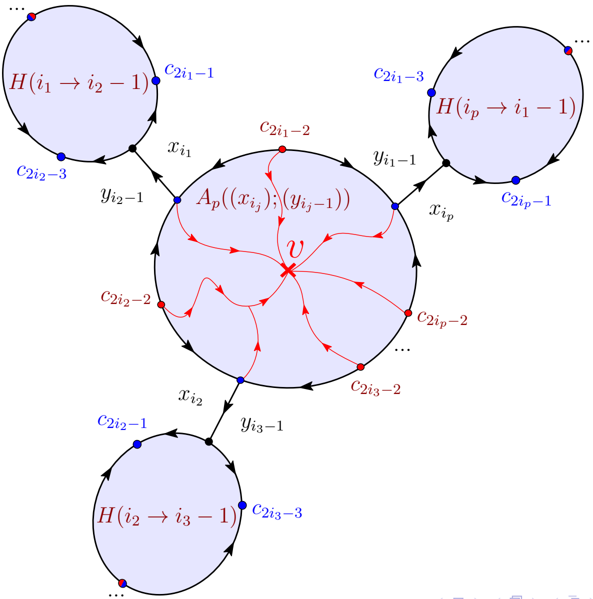


Now what happens between two consecutive outward corners that can be related to the marked vertex ?



k-alternating hypermap: Relation with $A_i, i \leqslant k$



k-alternating hypermap: Relation with $A_i, i \leqslant k$

Theorem 3 [TL, 2026]

The pointed k-alternating hypermap can be decomposed as follows:

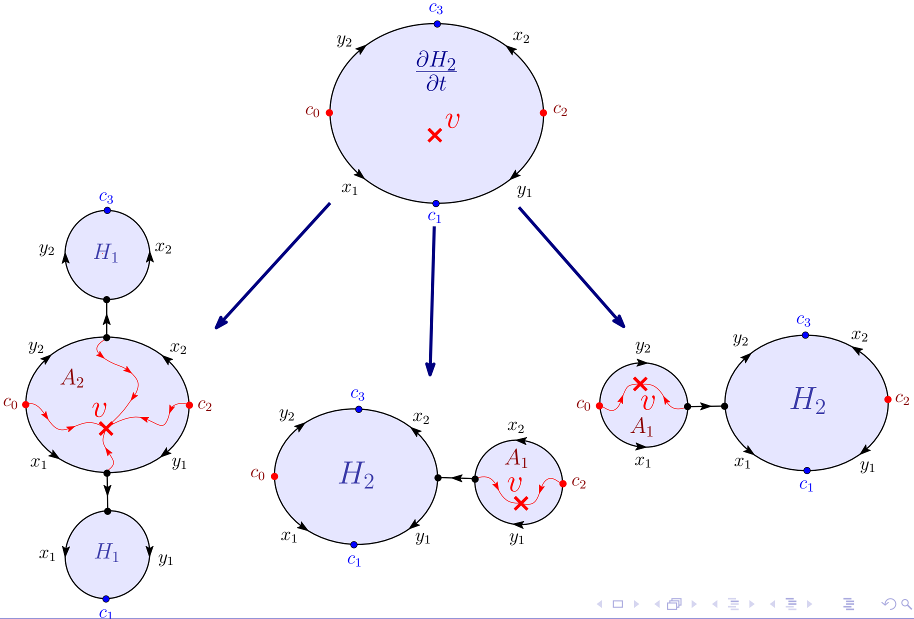
$$\frac{\partial H_k}{\partial t} = \sum_{p=1}^{k} \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant k} A_p(x_{i_1}, \ldots, x_{i_p}; y_{i_1 - 1}, \ldots, y_{i_p - 1}) \prod_{j=1}^{p} H(i_j \to i_{j+1} - 1)$$

Corollary 3.1 [B.Eynard + N. Orantin 05, L.Cantini 16]

The generating function of k-alternating hypermap can be expressed as a function of H_1 as follows:

$$H_k(x_1, \ldots, y_k; t, t_i^\bullet, t_j^\circ) = \sum_{\sigma \in S_k} C_\sigma(x_1, \ldots, y_k) \prod_{i=1}^{k} H_1(x_i; y_{\sigma(i)}; t, t_i^\bullet, t_j^\circ),$$

where the C_σ are universal functions: they do not depend on $t, (t_i^\bullet)_i, (t_i^\circ)_i$; and they are rational in the boundary variables x_i, y_i.


