

Modular symmetry models to explain leptonic flavour hierarchies

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IRN Terascale Heidelberg

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The **flavour puzzle(s)**:

- Why 3 generations?
- How neutrinos get their masses?
- Are neutrinos Majorana or Dirac?
- Do we really need so many parameters?
19 flavour parameters / 25 SM parameters

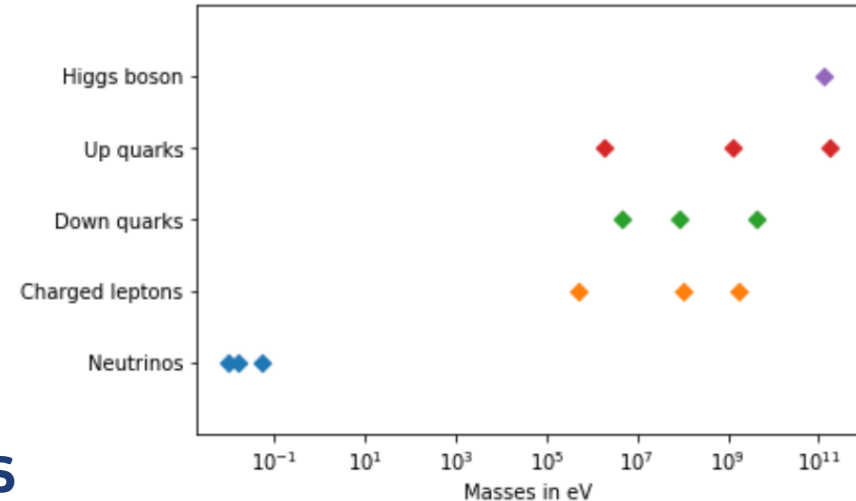
Open
questions

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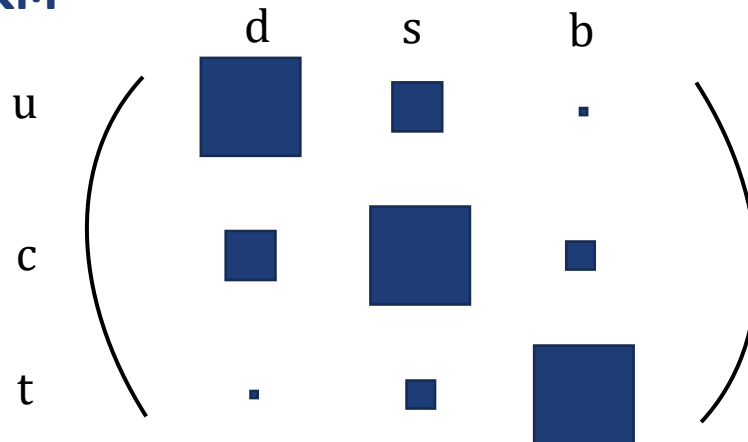
Open questions

Masses Hierarchy

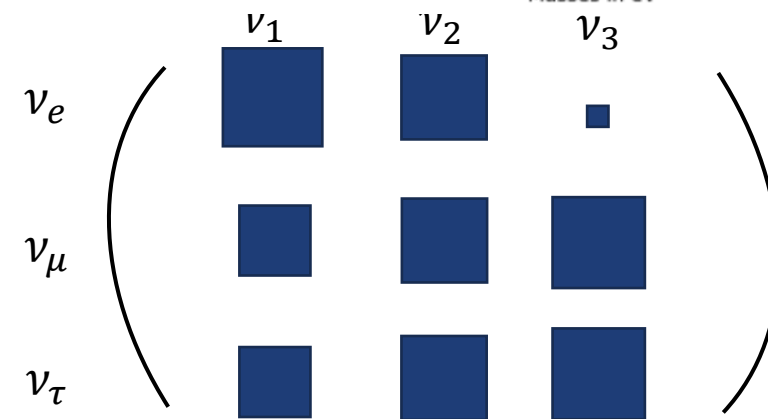


Open answers

CKM



PMNS



Suggests a broken flavour symmetry group:

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[Altarelli, Feruglio; 1002,0211]

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Tribimaximal pattern

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Exp. excluded: broken discrete symmetry group

