

IPVTs: convergence and face volume densities

Journée cartes, UMPA, Ens de Lyon

30 June 2026, 15h30 (CEST)

Matteo D'Achille



based (mostly) on

D'A, Christoph Thäle

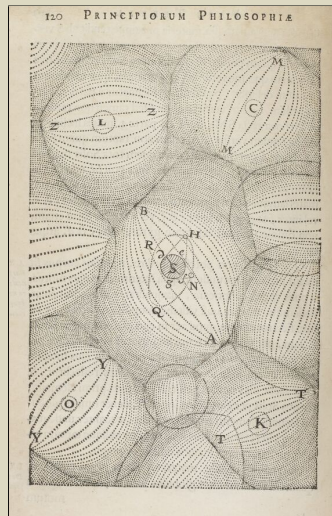
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D'A, Ali Khezeli

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Voronoi diagrams: a classic construction

*Principia
Philosophiæ*
Descartes 1644



Voronoi diagrams: a classic construction

Water pump Voronoi cell

vs

deaths distribution in
London's Broad Street
cholera outbreak of 1854

John Snow 1855



Voronoi diagrams: definition

One wants to compare **distances**.

- A metric **space** (E, d)
- **Nuclei**: a list of points $(x_i; i \geq 1) \in E$
- **Tiles**:

$$C_i := \{w \in E; d(w, x_i) \leq d(w, x_j), \forall j \geq 1\}.$$

The **Voronoi diagram** is $\text{Vor}(x_i : i \geq 1) = \{C_i; i \geq 1\}$.

$\text{Vor}(x_i : i \geq 1)$ is a **Voronoi tessellation** if it is locally-finite, normal and face-to-face.

Poisson Point Processes

In applications (wireless networks, . . .) it is natural to assume some complete randomness for the nuclei. A natural candidate is a **Poisson Point Process (PPP)** of nuclei.

Def. A **PPP** on a measurable space S is a random countable subset $\Pi \subset S$ s.t. the random variable $N(A) = \# \{A \cap \Pi\}$ satisfies:

- $\forall A \subset S, N(A) \sim \text{Pois}(\mu(A))$, for a measure μ (**intensity**);
- If $A, B \subset S$ are **disjoint** sets, $N(A) \perp N(B)$.

Poisson–Voronoi tessellations

(See Møller 1992)

Def. A **Poisson–Voronoi tessellation** is the Voronoi tessellation of a PPP of **nuclei** with intensity measure μ .



Today the **PPP** of nuclei has an **infinite Radon measure**

$$(\mu = \lambda \cdot \text{Leb}_{\mathbb{R}^2}, \lambda \cdot \text{Vol}_M \dots, \text{ for } \lambda > 0).$$

The proportionality constant λ is called **intensity**

Appearances of Poisson–Voronoi tessellations

☞: general case - s: λ small - L: λ large

☞ **Euclidean** space motivated by crystal growth (Meijering 1953, Gilbert 1961)

☞ **Hyperbolic** plane (with Bernoulli percolation on cells) (Benjamini–Schramm 2000)

s **Mean characteristics** in hyperbolic plane and 3-space (Isokawa 2000)

s Bernoulli–Voronoi **percolation** on regular trees, hyperbolic plane (Bhupatiraju 2019)

L “Euclidean” limit on **surfaces**, Gauss–Bonnet Theorem (Calka–Chapron–Enriquez 2019)

s, L Critical probabilities in **percolation** on the hyperbolic plane (Hansen–Müller 2021)

"Pointless" Poisson–Voronoi tessellation and $2/\pi$

(Budzinski–Curien–Petri Invent. Math.)

$\mathcal{S}_g$: a closed hyperbolic surface of genus g (large)

Cheeger's constant:

$$h(\mathcal{S}_g) \stackrel{\text{def}}{=} \inf_{A \subset \mathcal{S}_g} \left\{ \frac{\text{Perim}(A)}{\text{Area}(A)} ; 2 \text{Area}(A) \leq \text{Area}(\mathcal{S}_g) \right\}$$

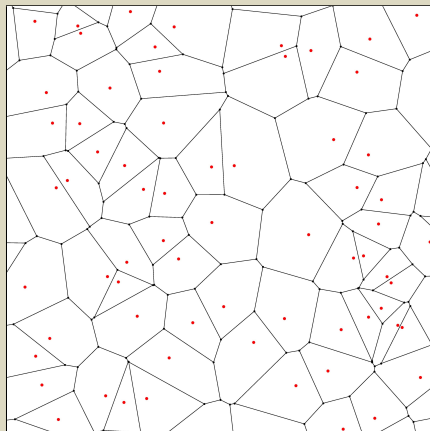
bounded from above by $2/\pi$.

: $\lim_{\lambda \downarrow 0}$ of Poisson–Voronoi tessellation of **hyperbolic plane**



What happens as $\lambda \downarrow 0$? And the $2/\pi$?

Euclidean plane $\mathbb{R}^2$



Ideal tessellations

Sketch of recipe

Tenet: one needs to compare “**differences**” of distances to nuclei.
However, these are ∞ quantities.

Two main deterministic **ingredients** as $\lambda \downarrow 0$:

 **Convergence** of the nuclei to the “boundary”;

 **Convergence** of difference of distances (**proto-delays**).

Probabilistic extension (convergence in distribution) if underlining space has **exponential** volume growth (tightness of delays).

