High-genus triangulations: local and global aspects

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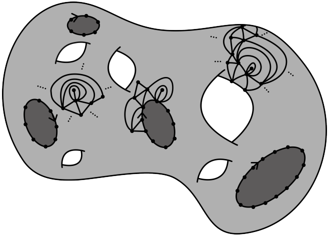


Introduction

- Random maps: discrete models of random geometry.
- The planar case has been extensively studied over the past twenty years, whereas the high-genus regime is much less understood due to the lack of precise combinatorial estimates.
- While random planar maps are "fractals", high-genus random maps behave more like random graphs having good expansion properties.
- Our goal in this talk is to understand the local structure of high-genus random maps. Then, we will estimate global distances using this local information.

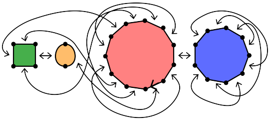
Definitions

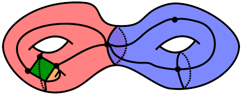
Definition

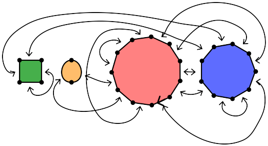
A **map** is given by:

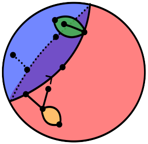
- A collection of oriented polygons.
- An orientation-preserving gluing of the edges of the polygons.

This construction yields a connected surface S endowed with a graph G embedded in S. The map is said to be **planar** if S has genus 0 (the sphere). We only consider maps rooted at a distinguished oriented edge.

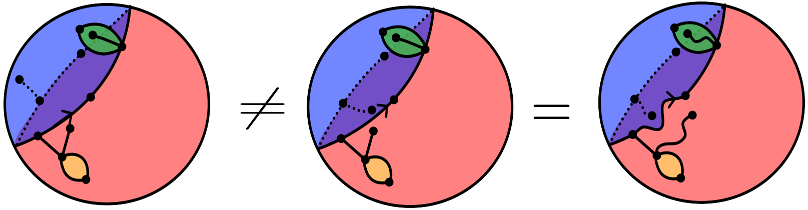








Definitions



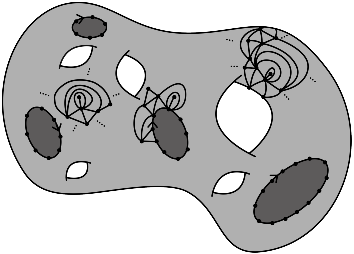
Triangulations

Definition

A **triangulation** is a map such that:

- There exist $\ell \geq 0$ distinguished simple faces, called the **boundaries**.
- All faces that are not boundaries have degree 3.

In this talk, we consider type-I triangulations, where loops and multiple edges are allowed.



Notations

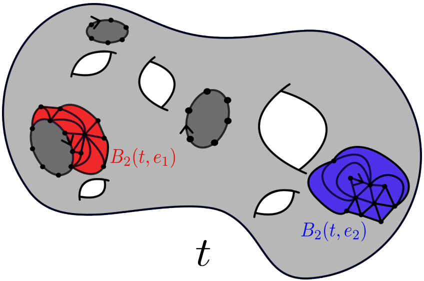
For $n, g, \ell \geq 0$ and $p_1, \ldots, p_\ell \geq 1$, we denote by:

- $\mathcal{T}(n, g; p_1, \ldots, p_\ell)$: the set of genus-$g$ triangulations (a torus with g holes), with ℓ boundaries of perimeters $p_1, \ldots, p_\ell$ and $2n - \sum_{i=1}^{\ell}(p_i - 2)$ triangles.
- $T_{n,g,p_1,\ldots,p_\ell}$: an element chosen uniformly at random in $\mathcal{T}(n, g; p_1, \ldots, p_\ell)$.