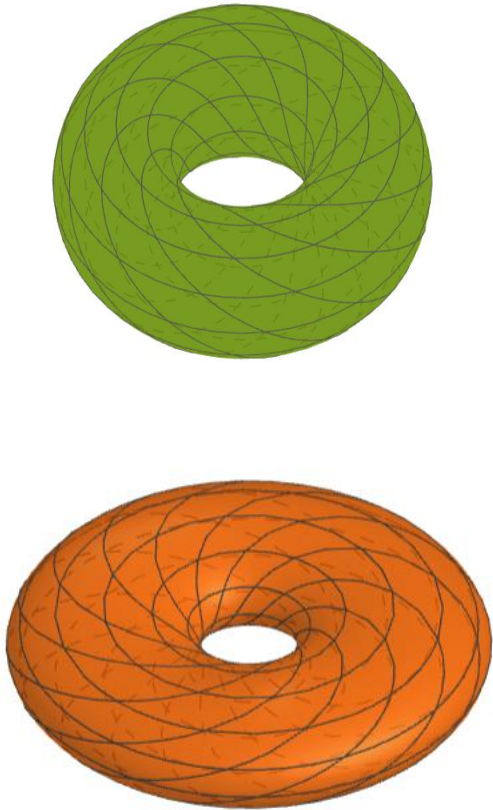
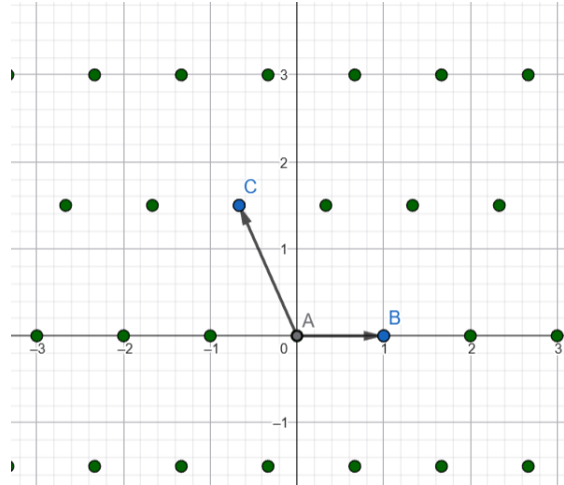
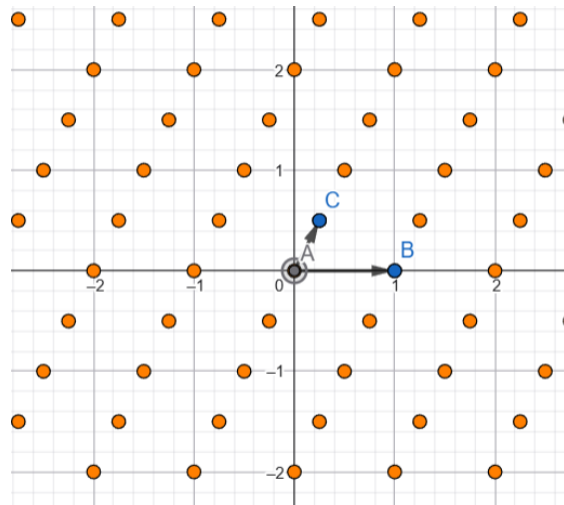
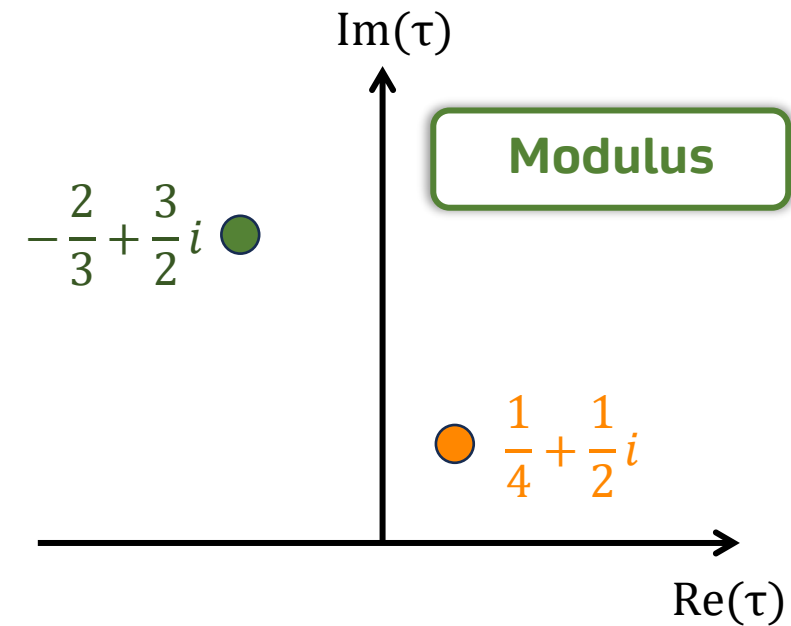
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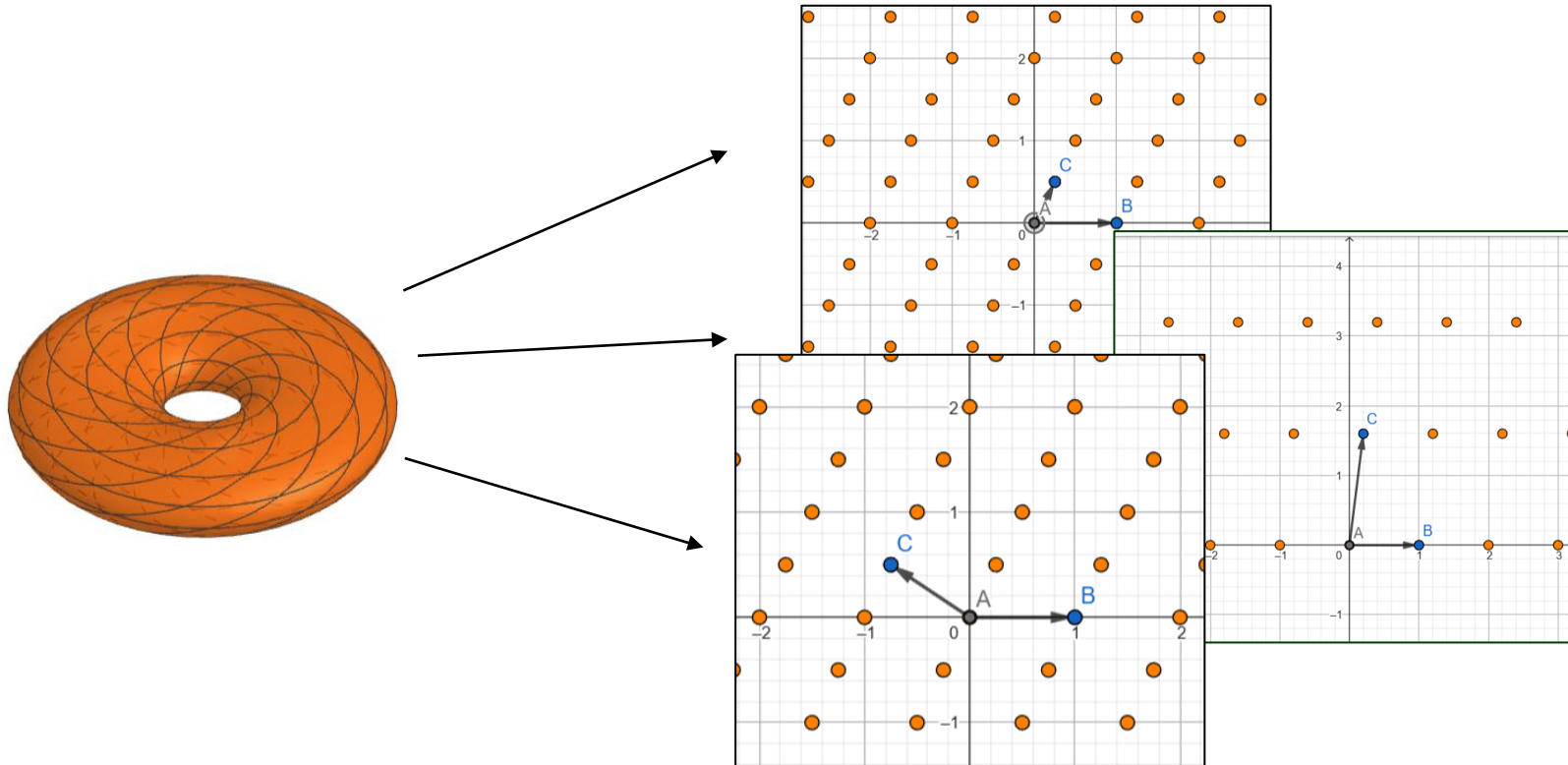
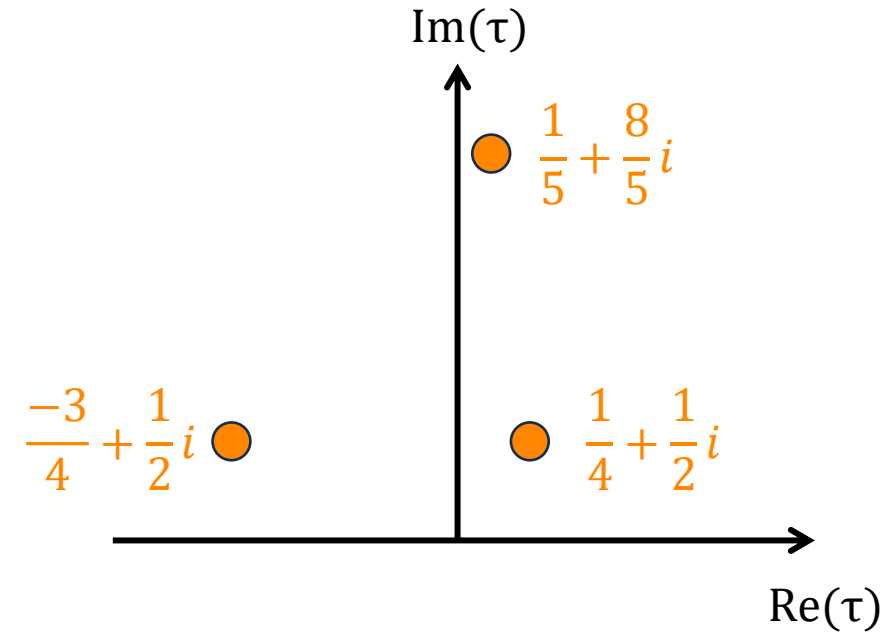
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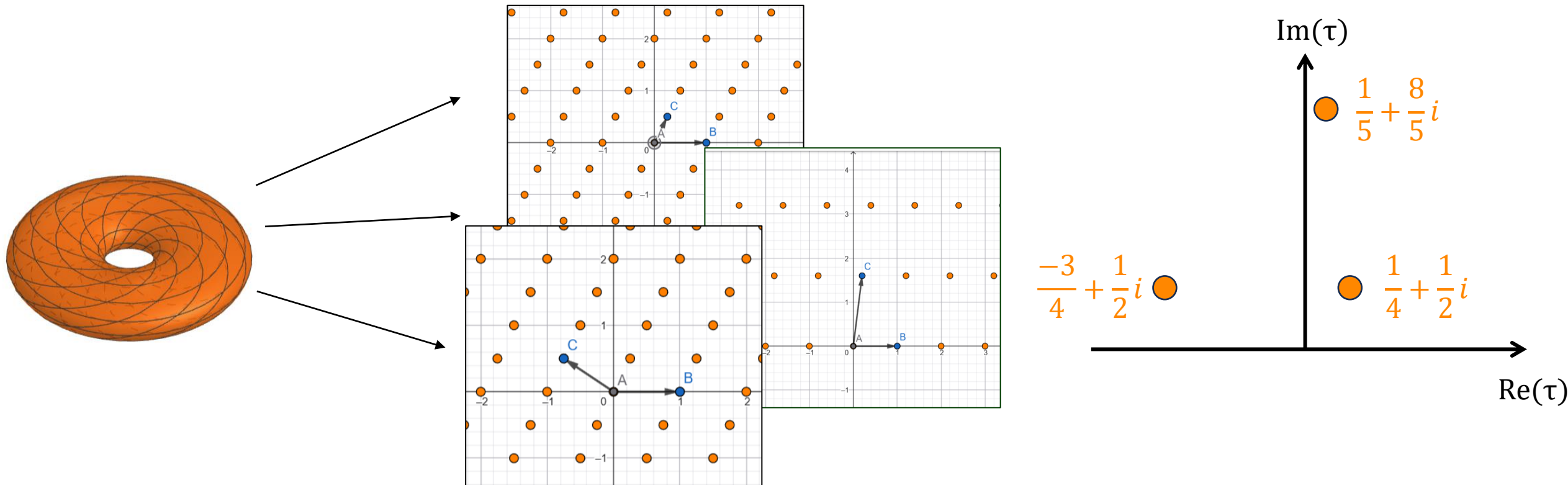
How to do in practice?