Ideal tessellations

Abstract deterministic theorem (**D'ACELÜ** AOP)

Let $(E, d, \mathbf{o}, \mu)$ be a locally compact metric space with a special point $\mathbf{o}$ (**origin**) and an infinite Radon measure μ . Let nuclei $x_i^{(\lambda)}$ be **sorted** according to $d(\mathbf{o}, x_i^{(\lambda)})$. If the following hold:

- I. $x_i^{(\lambda)} \xrightarrow[\lambda \downarrow 0]{\text{Gromov}} \theta_i \in \partial E$ (conv. to the boundary);
- II. $d(\mathbf{o}, x_i^{(\lambda)}) - d(\mathbf{o}, x_{\textcircled{1}}^{(\lambda)}) = \delta_i^{(\lambda)} \xrightarrow[\lambda \downarrow 0]{} \delta_i$ (conv. of proto-**delays**);

Then (**D'ACELÜ**):

$$\text{Vor}(x_i^{(\lambda)}) \xrightarrow[\lambda \downarrow 0]{\text{Fell}} \text{Vor}((\theta_i, \delta_i); i \geq 1) \quad (\text{Ideal tessellation})$$

Exponential volume growth: regular trees, hyperbolic space, etc...

Hyperbolic space $\mathbb{H}_d$

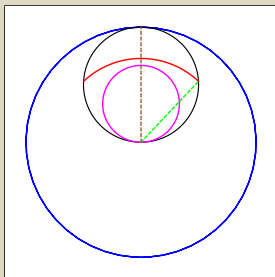
$\mathbb{H}_d$: d -dimensional Riemannian manifold of const. sectional curv. $= -1$

Möb_d : the group of **isometries** of $\mathbb{H}_d$

Ball model

$$\mathbb{B}_d = \{x \in \mathbb{R}^d; |x| < 1\}$$

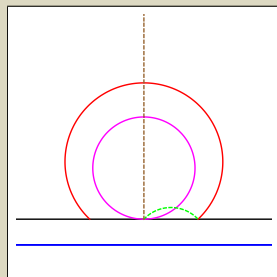
$$ds^2 = 4 \frac{dx_1^2 + \dots + dx_d^2}{(1 - |x|^2)^2}$$



Upper-half space model

$$\mathbb{U}_d = \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$$

$$ds^2 = \frac{dx_1^2 + \dots + dx_{d-1}^2 + dz^2}{z^2}$$

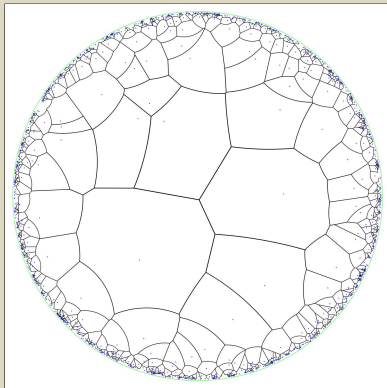


Nuclei and delays

Recipe of probabilistic extension for $\mathbb{H}_d$

- **Nuclei:**

$\mathbf{X}^{(\lambda)} = (X_i^{(\lambda)}; i \geq 1)$ a PPP of intensity measure $\lambda^{d-1} \cdot \text{Vol}_{\mathbb{H}_d}$



Animation credit: Russ Lyons

Nuclei and delays

Recipe of probabilistic extension for $\mathbb{H}_d$

- **Nuclei:**

$\mathbf{X}^{(\lambda)} = (X_i^{(\lambda)}; i \geq 1)$ a PPP of intensity measure $\lambda^{d-1} \cdot \text{Vol}_{\mathbb{H}_d}$

With this choice of **intensity measure**, the point closest to $\mathbf{o}$ is roughly at distance $\log(1/\lambda)$ as $\lambda \rightarrow 0$ in any dimension d .

- **Delays:**

$$D_i^{(\lambda)} \stackrel{\text{def}}{=} d_{\mathbb{H}_d}(\mathbf{o}, X_i^{(\lambda)}) - \log(1/\lambda), \quad i = 1, \dots$$

The delays $(D_i^{(\lambda)})_{i \geq 1}$ form an a.s. **strictly increasing** sequence.

Convergence of (proto-)delays



Convergence of (proto-)delays (D'ACELÜ AOP)

$(D_i^{(\lambda)})_{i \geq 1} \xrightarrow[\lambda \downarrow 0]{\text{law}} (D_i)_{i \geq 1}$, a **PPP** on $\mathbb{R}$ of intensity measure

$$c_d e^{(d-1)s} ds, \quad c_d = 4^{1-\frac{d}{2}} \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}.$$

Equivalently: the set $(R_i)_{i \geq 1} \stackrel{\text{def}}{=} \left(\frac{c_d}{d-1} e^{(d-1)D_i} \right)_{i \geq 1}$ is a rate-1 homogeneous PPP on $\mathbb{R}_+$.