Local distance

- Let $t \in \mathcal{T}(n, g; p_1, \ldots, p_\ell)$ and let e be an oriented edge of t, with origin x. Define $B_r(t, e)$ as the map obtained by keeping all the triangles having a vertex at distance at most $r - 1$ from x. We define the local distance:

$$d_{\mathrm{loc}}((t, e), (t', e')) = \frac{1}{1+\sup\{r \geq 0 : B_r(t,e) = B_r(t',e')\}}.$$



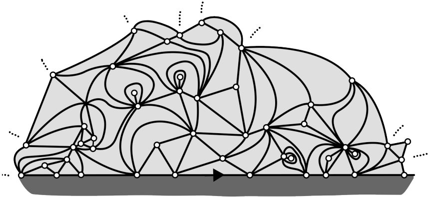
Local limits of high-genus triangulations

Does $T_{n,g}$ converge in distribution as $n \to +\infty$?

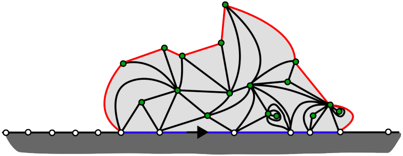
- Angel–Schramm (2003): $T_{n,0} \to \mathbb{T}$ where $\mathbb{T}$ is called the Uniform Infinite Planar Triangulation (UIPT). Balls of radius r grow like r^4, not like r^2 as in the Euclidean plane.
- Budzinski–Louf (2019): in the high-genus regime where $\frac{g}{n} \to \theta \in [0, \frac{1}{2})$, $T_{n,g} \to \mathbb{T}_\theta$, where $\mathbb{T}_\theta$ is an infinite planar triangulation.
- We have $\mathbb{T}_0 \overset{(d)}{=} \mathbb{T}$. For $\theta > 0$, the triangulations $\mathbb{T}_\theta$ behave very differently and exhibit a **hyperbolic behaviour**:
  - Balls of radius r grow like $\exp(D_\theta^{-1} r)$.
  - Transient random walk.
  - Positive isoperimetric constant.
- The limit $\mathbb{T}_\theta$ is planar!

Infinite triangulations of the half-plane

- Angel and Ray introduced in 2013 a one-parameter family $(\mathbb{H}_\theta)_{0\leq\theta<\frac{1}{2}}$ of half-plane triangulations (a single boundary of infinite perimeter).
- They are characterized by the following Markovian property: conditionally on $t \subset \mathbb{H}_\theta$, we have $\mathbb{H}_\theta \backslash t \overset{(d)}{=} \mathbb{H}_\theta$.
- Moreover, $\mathbb{P}(t \subset \mathbb{H}_\theta)$ is explicit.



$\mathbb{H}_\theta$



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Local limits: the case with large boundaries

Theorem (Lions, 2026)

Let $\theta \in [0, \frac{1}{2})$ and $\frac{g}{n} \to \theta$. Let also $\mathbf{p} = (p_1, \cdots, p_\ell) \in (\mathbb{N}^*)^\ell$ be such that $p_1 + \cdots + p_\ell = o(n)$. Let e_1 (resp. e_2) be a distinguished edge chosen uniformly in (resp. on the boundary of) $T_{n,g,\mathbf{p}}$. The following convergence holds for the local topology:

$$(T_{n,g,\mathbf{p}}, e_1) \xrightarrow{(d)} \mathbb{T}_\theta.$$

Moreover, assume that $\ell = o(p_1 + \cdots + p_\ell)$. Then we have

$$(T_{n,g,\mathbf{p}}, e_2) \xrightarrow{(d)} \mathbb{H}_\theta.$$

- The assumption $p_1 + \cdots + p_\ell = o(n)$ guarantees that the boundaries do not appear in the limit of $(T_{n,g,\mathbf{p}}, e_1)$.
- The assumption $\ell = o(p_1 + \cdots + p_\ell)$ guarantees that e_2 is typically chosen on a boundary of very large perimeter.