1. Choose a **discrete flavour group** G_f : $A_4, T', S_3, \Delta(96), \dots$
2. **Fix transformations** of the Standard Model fields: $L \xrightarrow{g \in G_f} \rho(g)L$
3. Add a field ϕ called **flavon** whose vev breaks the symmetry.
4. Write every **G_f -invariant operator** in the Lagrangian.
5. The **flavon takes a vev**, mass matrices are determined.
6. Find **a mechanism** such that at $\langle \phi \rangle \neq 0$ PMNS and m_e, m_μ, m_τ take the experimental values.
7. Possibility: when $\langle \phi \rangle = 0 \rightarrow$ **PMNS=Tribimaximal and $m_e = m_\mu = 0$.**

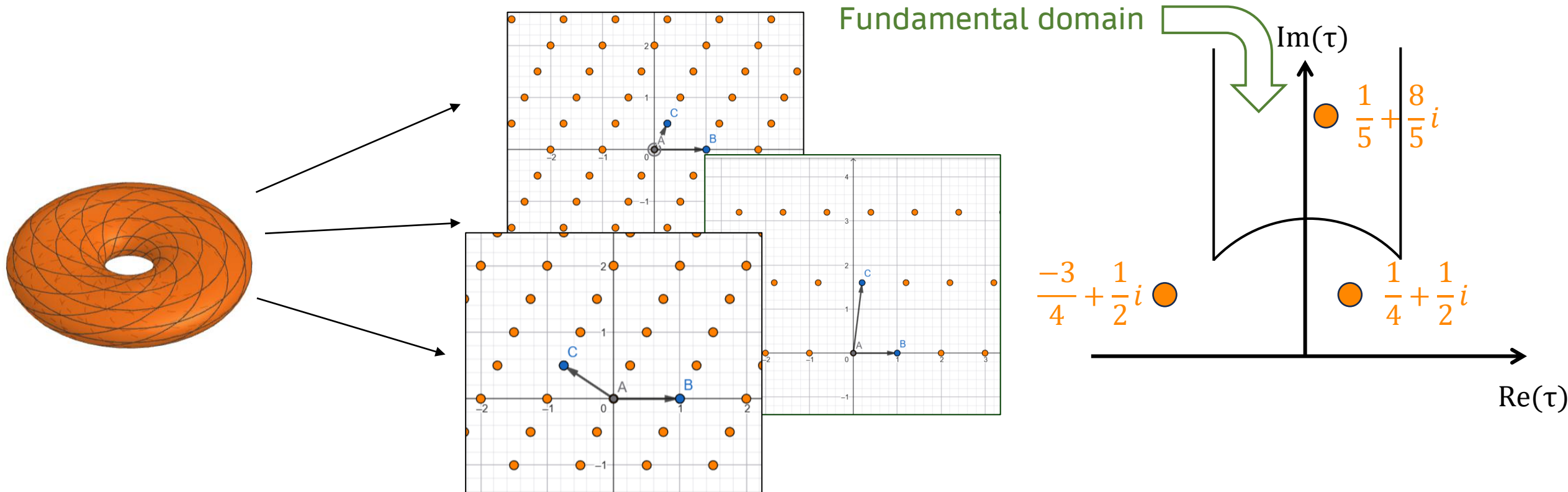
Modular symmetry is **inspired** by compactification of extra-dim:



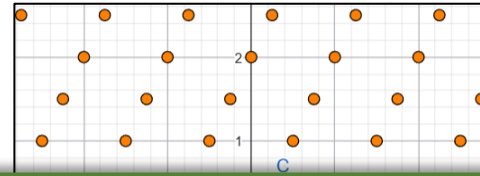




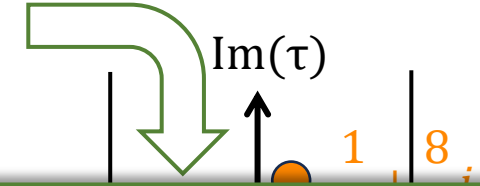
Two moduli τ and τ' describe the same torus $\Leftrightarrow \tau' = \frac{a\tau+b}{c\tau+d}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$



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Fundamental domain



Modular symmetry = Redundancy of parametrization of the torus

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Implementation of **Modular symmetry**:

1. Transformations of the SM fields under modular transformations:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \left\{ \begin{array}{l} \tau \rightarrow \frac{a\tau + b}{c\tau + d} \\ \phi \rightarrow (c\tau + d)^{-k_\phi} \rho_\phi(\gamma) \phi \end{array} \right. \quad \text{Non-linear realisation!}$$

ρ_ϕ representation of the **finite modular group** $G_d = SL(2, \mathbb{Z})/H$

2. Write couplings as functions of τ :

$$\mathcal{L} \supset Y_\rho^{k_Y}(\tau) L E^c H \quad \text{with } Y_\rho^{k_Y}(\gamma\tau) = (c\tau + d)^{k_Y} \rho(\gamma) Y_\rho^{k_Y}(\tau) \quad (1)$$

(1) + Holomorphy $\Rightarrow Y_\rho^{k_Y}$ are **modular forms**

3. Modular invariance of $\mathcal{L}$ requires:

$$\left\{ \begin{array}{l} k_Y(n) = k_L + k_{E^c} + k_H \\ 1 \subset \rho \otimes \rho_L \otimes \rho_{E^c} \otimes \rho_H \end{array} \right. \quad + \text{Modular forms} \rightarrow \text{Very constrained: few Yukawa couplings allowed}$$

Very first model:

[Feruglio; 1706.08749]

- Based on $A_4 = SL(2, \mathbb{Z})/\Gamma(3)$ irreps are: $\mathbf{1}, \mathbf{1}', \mathbf{1}'', \mathbf{3}$
- Neutrino masses generated by **seesaw mechanism**

	E^c	L	N^c	H
ρ_ϕ	$\mathbf{1} \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}$
k_ϕ	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$

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k_ϕ	1	1	1	0

Invariant operators:

Yukawa terms

$$\mathcal{L}_e \supset \alpha \left(E_1^c L Y_3^{(2)} \right)_1 H + \beta \left(E_2^c L Y_3^{(2)} \right)_1 H + \gamma \left(E_3^c L Y_3^{(2)} \right)_1 H$$

$$\mathcal{L}_\nu \supset g_1 \left((N^c L)_{3,S} Y_3^{(2)} \right)_1 H + g_2 \left((N^c L)_{3,A} Y_3^{(2)} \right)_1 H + \Lambda \left(N^c N^c Y_3^{(2)} \right)_1$$

Dirac terms

Majorana terms

$$k_Y(n) = k_L + k_{E^c} + k_H$$

$$1 \subset \rho \otimes \rho_L \otimes \rho_{E^c} \otimes \rho_H$$

Only one

$$Y_3^{(2)}(\gamma\tau) = (c\tau + d)^2 \rho_3(\gamma) Y_3^{(2)}(\tau)$$

Mass matrices:

[Feruglio; 1706.08749]

$$M_e = \begin{pmatrix} \alpha Y_1 & \alpha Y_3 & \alpha Y_2 \\ \beta Y_2 & \beta Y_1 & \beta Y_3 \\ \gamma Y_3 & \gamma Y_2 & \gamma Y_1 \end{pmatrix} \nu \quad M_D = \begin{pmatrix} 2g_1 Y_1 & (-g_1 + g_2) Y_3 & (-g_1 - g_2) Y_2 \\ (-g_1 - g_2) Y_3 & 2g_1 Y_2 & (-g_1 + g_2) Y_1 \\ (-g_1 + g_2) Y_2 & (-g_1 - g_2) Y_1 & 2g_1 Y_3 \end{pmatrix} \nu \quad M_N = \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix} \Lambda$$

Seesaw formula:

$$M_\nu = -M_D^T M_N^{-1} M_D$$

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$$M_e, M_\nu \rightarrow \left(\frac{m_e}{m_\mu}, \frac{m_\mu}{m_\tau}, \frac{\Delta m_{21}^2}{\Delta m_{31}^2}, \sin^2(\theta_{12}), \sin^2(\theta_{13}), \sin^2(\theta_{23}), \delta_{CP} \right) \text{ depend on } \tau, \frac{\alpha}{\gamma}, \frac{\beta}{\gamma}, \frac{g_2}{g_1}$$

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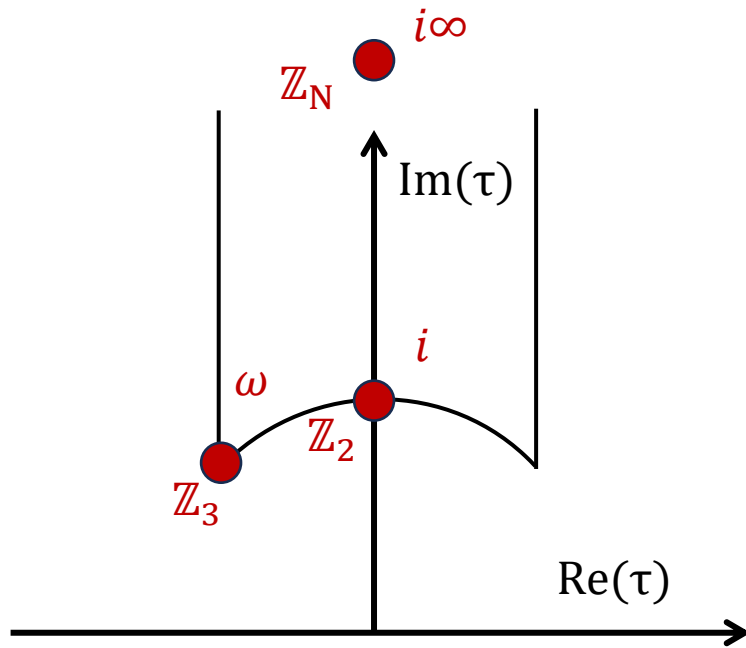
Best fit point in parameter space:

$$\begin{aligned} \tau &= 0.008 + 0.98i \\ \alpha/\gamma &= 15.3 \\ \beta/\gamma &= 0.054 \\ g_2/g_1 &= 0.029i \end{aligned}$$