Extended ideal boundary and corona

Let us work in the ball model $\mathbb{B}_d$ in “spherical coordinates”:

$$X_i^{(\lambda)} = \left(\underbrace{\Theta_i^{(\lambda)}}_{d-1 \text{ angles}}, d_{\mathbb{H}_d}(\mathbf{o}, X_i^{(\lambda)}) - \log(1/\lambda) = D_i^{(\lambda)} \right)$$

- **Conditionally** on $D_i^{(\lambda)}$, $(\Theta_i^{(\lambda)})_{i \geq 1}$ are i.i.d. **uniform** on $\partial\mathbb{B}_d = \mathbb{S}_{d-1}$.
- A.s. converg. $(\Theta_i^{(\lambda)}, D_i^{(\lambda)}) \rightarrow (\Theta_i, D_i)$ via Skorokhod (or “dilations”).

$\text{Vor}(\mathbf{X}^{(\lambda)}) \xrightarrow[\lambda \rightarrow 0]{\text{dist.}} \text{Vor}(\Theta, \mathbf{D}) \stackrel{\text{def}}{=} \mathcal{V}_d$ **ideal Poisson–Voronoi tessellation**

Equivalently, $(\Theta, \mathbf{R})$ (recall $R_i = \frac{c_d}{d-1} e^{(d-1)D_i}$ for all i) is a PPP on $\mathbb{S}_{d-1} \times \mathbb{R}_+$ (**corona**) of intensity measure

$$\mu_d = \text{Unif}_{\mathbb{S}_{d-1}} \otimes \text{Leb}_{\mathbb{R}_+}.$$

Möbius action extended to the corona

Let $\phi \in \text{Möb}_d$ and (θ_i, r_i) be a point in the corona $\mathbb{S}_{d-1} \times \mathbb{R}_+$.



Equivariant action of Möb_d (D'ACELÜ AOP)

Let $\phi : \mathbb{S}_{d-1} \times \mathbb{R}_+ \rightarrow \mathbb{S}_{d-1} \times \mathbb{R}_+$ be defined by

$$\phi(\theta_i, r_i) \stackrel{\text{def}}{=} \left(\phi(\theta_i), \frac{r_i}{K(\phi^{-1}(\mathbf{o}), \theta_i)} \right)$$

where $K(z, \theta) = \left(\frac{1-|z|^2}{|z-\theta|^2} \right)^{d-1}$ is hyperbolic **Poisson kernel**.

Then ϕ provides a transitive group action that leaves the measure μ_d **invariant** under the action of Möb_d .

The IPVT $\mathcal{V}_d$: a pointless tessellation

We can **canonically** assign a $z \in \mathbb{H}_d$ to a cell without a nucleus!

Given two points $(\theta_1, r_1), (\theta_2, r_2)$ in the **corona**, solve

$$\frac{r_1}{K(z, \theta_1)} \stackrel{?}{\leq} \frac{r_2}{K(z, \theta_2)} .$$

Perpendicular bisectors are Euclidean semi-spheres:

$$|z - \theta_1| r_2^{\frac{1}{2(d-1)}} = |z - \theta_2| r_1^{\frac{1}{2(d-1)}} , \quad z \in \mathbb{B}_d .$$

Topological properties of $\mathcal{V}_d$



$\mathcal{V}_d$ is a tessellation (D'ACELÜ AOP)

Almost surely, $\mathcal{V}_d$ is:

- locally finite;
- face-to-face: the intersection of two cells will always be a face of each cell, or empty;
- normal: each k -face is the intersection of exactly $d - k + 1$ cells.

Geometry of perpendicular bisectors

Let us focus on the cell $\mathcal{C}_d \subset \mathcal{V}_d$ **containing the origin** $\mathbf{o} \in \mathbb{H}_d$.



Perpendicular bisector in $\mathbb{U}_d$ (D'ACELÜ AOP)

In $\mathbb{U}_d$ where θ_1 is put at ∞ and θ writes $(C, 0, \dots, 0)$, the $(d-1)$ -hyperplane bisecting (θ_1, r_1) and (θ, r) is a **half-sphere** centered at $(C, 0, \dots, 0)$ with radius

$$\rho = \sqrt{1 + C^2} \left(\frac{r_1}{r} \right)^{\frac{1}{2(d-1)}}.$$

The deposition model for $\mathcal{C}_d$



Description of $\mathcal{C}_d$ (D'ACELÜ AOP)

$\mathcal{C}_d$ is the complement of all **open half-balls** whose **centers** and **radii** are a **PPP** of intensity measure

$$\mu_d = 2 \cdot \mathcal{E}_d \cdot dx \, \rho^{1-2d} d\rho \, \mathbf{1}_{\rho \leq \sqrt{1+x^2}} \quad \text{on} \quad \mathbb{R}^{d-1} \times \mathbb{R}_+ .$$

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- $\mathcal{E}_d$: random variable of law $\text{Exp}(c'_d)$, $c'_d = \frac{4^{1-d/2} \pi^{d/2}}{\Gamma(d/2)}$;

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- dx : **Lebesgue** on $\mathbb{R}^{d-1}$ for the **centers** of the **half-balls**;

The deposition model for $\mathcal{C}_d$



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- $\rho^{1-2d} d\rho$ on $\mathbb{R}_+$ for the **radii** of the half-balls;