c_3
y_2
x_2
$\frac{\partial H_2}{\partial t}$
c_0
c_2
v
x_1
y_1
c_1
H_1
A_2
H_2
A_1

Zoom on case $k = 2$: proof of the corollary 2.1

The previous proof leads to the relation :

Theorem 4 [J. Bouttier, B. Eynard, TL, 2026]

$$\frac{\partial H_2(x_1, x_2; y_1, y_2)}{\partial t} = A_2(x_1, x_2; y_1, y_2)\, H_1(x_1; y_1)\, H_1(x_2; y_2) + \left(A_1(x_1; y_2) + A_1(x_2; y_1)\right) H_2(x_1, x_2; y_1, y_2).$$

How can we integrate this relation to restaure the algebricity ?

Zoom on case $k = 2$: proof of the corollary 2.1

Now using Theorem 2:

$$\frac{\partial H_1}{\partial t}(x, y) = A_1(x, y)\, H_1(x, y),$$

Theorem 4:

$$\begin{aligned}\frac{\partial H_2(x_1, x_2; y_1, y_2)}{\partial t} &= A_2(x_1, x_2; y_1, y_2)\, H_1(x_1; y_1)\, H_1(x_2; y_2) \\ &+ \left(A_1(x_1; y_2) + A_1(x_2; y_1)\right)\, H_2(x_1, x_2; y_1, y_2).\end{aligned}$$

And Theorem 1:

$$\begin{aligned}A_2 &= [z^0]\frac{1}{(x_1 - x(z))(x_2 - x(z))(y_1 - y(z))(y_2 - y(z))} \\ &\overset{PFD}{=} \frac{1}{(x_1 - x_2)(y_1 - y_2)}\left(A_1(x_1; y_1) + A_1(x_2; y_2) - A_1(x_1; y_2) - A_1(x_2; y_1)\right)\end{aligned}$$

Zoom on case $k = 2$: proof of the corollary 2.1

Now using Theorem 2

$$\frac{\partial H_1}{\partial t}(x,y,t) = A_1(x,y)\,H_1(x,y),$$

Theorem 4

$$\begin{aligned}\frac{\partial H_2(x_1,x_2;y_1,y_2)}{\partial t} &= A_2(x_1,x_2;y_1,y_2)\,H_1(x_1;y_1)\,H_1(x_2;y_2) \\ &+ \left(A_1(x_1;y_2)+A_1(x_2;y_1)\right)\,H_2(x_1,x_2;y_1,y_2),\end{aligned}$$

And a corollary of Theorem 1

$$A_2 \overset{PFD}{=} \frac{1}{(x_1-x_2)(y_1-y_2)}\left(A_1(x_1;y_1)+A_1(x_2;y_2)-A_1(x_1;y_2)-A_1(x_2;y_1)\right)$$

We finally get another proof of

Corollary 4.1 [B. Eynard 2002, N. Orantin 2005, L. Cantini, 2016]

$$H_2(x_1,x_2;y_1,y_2) = \frac{H_1(x_1;y_1)\,H_1(x_2;y_2) - H_1(x_2;y_1)\,H_1(x_1;y_2)}{(x_1-x_2)(y_1-y_2)}$$

Conclusion: Pros and Cons

Accessibly pointed hypermaps are effective tools to give bijective proofs for hypermaps. They make a bridge between general hypermaps and slices, that are already well known.

It allows us to find new equations than Tutte's which enables new perspectives for computations.

However this approach leads to differential equations which are not (a priori) algebraic. A general method to integrate them is still missing, however we've been able to integrate any equation we were confronted to for now.

Conclusion: What next ?

Hereafter we hope that this new method will lead us to:

- Use for specific boundary conditions (with A. Carrance + B. Eynard $\rightarrow$ *alternating boundary*).
- Generalize to general genus.
- Understand other operators/observables than traces in matrix models (correlation functions, boundary operators, geodesic distances ...).
- Keep track of distances between vertices and compute the asymptotics as local and continuous limits (*peeling ?*).
- Generalize our approach to other models ($O(n)$, Potts, *Matrix chain*...).
- *Understand the integrable combinatorics lexicon via this language* (for now: slices $\leftrightarrow$ spectral curve parametrization).

And maybe so much more !

Thank you for your attention !