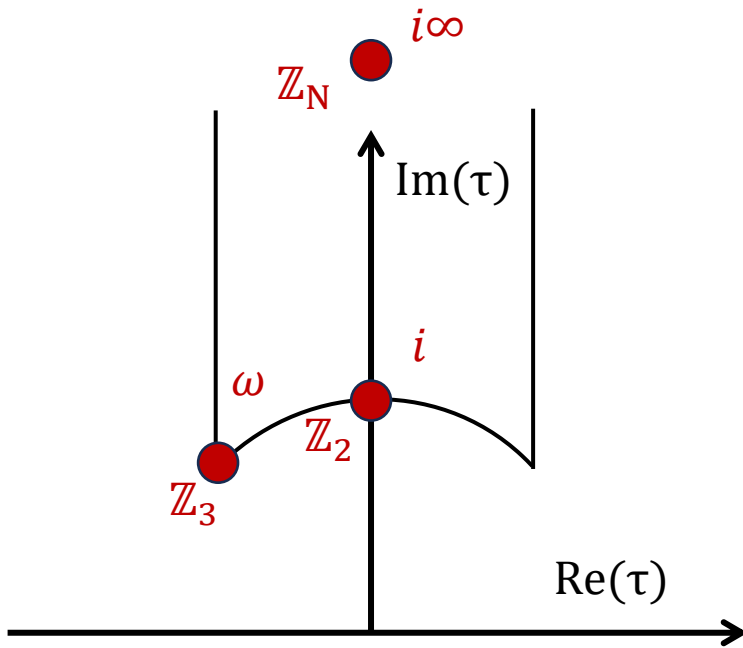


$$\begin{aligned} m_e/m_\mu &= 0.0048 & \sin^2(\theta_{12}) &= 0.459 \\ m_\mu/m_\tau &= 0.0567 & \sin^2(\theta_{13}) &= 0.001 \\ \Delta m_{21}^2/\Delta m_{31}^2 &= 0.0292 & \sin^2(\theta_{23}) &= 0.749 \\ & & \delta_{CP} &= 1.25\pi \end{aligned}$$