The deposition model for $\mathcal{C}_d$



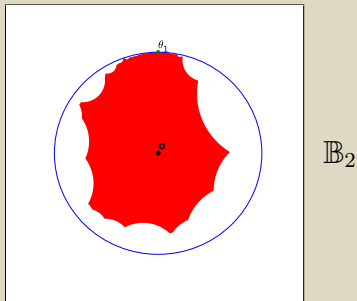
Description of $\mathcal{C}_d$ (D'ACELÜ AOP)

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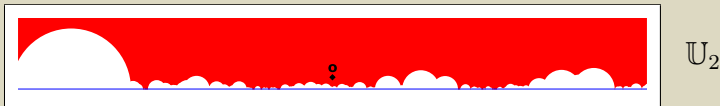
$$\mu_d = 2 \cdot \mathcal{E}_d \cdot dx \, \rho^{1-2d} d\rho \, \mathbf{1}_{\rho \leq \sqrt{1+x^2}} \quad \text{on} \quad \mathbb{R}^{d-1} \times \mathbb{R}_+.$$

- $\mathcal{E}_d$: **random variable** of law $\text{Exp}(c'_d)$, $c'_d = \frac{4^{1-d/2} \pi^{d/2}}{\Gamma(d/2)}$;
- dx : Lebesgue on $\mathbb{R}^{d-1}$ for the **centers** of the half-balls;
- $\rho^{1-2d} d\rho$ on $\mathbb{R}_+$ for the **radii** of the half-balls;
- $\mathbf{1}_{\rho \leq \sqrt{1+x^2}}$: condition of **origin exclusion**.

Portrait of $\mathcal{C}_2$

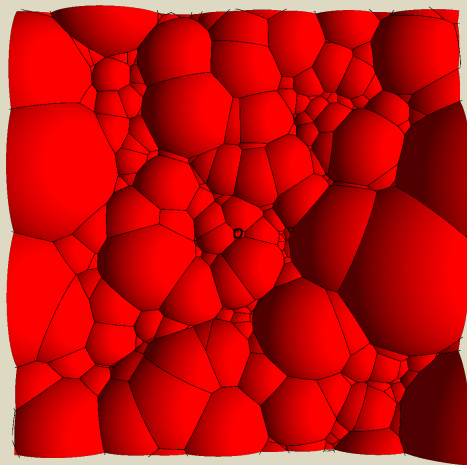


$$\varphi(z) = i \frac{i+z}{i-z} \quad \Downarrow \quad \Uparrow \quad \varphi^{-1}(w) = i \frac{w-i}{w+i}$$



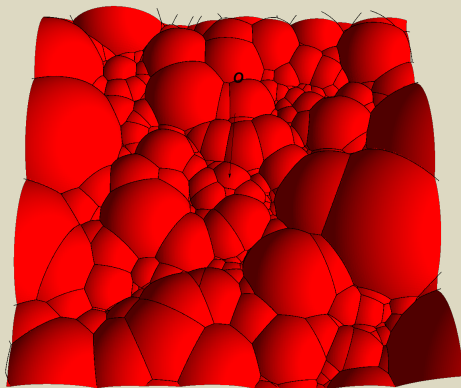
Portrait of $\partial\mathcal{C}_3$ in $\mathbb{U}_3$ (view from above)

Interactive version at sketchfab.com/foam



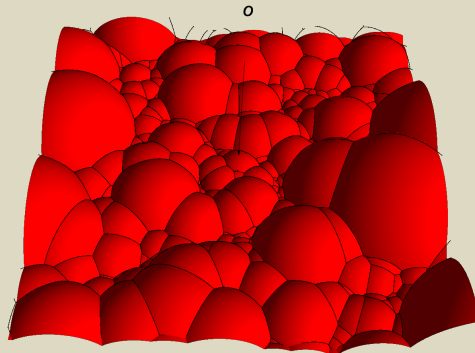
Portrait of $\partial\mathcal{C}_3$ in $\mathbb{U}_3$

Interactive version at sketchfab.com/foam



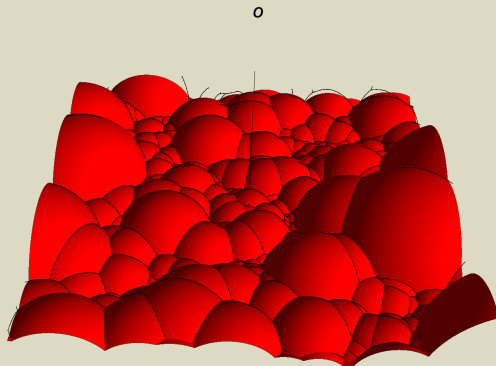
Portrait of $\partial\mathcal{C}_3$ in $\mathbb{U}_3$

Interactive version at sketchfab.com/foam



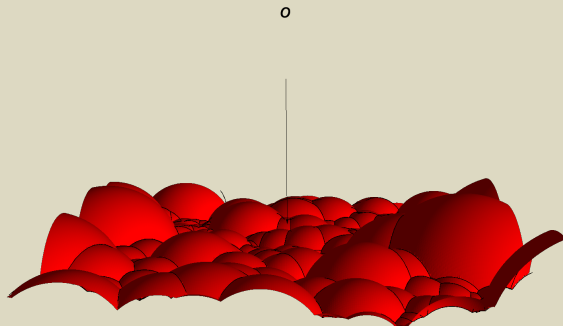
Portrait of $\partial\mathcal{C}_3$ in $\mathbb{U}_3$