Application: typical distances

- High-genus random maps have been shown to have distances (diameter and distances between two uniform points) of logarithmic order:
  - Ray (2014): for unicellular maps.
  - Budzinski–Chapuy–Louf (2023): for triangulations.
- Mirzakhani (2010): In a continuous setting, similar estimates hold for random hyperbolic surfaces under the Weil–Petersson measure.
- These results only give the order of magnitude for distances but not the sharp asymptotics.

Application: typical distances

Theorem (Lions, 2026)

Assume $\frac{g}{n} \to \theta \in (0, \frac{1}{2})$. Let e_n^1, e_n^2 be two oriented edges chosen uniformly and independently in $T_{n,g}$. Let us denote by x_n^1, x_n^2 their starting points. There exists a constant $D_\theta > 0$ such that

$$\frac{d_{T_{n,g}}(x_n^1, x_n^2)}{\log n} \xrightarrow[n\to+\infty]{(\mathbb{P})} D_\theta.$$

- The constant D_θ is such that the ball of radius r in $\mathbb{T}_\theta$ (the local limit of $T_{n,g}$) grows like $\exp(D_\theta^{-1} r)$.

High-genus triangulations are expander-like

- An **expander** is a very well-connected graph: exponential volume growth, logarithmic diameter, positive isoperimetric constant.

	Planar $(g = 0)$	**High genus** $(\frac{g}{n} \to \theta > 0)$
Local limit (typical edge)	UIPT $= \mathbb{T}_0$	PSHT $\mathbb{T}_\theta$ $(\theta > 0)$
Volume growth $\lvert B_r \rvert$	$\asymp r^4$	$\asymp \exp(D_\theta^{-1} r)$
Simple random walk	recurrent	transient
Isoperimetry	$\exists (A_n),\ \frac{\lvert\partial A_n\rvert}{\lvert A_n\rvert} \to 0$	$\forall A,\ \lvert\partial A\rvert \geq c_\theta \lvert A\rvert$
Typical distance & diameter	$\asymp n^{1/4}$	$\asymp \log n$
Scaling limit	Brownian sphere	no scaling limit

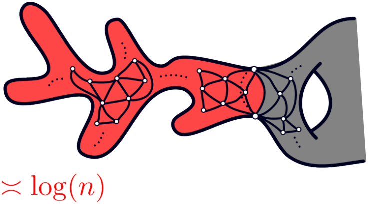
- In short: high-genus triangulations have good expansion properties like random graphs.

A useful tool: isoperimetric inequality

We fix $\frac{g}{n} \to \theta \in (0, \frac{1}{2})$.

- Budzinski–Chapuy–Louf (2023) : there exist K_θ, c_θ such that, with high probability, for any $A \subset T_{n,g}$, as soon as $K_\theta \log(n) \leq |A| \leq n$, then $|\partial A| \geq c_\theta |A|$.

- With high probability, $T_{n,g}$ has "defects" of logarithmic size...



- In this talk, we will cheat and use the isoperimetric inequality as soon as $|A| \leq n$.

The proof: structure and order

2 **Definition of the hulls $B_r^\bullet$**

$\downarrow$

4 **Convergence of the hull boundary perimeter:** $\left|\partial B^\bullet_{(1-\varepsilon)D_\theta \log n}\right| \asymp n^{1-\varepsilon}$

$\downarrow$

3 **Volume of the balls:** $\left|B_{(1-\varepsilon)D_\theta \log n}\right| \asymp n^{1-\varepsilon}$

$\downarrow$

1 **Main theorem:** $d_{T_{n,g}}(x_n^1, x_n^2) \approx D_\theta \log n$

1–**4**: order in which the points are treated in this talk.

Convergence of the volume of radius balls

For any $r \geq 0$, let us denote by $B_r(T_{n,g}, x_n^1)$ (or just B_r for short) the ball of radius r centered at x_n^1. We denote by $|B_r|$ the number of triangles of B_r.