- τ fully breaks modular symmetry
- τ at **special points** = residual symmetry

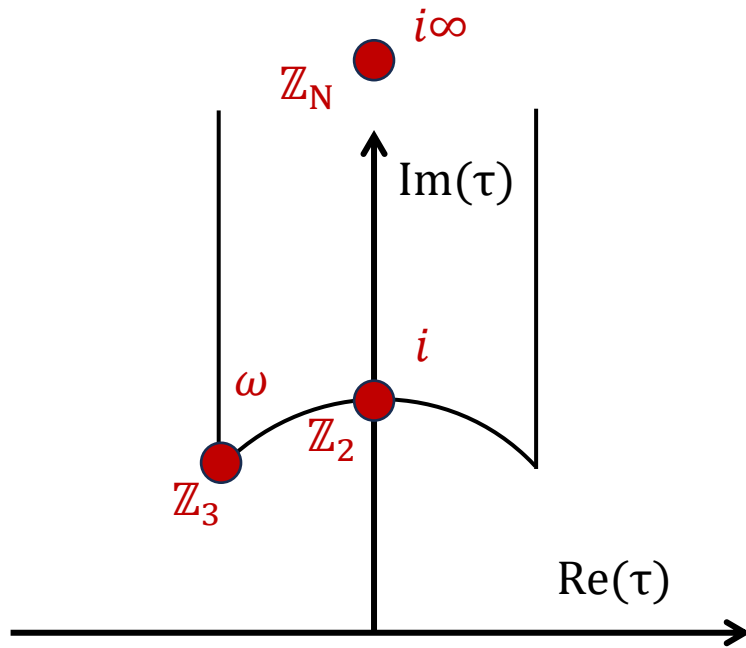


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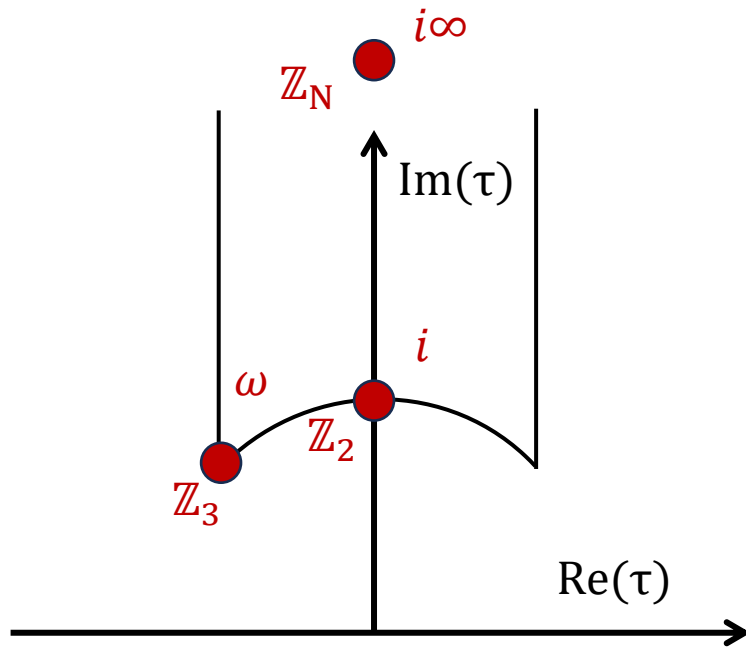
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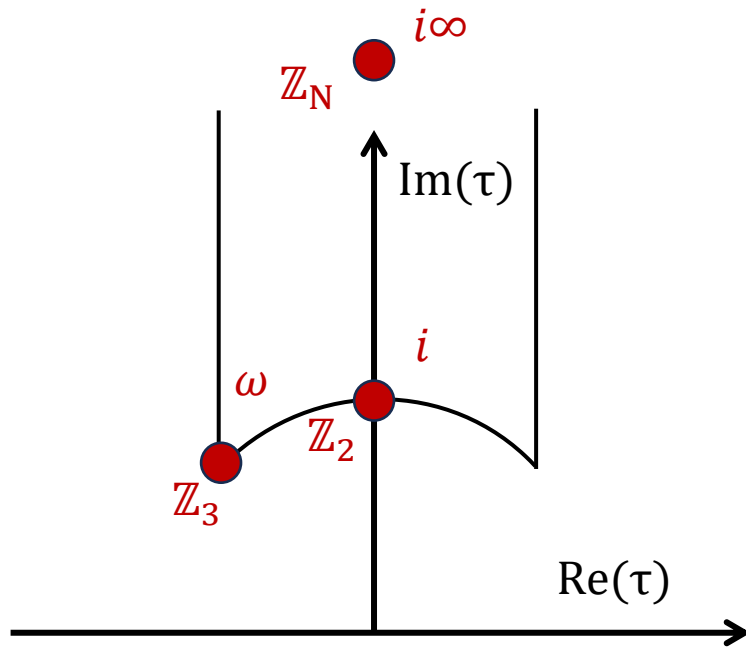
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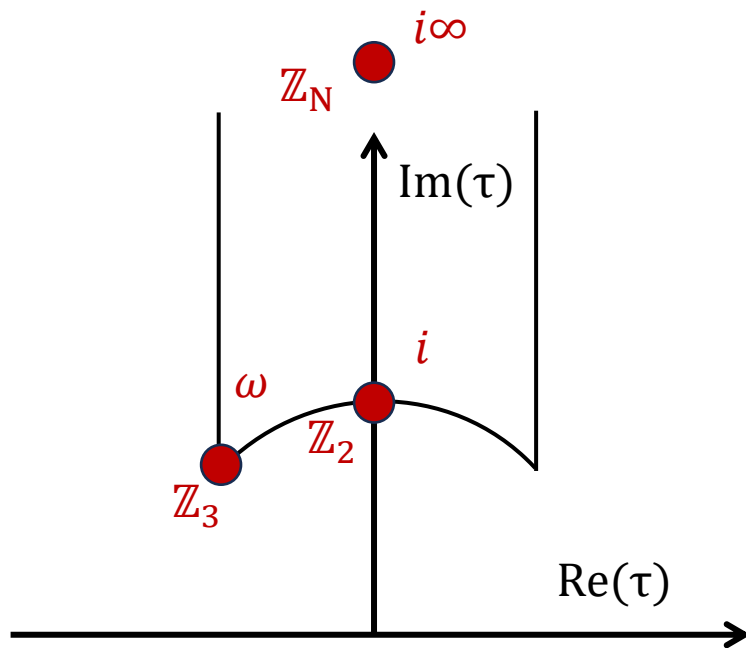
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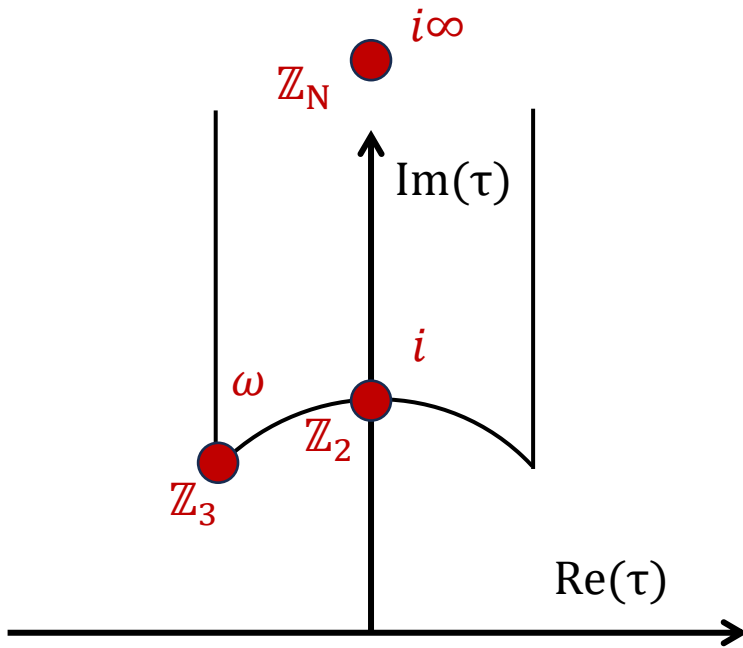
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$$\rho = 1_{q_1} \oplus 1_{q_2} \oplus 1_{q_3}, \quad \rho^c = 1_{q_1^c} \oplus 1_{q_2^c} \oplus 1_{q_3^c}$$

Modular Hierarchy

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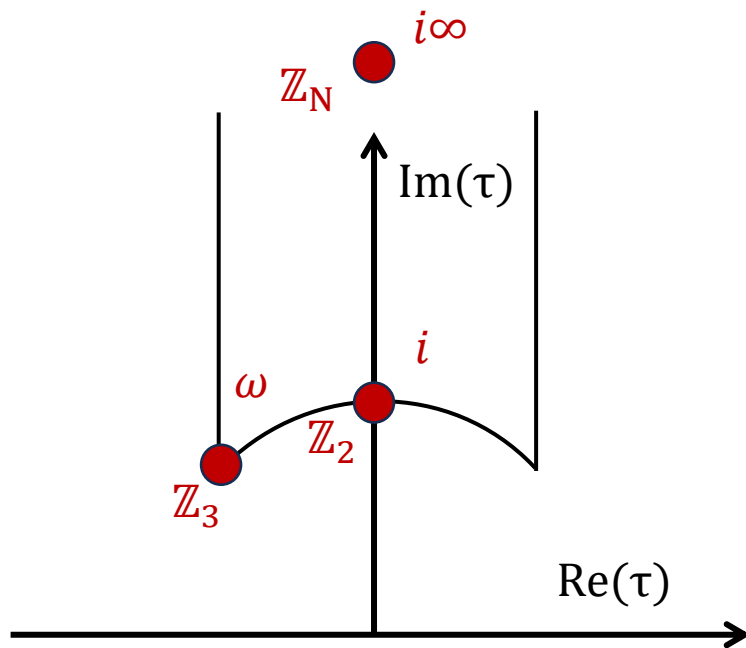
[Novichkov, Penedo, Petcov; 2102.07488]

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- Order of each entry of mass matrix $\rightarrow (m_1, m_2, m_3) \sim (\epsilon^a, \epsilon^b, \epsilon^c)$ where $0 \leq a, b, c \leq N - 1$.