Interactive version at sketchfab.com/foam



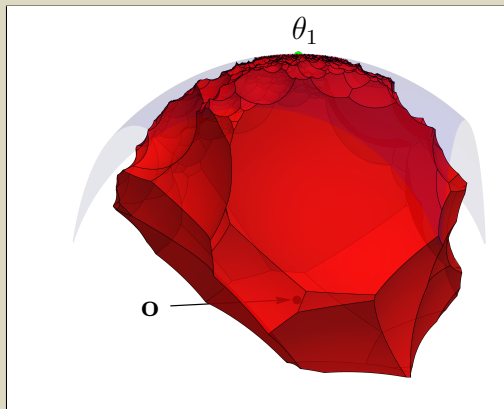
Portrait of $\partial\mathcal{C}_3$ in $\mathbb{U}_3$ (view from the side)

Interactive version at sketchfab.com/foam



Portrait of $\mathcal{C}_3$ in $\mathbb{B}_3$

Interactive version at sketchfab.com/jewel (or in your hands)



Local properties of $\mathcal{C}_d$: hole probability

Hole probability: the law of the distance of $\mathbf{o}$ to $\partial\mathcal{V}_d$

(i) **Conditional** on $R_1 = s$,

$$\mathbb{P}(B_r(\mathbf{o}) \subset \mathcal{C}_d \mid R_1 = s) = \exp(-sI_d(r)) ;$$

(ii) **Averaging** on the value of R_1 , (recall $R_1 \sim \text{Exp}(1)$)

$$\mathbb{P}(B_r(\mathbf{o}) \subset \mathcal{C}_d) = \frac{1}{1 + I_d(r)} ,$$

where

$$1 + I_d(r) = \frac{1}{c_d} \int_{\mathbb{R}^{d-1}} \frac{dx}{\left(\sqrt{\cosh^2 r + |x|^2} - \sinh r \right)^{2d-2}} .$$

Local properties of $\mathcal{C}_d$: hole probability

- $d = 2$

$$\mathbb{P}(B_r(\mathbf{o}) \subset \mathcal{C}_2) = \frac{\pi}{4 \left(\arctan(e^r) \cosh^2 r + 2 \sinh r \right)}$$

- $d > 2$ odd: a rational function of e^r , e.g. for $d = 3$

$$\mathbb{P}(B_r(\mathbf{o}) \subset \mathcal{C}_3) = \frac{3e^{-2r}}{2 + e^{2r}}$$

Counting face intensities and face densities

Let $\mathcal{V}_d = \text{IPVT}(\mathbb{H}_d)$.

Let $\mathcal{V}_{d,k}$ = collection of k -faces of cells of $\mathcal{V}_d$. Consider the following random measures

$$\mu_{d,k} := \sum_{\mathfrak{s} \in \mathcal{V}_{d,k}} \delta_{\text{Cen}(\mathfrak{s})} \quad \text{and} \quad \tilde{\mu}_{d,k} := \sum_{\mathfrak{s} \in \mathcal{V}_{d,k}} \nu_{\mathfrak{s}}.$$

where $\nu_{\mathfrak{s}}$ is the k -dimensional hyperbolic measure on $\mathfrak{s}$.

Since $\mathcal{V}_d$ is isometry invariant in law,

$$\mathbb{E}[\mu_{d,k}] = \text{I}_{d,k} \cdot \text{Vol}_{\mathbb{H}_d} \quad \text{and} \quad \mathbb{E}[\tilde{\mu}_{d,k}] = D_{d,k} \cdot \text{Vol}_{\mathbb{H}_d}$$

$$(D_{d,k} \equiv \tilde{\text{I}}_{d,k} \text{ of D'ACELÜ AOP/measure face intensities})$$

Similar definitions hold for $\text{I}_{d,k}(\lambda)$ and $D_{d,k}(\lambda)$

Isokawa and rolling balls

- Isokawa computed $D_{d,k}(\lambda)$ for $d = 2, 3$ and all $k \in \{0, \dots, d-1\}$ via expert knowledge of hyperbolic trigonometry.



Measure face intensities $D_{d,d-1}$ (D'ACELÜ AOP)

For all $d \geq 2$,

$$D_{d,d-1} = -\frac{1}{2} \cdot \frac{d}{d\varepsilon} \mathbb{P}\left(B_\varepsilon(\mathbf{o}) \subset \mathcal{C}_d\right) \Big|_{\varepsilon=0}.$$

Completing the picture, for all $\lambda > 0$



All face volume densities for λ -PVT ($\mathbb{H}_d$) (D'A-Thäle 26)

For $d \geq 2$ and $k \in \{0, 1, \dots, d-1\}$, with $q = d - k$,

$$D_{d,k}(\lambda) = \frac{\lambda^{q+1}}{(q+1)!} A_{d,q} \int_0^\infty e^{-\lambda b_d(r)} \sinh^{(d-1)(q+1)}(r) \, dr,$$

where

$$A_{d,q} := \frac{\omega_{d(q+1)-q} \cdot \omega_{d+1}^q \cdot \omega_{d-q+1}}{\omega_{d(q+1)-q+1}}.$$

$(A_{d,q} \propto \mathbb{E}[q\text{-dim Euclidean vol of the simplex spanned by } q+1 \text{ i.i.d. Unif}(\mathbb{S}_{d-1}) \text{ points}])$

**Face volume densities for $\text{IPVT}(\mathbb{H}_d)$ (D'A-Thäle 26)**

For $d \geq 2$ and $k \in \{0, 1, \dots, d-1\}$, with $q = d - k$,

$$D_{d,k} = \frac{A_{d,q}}{q+1} \frac{(d-1)^q}{\omega_d^{q+1}}.$$



Proof: Laplace method as $\lambda \downarrow 0$.