Proposition

For any $\varepsilon > 0$, we have

$$\frac{\log |B_{(1-\varepsilon)D_\theta \log(n)}|}{(1-\varepsilon)\log(n)} \xrightarrow[n\to+\infty]{(\mathbb{P})} 1.$$

- A direct consequence is the lower bound for the typical distances. That is $\mathbb{P}(d_{T_{n,g}}(x_n^1, x_n^2) > (1-\varepsilon)D_\theta \log(n)) \xrightarrow[n\to+\infty]{} 1$.

Proof of the main theorem assuming the proposition

- It remains to prove

$$\mathbb{P}(d_{T_{n,g}}(x_n^1, x_n^2) \le (1+\varepsilon)D_\theta \log(n)) \to 1.$$

- We fix $\delta > 0$. Using isoperimetric inequalities, as long as $|T_{n,g} \setminus B_r| \ge \delta n$, we have $|B_{r+1}| \ge |B_r| + |\partial B_r| \ge (1+c_{\theta,\delta})|B_r|$ for some $c_{\theta,\delta} > 0$.
- By induction as long as $|T_{n,g} \setminus B_{(1-\varepsilon)D_\theta \log(n)+i}| \ge \delta n$:

$$|B_{(1-\varepsilon)D_\theta \log(n)+i}| \ge (1+c_{\theta,\delta})^i n^{1-\varepsilon+o(1)}.$$

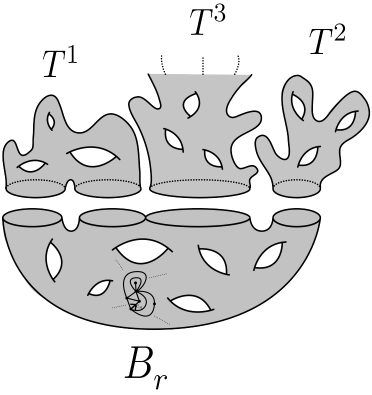
- We deduce $|T_{n,g} \setminus B_{(D_\theta+\varepsilon C_{\theta,\delta})\log(n)}| \le \delta n$ for some $C_{\theta,\delta} > 0$.
- We deduce

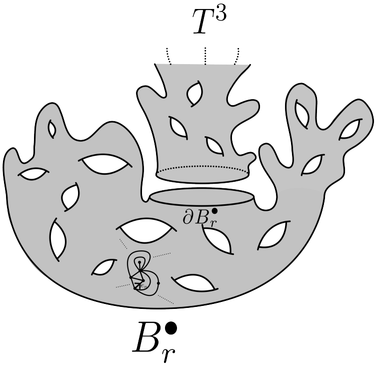
$$\mathbb{P}\left(d_{T_{n,g}}(x_n^1, x_n^2) \ge (1+\varepsilon C_{\theta,\delta})D_\theta \log(n)\right) \le \delta \text{ for } n \text{ large enough}.$$

This concludes the proof letting $\varepsilon \to 0$ and then $\delta \to 0$.

The proof: the hull

For $r \geq 0$, we write $T_{n,g} \setminus B_r = T^1, \dots, T^k$. We define the **hull of radius** r denoted by $B_r^\bullet$ as the ball B_r to which we glue the components T^i having at most n triangles. Since $T_{n,g}$ has $2n$ triangles, $T_{n,g} \setminus B_r^\bullet$ is either empty or consists of a single connected component.





The hull is a "good" approximation for the ball

We assume $|B_r| \leq n^{1-\varepsilon}$.

- Using the isoperimetric inequality, with high probability:

$$\sum_{i=1}^{k} |T^i| \mathbb{1}_{|T^i| \leq n} \leq c_\theta^{-1} \sum_{i=1}^{k} |\partial T^i| = c_\theta^{-1} |\partial B_r| \leq c_\theta^{-1} n^{1-\varepsilon}.$$

  In particular, we have $B_r^\bullet \neq T_{n,g}$.