[Novichkov, Penedo, Petcov; 2102.07488]

- Finite modular group: $G_d = \Gamma'_5 = SL(2, \mathbb{Z})/\Gamma(5)$ (N=5)

$$L \sim \mathbf{3}, E^c \sim \mathbf{3}'$$

- Decomposition under $\mathbb{Z}_5$:

$$L = 1_0 \oplus 1_1 \oplus 1_4; \quad E^c = 1_0 \oplus 1_2 \oplus 1_3$$

- Charge Conservation:

$$M_e(\tau) \sim \begin{pmatrix} 1 & \epsilon^4 & \epsilon \\ \epsilon^3 & \epsilon^2 & \epsilon^4 \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix} \quad (\epsilon = e^{-2\pi \text{Im}(\tau)/5})$$

$$\rightarrow (m_\tau, m_\mu, m_e) \sim (1, \epsilon, \epsilon^4)$$

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$$\rightarrow (m_\tau, m_\mu, m_e) \sim (1, \epsilon, \epsilon^4)$$

$$(m_\tau, m_\mu, m_e) \sim (1, \epsilon^a, \epsilon^b) \xrightarrow{\text{Exp.values}} \frac{b}{a} \approx 2,89$$

$$a \geq 1, b \geq 3$$

Modular Hierarchy

[Novichkov, Penedo, Petcov; 2102.07488]

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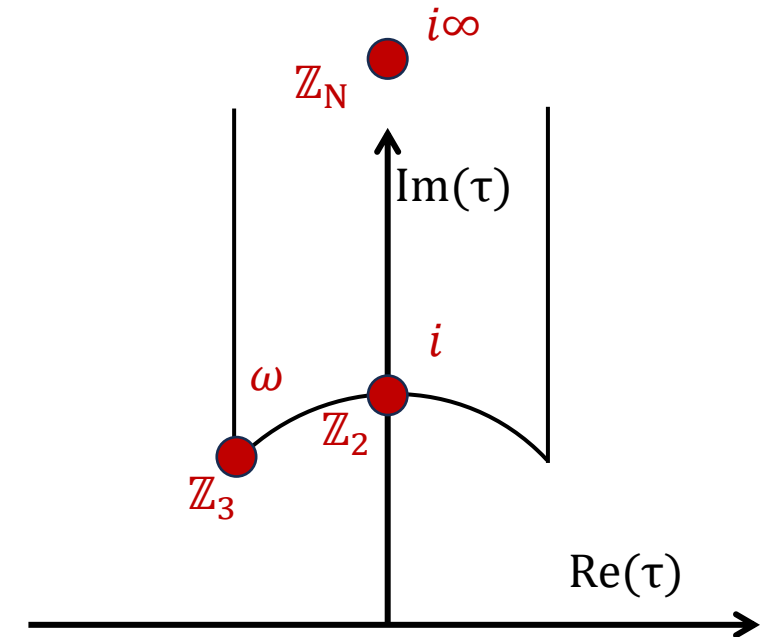
$$L = 1_0 \oplus 1_1 \oplus 1_4; \quad E^c = 1_0 \oplus 1_2 \oplus 1_3$$

- **Charge Conservation:**

$$M_e(\tau) \sim \begin{pmatrix} 1 & \epsilon^4 & \epsilon \\ \epsilon^3 & \epsilon^2 & \epsilon^4 \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix} \quad (\epsilon = e^{-2\pi \text{Im}(\tau)/5})$$

$$\rightarrow (m_\tau, m_\mu, m_e) \sim (1, \epsilon, \epsilon^4)$$

Same game when $\tau \sim i, \omega$ with some subtleties (depend on the weight) and $0 \leq a, b, c \leq 1$ (resp. 2)



$$(m_\tau, m_\mu, m_e) \sim (1, \epsilon^a, \epsilon^b) \xrightarrow[a \geq 1, b \geq 3]{\text{Exp.values}} \frac{b}{a} \approx 2,89$$

[Novichkov, Penedo, Petcov; 2102.07488]

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N	Γ'_N	Pattern	Sym. point	Viable $r \otimes r^c$
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[2 \oplus 1^{(r)}] \otimes [1 \oplus 1^{(r)} \oplus 1']$
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[1_a \oplus 1_a \oplus 1'_a] \otimes [1_b \oplus 1_b \oplus 1''_b]$
			$\tau \simeq i\infty$	$[1_a \oplus 1_a \oplus 1'_a] \otimes [1_b \oplus 1_b \oplus 1''_b]$ with $1_a \neq (1_b)^*$
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[3_a, \text{ or } 2 \oplus 1^{(r)}, \text{ or } \hat{2} \oplus \hat{1}^{(r)}] \otimes [1_b \oplus 1_b \oplus 1'_b]$
		$(1, \epsilon, \epsilon^3)$	$\tau \simeq i\infty$	$3 \otimes [2 \oplus 1, \text{ or } 1 \oplus 1 \oplus 1'], 3' \otimes [2 \oplus 1', \text{ or } 1 \oplus 1' \oplus 1'],$ $\hat{3}' \otimes [\hat{2} \oplus \hat{1}, \text{ or } \hat{1} \oplus \hat{1} \oplus \hat{1}'], \hat{3} \otimes [\hat{2} \oplus \hat{1}', \text{ or } \hat{1} \oplus \hat{1}' \oplus \hat{1}']$
5	A'_5	$(1, \epsilon, \epsilon^4)$	$\tau \simeq i\infty$	$3 \otimes 3'$

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42	—	$T^6, (ST^{-1}S)^2TST^{-1}ST^2$	$Z_7 \rtimes Z_6$	[42, 1]
	—	$T^6, ST^{-1}ST(ST^{-1}S)^2T^2$		
48	—	S^2T^4	$2O$	[48, 28]
	—	T^8, ST^4ST^{-4}	$GL(2, 3)$	[48, 29]
	$\Gamma(4)$	T^4	$A_4 \rtimes Z_4 \cong S'_4$	[48, 30]
	—	$(ST^{-1}ST)^2$	$A_4 \times Z_4$	[48, 31]
	—	$S^2(ST^{-1}ST)^2$	$T' \times Z_2$	[48, 32]
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[Liu, Ding; 2112.14761]

[Bonnefoy, Rausch de Traubenberg, **V.S.**; 26xx.xxxxx]

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4 representations from $(\mathbb{Z}_3 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_3$ and $A_4 \times \mathbb{Z}_4$

$$\text{Exp.values: } \frac{\Delta m_{21}^2}{\Delta m_{31}^2} \approx \frac{m_\mu}{m_\tau}$$

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No irreducible triplet:

- **General Fact?** PMNS impossible at ω
[Feruglio; 2302.11580]
- Depends on our hypothesis?

CP phase:

- Most models predict $\delta_{CP} = 0, \pi \rightarrow$ Exceed 3σ
- Predictable? Criterion on the choice of representations?

Inverted Hierarchy:

- **Latest JUNO result:** Normal hierarchy is favoured.
[JUNO collaboration; 2609.37895]
- Weinberg operator? Same result if seesaw mechanism?

Conclusion:

- The flavour puzzle suggests a **broken discrete symmetry**.
- **Modular symmetry** is a constraining candidate → few parameters predict the observables.
- Near fixed points, the mass hierarchy is **naturally produced**.
- **Classification** of models predicting all observed hierarchies in the leptonic sector of SM.
- Constructed model fit the observables with **order 1 coefficients**.
- **Latest experiments** suggest hypothesis revisions.

Conclusion:

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Thank you for your attention!