Proof for finite intensity face densities



Sketch of Proof:

- New hyperbolic Blaschke–Petkantschin formula applied to equidistant set $L(\mathbf{x})$ of $(q + 1)$ (generic) points;
- Multivariate Mecke equation applied onto a ball around any point of $L(\mathbf{x})$.
- Change of variables ...



Remarks on $D_{d,k}$, link with Poisson-Voronoi perco

- $d = 2$

$$D_{2,0} = \frac{1}{\pi} < \frac{2}{\pi} = D_{2,1}$$

- $d = 3$

$$D_{3,0} = \frac{16\pi}{35} < D_{3,1} = \frac{8\pi}{15} > D_{3,2} = \frac{4}{3}$$

- $d \geq 4$

$$D_{d,0} > D_{d,1} > \dots > D_{d,d-1}.$$

These constants appear in the c of $p_c(\lambda; \mathbb{H}_2) = c \cdot \lambda + o(\lambda)$ (Hansen–Muller AOP).

 Is that **true** for $\forall d \geq 3$?

Ideal tessellations

Abstract deterministic theorem (D'ACELÜ AOP)

Let $(E, d, \mathbf{o}, \mu)$ be a locally compact metric space with a special point $\mathbf{o}$ (**origin**) and an infinite Radon measure μ . Let nuclei $x_i^{(\lambda)}$ be **sorted** according to $d(\mathbf{o}, x_i^{(\lambda)})$. If the following hold:

- I. $x_i^{(\lambda)} \xrightarrow[\lambda \downarrow 0]{\text{Gromov}} \theta_i \in \partial E$ (conv. to the boundary);
- II. $d(\mathbf{o}, x_i^{(\lambda)}) - d(\mathbf{o}, x_{\textcircled{1}}^{(\lambda)}) = \delta_i^{(\lambda)} \xrightarrow[\lambda \downarrow 0]{} \delta_i$ (conv. of proto-**delays**);

Then (D'ACELÜ AOP):

$$\text{Vor}(x_i^{(\lambda)}) \xrightarrow[\lambda \downarrow 0]{\text{Fell}} \text{Vor}((\theta_i, \delta_i); i \geq 1) \quad (\text{Ideal tessellation})$$

$\text{IPVT}(\mathbb{H}_2 \times \mathbb{H}_2, L^1)$

D'A 24⁺

Fraczyk–Mellick–Wilkins 2023: let $\mathcal{X}$ an higher-rank symmetric space or a product of trees. The cells of $\text{IPVT}(\mathcal{X})$ share an unbounded border a.s. !

Let $\mathcal{M} = (\mathbb{H}_2 \times \mathbb{H}_2, d)$ with distance d

$$d(x, y) = d_{\mathbb{H}_2}(x_1, y_1) + d_{\mathbb{H}_2}(x_2, y_2)$$

and origin $\mathbf{o} = (\mathbf{o}_{\mathbb{H}_2}, \mathbf{o}_{\mathbb{H}_2})$.



Convergence towards $\text{IPVT}(\mathcal{M})$ (D'A 24)

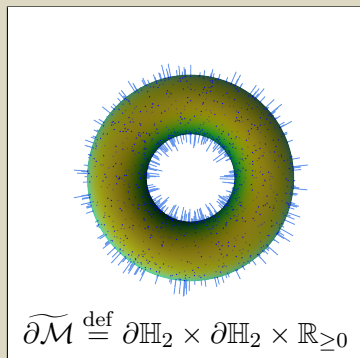
Let $\mathbf{X}^{(\lambda)} = (X_i^{(\lambda)} ; i \geq 1)$ be a $\text{PPP}(\lambda \cdot \text{Vol}_{\mathcal{M}})$ (points of $\mathbf{X}^{(\lambda)}$ ranked by $d(X_i^{(\lambda)}, \mathbf{o})$). Then

$$\text{Vor}(\mathbf{X}^{(\lambda)}) \xrightarrow[\lambda \downarrow 0]{\text{law}} \text{Vor}(\Theta, \Phi, \mathbf{D}) \equiv \text{IPVT}(\mathcal{M})$$

IPVT($\mathcal{M}$)

D'A 2412.00822

$\Theta = (\Theta_1, \dots)$ and $\Phi = (\Phi_1, \dots)$ are i.i.d. uniform on $\partial\mathbb{H}_2 \times \partial\mathbb{H}_2 \simeq \mathbb{S}_1 \times \mathbb{S}_1 \simeq \mathbb{T}_2$ and $\mathbf{D} = (D_1, \dots)$ is s.t. $(\pi^2 e^{D_i})_{i \geq 1}$ is a rate-1 homogeneous point process on $\mathbb{R}_{\geq 0}$.