- Furthermore

$$1 \leq \frac{|B_r^\bullet|}{|B_r|} \leq \frac{|B_r^\bullet|}{|\partial B_r^\bullet|} \leq c_\theta^{-1}.$$

- It suffices to prove

$$\frac{\log |\partial B^\bullet_{(1-\varepsilon) D_\theta \log(n)}|}{(1-\varepsilon)\log(n)} \xrightarrow[n \to +\infty]{(\mathbb{P})} 1.$$

Discovering the neighbourhood of the hull

We fix $r \gg 1$. We work conditionally on $B_r^\bullet$ and assume $|B_r^\bullet| \leq n^{1-\varepsilon}$.

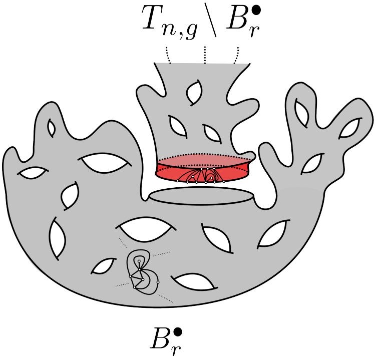
- We write $k(r), h(r), p_1(r), \dots, p_\ell(r)$ for the size of $B_r^\bullet$, its genus and the lengths of the boundaries. Then

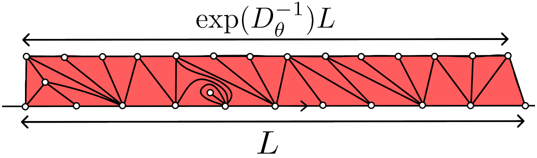
$$T_{n,g} \setminus B_r^\bullet \in \mathcal{T}(n-k(r), g-h(r)-\ell+1; p_1(r), \dots, p_\ell(r))$$

  and is uniformly distributed!
- The ratio $\frac{|\partial B_{r+1}^\bullet|}{|\partial B_r^\bullet|}$ can be read in the neighbourhood of the boundary of $T_{n,g} \setminus B_r^\bullet$.
- Moreover, $\frac{g-h(r)-\ell+1}{n-k(r)} \to \theta$ and $p_1(r), \dots, p_\ell(r) = o(n)$ uniformly in r. We can apply our local limit result for high-genus triangulations with boundaries to $T_{n,g} \setminus B_r^\bullet$.

Growth of the hull perimeter in pictures!

- In the half-plane limit $\mathbb{H}_\theta$, the length of a large segment of length $L \gg 1$ on the boundary typically "grows" by a factor $\exp(D_\theta^{-1})$.
- Spanning $\partial B_r^\bullet$ with large segments and using this estimate for each segment yields the estimate $\frac{|\partial B_{r+1}^\bullet|}{|\partial B_r^\bullet|} \approx \exp(D_\theta^{-1})$.





Uniform convergence of the growth of the hull perimeter

We have the following proposition.

Proposition

For any $\varepsilon, \delta > 0$, we have

$$\sup_r \mathbb{P}\left(|B_r^\bullet| \leq n^{1-\varepsilon} \text{ and } \left| \frac{|\partial B_{r+1}^\bullet|}{|\partial B_r^\bullet|} - \exp(D_\theta^{-1}) \right| \geq \delta \right) \xrightarrow[n \to +\infty]{} 0.$$

- We deduce that the perimeter of the hull of radius r grows like:

$$|\partial B_r^\bullet| = \exp\left((1 + o(1)) D_\theta^{-1} r \right).$$

- Since we do not have a quantitative estimate, it could happen that for exceptional values of r, the ratio $\frac{|\partial B_{r+1}^\bullet|}{|\partial B_r^\bullet|}$ is arbitrarily large...

A first moment bound

Proposition

There exists a constant $C_\theta > 0$ such that for any $\varepsilon > 0$, for n large enough, for any $r \geq 0$, we have

$$\mathbb{E}\left[\mathbb{1}_{|B_r^\bullet| \leq n^{1-\varepsilon}} \left| \frac{|\partial B_{r+1}^\bullet|}{|\partial B_r^\bullet|} \right| \right] \leq C_\theta,$$

We omit its proof. This is obtained using only crude combinatorial estimates.

The convergence of $\log \left|\partial B^{\bullet}_{(1-\varepsilon)D_\theta \log(n)}\right|$.

- We have

$$\frac{\log \left|\partial B^{\bullet}_{(1-\varepsilon)D_\theta \log(n)}\right|}{(1-\varepsilon)\log(n)} = \frac{1}{(1-\varepsilon)\log(n)} \sum_{r=0}^{(1-\varepsilon)D_\theta \log(n)-1} \underbrace{\log \frac{|\partial B^{\bullet}_{r+1}|}{|\partial B^{\bullet}_{r}|}}_{X_n(r)}$$

- We proved that $X_n(r)$ converges (uniformly in r) in probability to D_θ^{-1}.
- We proved that $\mathbb{E}[\exp(X_n(r))]$ is bounded (uniformly in r).
- Thus, $X_n(r)$ converges (uniformly in r) in L^1 to D_θ^{-1} and we conclude by the law of large numbers.

Perspective: local limit with a macroscopic boundary

- In our previous local limit result, we have studied the regime $p = o(n)$. What happens when $\frac{p}{n} \to \alpha > 0$?

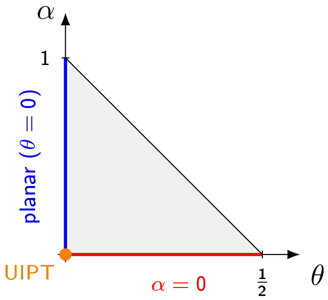
Conjecture

Let $\frac{g}{n} \to \theta \in [0, \frac{1}{2})$ and $\frac{p}{n} \to \alpha \in [0, 1 - 2\theta]$. Let e be an edge chosen uniformly on $T_{n,g,p}$. We have

$$(T_{n,g,p}, e) \to \mathbb{T}_{\theta,\alpha},$$

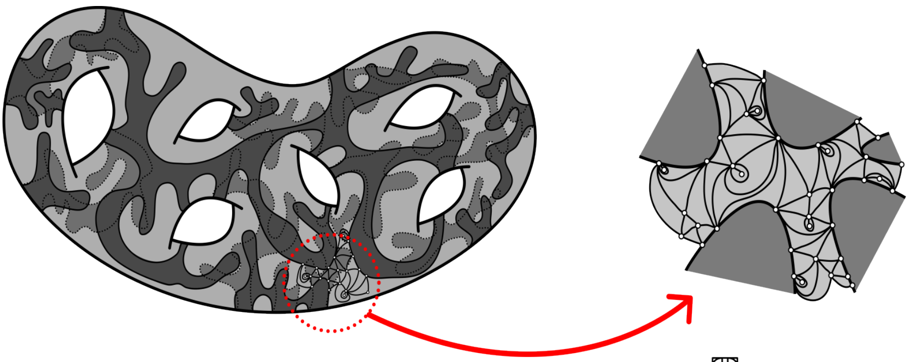
where $\mathbb{T}_{\theta,\alpha}$ is a map with infinitely many infinite faces.

Perspective: the conjectural phase diagram



- $\alpha = 0$: already treated in this talk. The limit is the PSHT $\mathbb{T}_\theta$.
- $\theta = 0$, planar: treated by Angel–Ray. The boundary folds on itself and the limit is a half-plane triangulation that looks like a Kesten tree.
- $\alpha > 0$ and $\theta > 0$ (grey part): the new cases.

Perspective: when the boundary is “very large”



$T_{n,g,p}$ $\qquad$ $\mathbb{T}_{\theta,\alpha}$