Measure on $\widetilde{\partial\mathcal{M}}$:

$$\mu = \text{Unif} \otimes \text{Unif} \otimes \text{Leb}$$

On IPV $T(\mathcal{M})$



Transitive group action of isometries (D'A 24)

Let $g = (g_1, g_2)$ be an isometry of $(\mathbb{H}_2 \times \mathbb{H}_2, L^1)$. Then

$$g(\theta, \phi, r) \stackrel{\text{def}}{=} \left(g_1(\theta), g_2(\phi), \frac{r}{K(g_1^{-1}(\mathbf{o}), \theta) \cdot K(g_2^{-1}(\mathbf{o}), \phi)} \right)$$

is a **transitive group action** that leaves the corona measure μ invariant. Consequently, the law of IPV $T(\mathcal{M})$ is invariant under **all** isometries of $\mathcal{M}$.



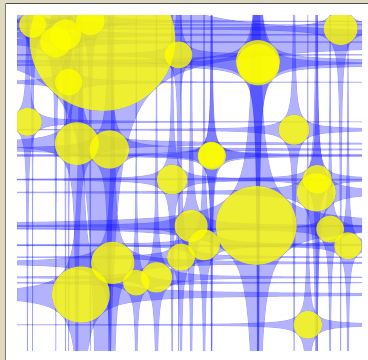
Topology of cells at $\partial\mathcal{M}$ (D'A 24)

Almost surely, each tile of IPV $T(\mathcal{M})$ **does not cover** the union of the two orthogonal circles through its ideal nucleus.

On $\text{IPVT}(\mathbb{H}_2 \times \mathbb{H}_2, L^1)$

Topology of IPVT cells

Sketch of proof:

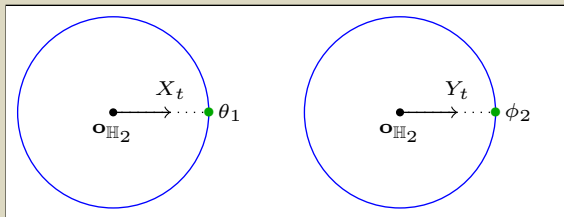


$$\tilde{\mu}_{2,2} = \text{cst} \cdot \mathcal{E}_2 \cdot dx \frac{1}{\rho^5} d\rho \mathbf{1}_{\rho^4 \leq (1+|x|^2)(1+|y|^2)} \quad \text{on} \quad \mathbb{R}^2 \times \mathbb{R}_+$$

Comparison with Bierné–Estrade Poisson random ball model (yellow) ■

The “double quarter-past-three” argument

D'A 24



Put $\varepsilon := 1 - \tanh(t/2)$. Then the separation to a generic corona point (θ, ϕ, r) is

$$\mathbf{d}\left((X_t, Y_t), (\theta, \phi, r)\right) = r \cdot [f_1(\varepsilon) + 2 \cdot f_2(\varepsilon)(2 - \cos \theta - \cos \phi) + 4 \cdot f_3(\varepsilon)(1 - \cos \theta)(1 - \cos \phi)]$$

where $f_1(\varepsilon) = O(\varepsilon^2)$ **BUT** $f_2(\varepsilon) = f_3(\varepsilon) = O(1)$ when $\varepsilon \rightarrow 0$.

Corollary: the tie break at infinity



Tie-break at infinity (D'A 24)

Let (Θ_1, Φ_1, R_1) and (Θ_2, Φ_2, R_2) be the 1st and 2nd corona points. Let $\mathcal{C}_{2,2}$ be the cell of (Θ_1, Φ_1, R_1) . Then

$$\mathbb{P}((\Theta_1, \Phi_2) \in \mathcal{C}_{2,2}) = \frac{1}{2} + \frac{2}{\pi^2} = 0.70264^+.$$

Proof. $(\Theta_1, \Phi_2) \in \mathcal{C}_{2,2} \iff \frac{d((\Theta_1, \Phi_2), (\Theta_1, \Phi_1, R_1))}{d((\Theta_1, \Phi_2), (\Theta_2, \Phi_2, R_2))} \leq 1.$

In this ratio of separations use:

- $X, Y \sim \text{Exp}(1) \implies \frac{X}{X+Y} \sim \text{Unif}([0, 1]);$
- $\Theta \sim \text{Unif}([-\pi, \pi]) \implies \frac{1}{2} (1 - \cos \Theta) \sim \text{Beta}\left(\frac{1}{2}, \frac{1}{2}\right).$



IPVT of higher rank symmetric spaces

Fraczyk–Mellick–Wilkins Ann. Math. proved that every pair of cells shares an **unbounded boundary** and used this to answer Gaboriau's fixed price question in some cases.



**Vanishing unicity threshold for Poisson–Voronoi perco.
(D'A-Grebik-Khezeli-Recke-Wilkins 25)**

Let M be the symmetric space of a connected and simply connected higher rank semisimple real Lie group G . Then $p_u(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$.

Related ideas applied to (non-amenable, Cayley) graphs provided many answers to a question of Pete–Rokob 25!

IPVT of higher rank symmetric spaces

Fraczyk–Mellick–Wilkins Ann. Math. do not prove convergence of tessellations in the low-intensity limit, see also Wilkens' lecture notes at IHES summer school 2025.



Convergence to a unique IPVT on symmetric spaces (D'A-Khezeli 26)

In symmetric spaces, low-intensity Poisson–Voronoi diagrams converge to a unique IPVT.

Full convergence of nuclei

Let $F(r) := \mu(B_r(\mathbf{o}))$ be the volume function. F is **log-regularly-varying** if $F \circ \log$ is regularly-varying:

$\forall r_0 \in \mathbb{R} \lim_{r \rightarrow +\infty} F(r + r_0)/F(r)$ exists ($= e^{br_0}$).



Full convergence of nuclei (D'A-Khezeli 26)

$X^{(\lambda)}$ converges weakly (after a suitable shift) to a nontrivial point process $(\Theta_i, \Delta_i)_{i \geq 1}$ in the extended boundary iff:

- (i) F is log-regularly-varying with index $b > 0$;
- (ii) $F(r)^{-1} \mu|_{B_r(\mathbf{o})}$ (as a proba measure on $\bar{E}$) converges weakly to a proba measure ∂E .

$\text{Vor}(X^{(\lambda)})$ converges to $\text{Vor}((\Theta_i, \Delta_i)_{i \geq 1})$.

Convergence towards $\text{IPVT}(\mathbb{H}_d)$, again

In $\mathbb{H}_d$:

$F(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^r (\sinh \rho)^{d-1} d\rho$ is log-regularly-varying with index $b = (d-1)$;

The uniform probability measure on $B_R(\mathbf{o})$ converges to the harmonic measure on $\partial\mathbb{H}_d$.

“New” proof for convergence to $\text{IPVT}(\mathbb{H}_d)$ via low-intensity limit!

Elements of proof

$SD(E)$ = space of shifted distance func. on E . Then $x \mapsto (y \mapsto d(x, y))$ defines an homeo from E to $SD(E)/\mathbb{R}$, denoted by ι_t .



Sketch of Proof:

- FMW Ann. Math. prove that $\lim_{t \rightarrow \infty} (F(t))^{-1} \iota_t^* \mu$ has a unique subsequential limit;
- The limiting measure is diffuse under (i) - (ii);
- These are non-branching CAT(0) spaces $\implies$ Vor is non-degenerate.



Some perspectives



Many, actually!

- $\text{Var}[\text{Vol}_{\mathbb{H}_d}(\mathcal{C}_d)] = \text{cst}_d$, via $\lambda \downarrow 0$ limit (w.i.p. with Anna Gusakova)
- Nuclei: other point processes (α -hyperbolic ens. as $\alpha \downarrow 0$)
- Limits of $p_c(\lambda)$, $p_u(\lambda)$ for Poisson–Voronoi percolation of $\mathbb{H}_d$?
- 1-skeleton of $\text{IPVT}(\mathbb{H}_d)$ vs Degree Constr. MST (Russ Lyons)
- $\text{IPVT}(\text{Complex hyperbolic spaces})$ (Greg Kuperberg)
- Dynamic version of $\text{IPVT}(\mathbb{H}_d)$ (Itai Benjamini)

StokaChic : haute bijouterie mathématique

w Nicolas Curien, Lucas Moissonnier et Vasilisa Nikiforova

POLY MATH Le langage de la géométrie  **Exposition**
1 - 19 décembre 2025

StoKaChic : haute bijouterie mathématique



Plongez dans une expérience immersive qui questionne et valorise différentes formes d'intelligence humaine : percevoir la beauté, comprendre la complexité, et relier l'une à l'autre.

Découvrez une nouvelle tendance : les bijoux mathématiques, et explorez, de façon ludique et interactive, les concepts mathématiques qui les sous-tendent.

Exposition à la Bibliothèque Mathématique Jacques Hadamard (IMO).
Adresse : Bâtiment 307, Université Paris-Saclay.

Vernissage le 04/12/2025 à 17h (amphithéâtre Yoccoz) : **conférence** de Matteo D'ACHILLE, mathématicien à l'Institut Elie Cartan de Lorraine, suivie d'une visite commentée de Vasilisa NIKIFOROVA, créatrice de l'exposition (Bibliothèque).

Entrée libre et gratuite. Ouvert à tou-te-s. Adressé aux étudiant-e-s.

UNIVERSITÉ PARIS-SACLAY   



 **Exhibition**
4 - 26 February, 2026

StoKaChic: Mathematical Haute Joaillerie



Mathematical jewellery is a new trend. Research at the Laboratoire de Mathématiques d'Orsay on stochastic geometry inspired the creation of jewellery pieces shaped like cells of ideal Poisson-Voronoi tessellation in 3-dimensional hyperbolic space.

Embark on a fascinating journey that highlights these unique forms and reveals the mathematics behind them. Experience three facets of human intelligence: perceiving beauty, understanding mathematics, and bridging the two.

Opening: February 4, 2026, at 18:00

- **lecture** by Matteo D'Achille, mathematician at the Institut Elie Cartan de Lorraine (Léon Motchane Amphitheater).
- **guided tour** by Vasilisa Nikiforova, creator of the installation (Library).

Exhibition at the IHES Library.

Thanks !