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

$$= \langle S, T \mid S^4 = (ST)^3 = S^2TS^{-2}T^{-1} = 1 \rangle$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$S\tau = -\frac{1}{\tau}$$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$T\tau = \tau + 1$$

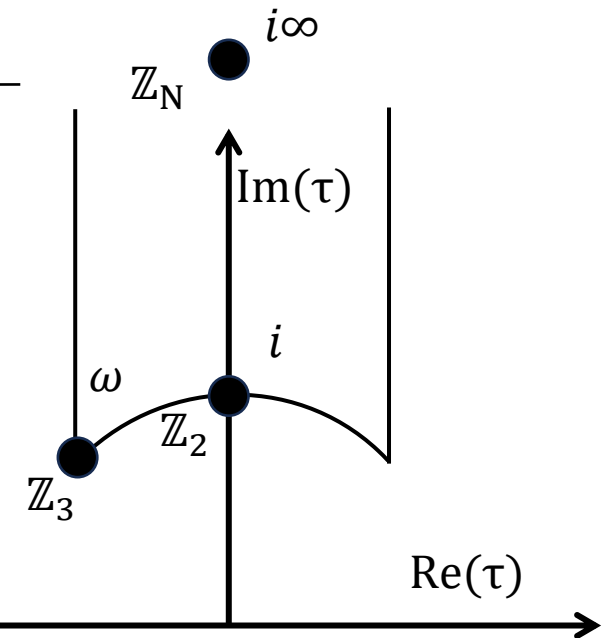
Finite modular group: $G_d = SL(2, \mathbb{Z})/H = \langle S, T \mid S^4 = (ST)^3 = S^2TS^{-2}T^{-1} = 1 + \text{other relations} \rangle$

G_d finite $\rightarrow$ N such that $T^N = 1$

Fixed points: τ such that $\gamma\tau = \tau$

In the fundamental domain:

- $Ti\infty = i\infty$
- $Si = i$
- $ST\omega = \omega$



Unique transformation of τ under CP:

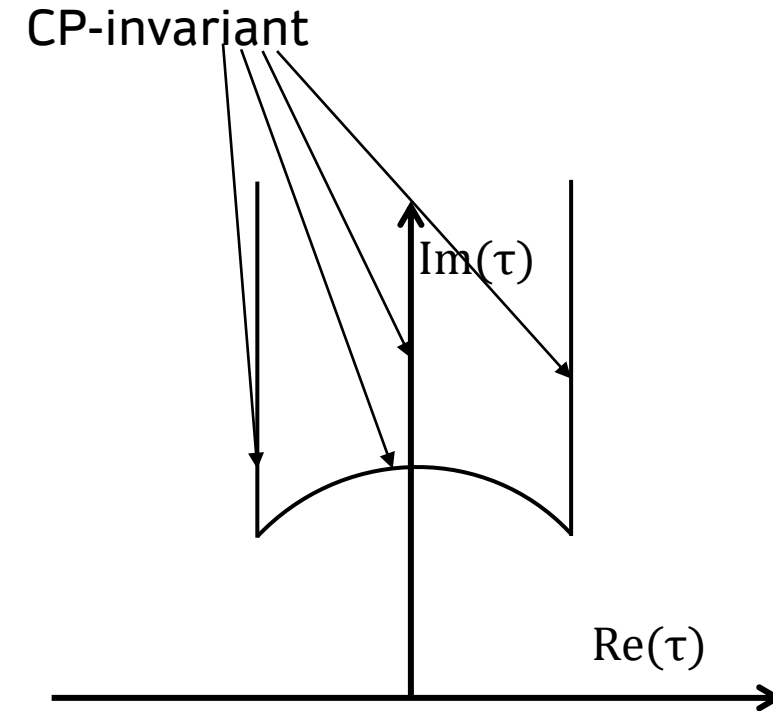
$$\tau \rightarrow -\bar{\tau}$$

Then if $\text{Re}(\tau) = 0, 1/2, -1/2$ or $|\tau| = 1$, τ is CP-invariant

Build models which spontaneously break CP!

- Impose CP-symmetry: free parameters of the Lagrangian are reals
- τ takes a value not CP-invariant $\rightarrow$ spontaneously breaks CP
- δ_{CP} is not trivial

[Novichkov, Penedo, Petcov, Titov; 1905.11970]



A_4 the group of even permutations of even elements

$$= SL(2, \mathbb{Z})/\Gamma(3) = \langle S, T | S^4 = (ST)^3 = S^2TS^{-2}T^{-1} = T^3 = 1 \rangle$$

Four irreducible representations: $\mathbf{1}, \mathbf{1}', \mathbf{1}'', \mathbf{3}$ ($1+1+1+3^2=12=|A_4|$)

	S	T
1	1	1
1'	1	ω
1''	1	ω^2
3	$\frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$

Clebsch-Gordan coefficients:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_3 \otimes \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}_3 = (\phi_1\psi_1 + \phi_2\psi_3 + \phi_3\psi_2)_1 \oplus (\phi_3\psi_3 + \phi_1\psi_2 + \phi_2\psi_1)_{1'} \oplus (\phi_2\psi_2 + \phi_3\psi_1 + \phi_1\psi_3)_{1''} \\ \oplus \frac{1}{\sqrt{3}} \begin{pmatrix} 2\phi_1\psi_1 - \phi_2\psi_3 - \phi_3\psi_2 \\ 2\phi_3\psi_3 - \phi_1\psi_2 - \phi_2\psi_1 \\ 2\phi_2\psi_2 - \phi_3\psi_1 - \phi_1\psi_3 \end{pmatrix}_3 \oplus \begin{pmatrix} \phi_2\psi_3 - \phi_3\psi_2 \\ \phi_1\psi_2 - \phi_2\psi_1 \\ \phi_3\psi_1 - \phi_1\psi_3 \end{pmatrix}_3$$

Irreps of $A_4 \times \mathbb{Z}_4$: $\mathbf{1}_k, \mathbf{1}_k', \mathbf{1}''_k, \mathbf{3}_k, k = 0, 1, 2, 3$

Model predicting correct hierarchies:

$$L = \mathbf{1}_1' \oplus \mathbf{1}_1' \oplus \mathbf{1}_3 \quad E^c = \mathbf{1}_1 \oplus \mathbf{1}_3' \oplus \mathbf{1}_3'$$

Mass matrices:

$$M_e = \begin{pmatrix} \alpha_1^i Y_{12,i}^{k_{L1}+k_{E1}^c} & \alpha_4^i Y_{12,i}^{k_{L2}+k_{E1}^c} & \alpha_7^i Y_{10,i}^{k_{L3}+k_{E1}^c} \\ \alpha_2^i Y_{10,i}^{k_{L1}+k_{E2}^c} & \alpha_5^i Y_{10,i}^{k_{L2}+k_{E2}^c} & \alpha_8^i Y_{12,i}^{k_{L3}+k_{E2}^c} \\ \alpha_3^i Y_{10,i}^{k_{L1}+k_{E3}^c} & \alpha_6^i Y_{10,i}^{k_{L2}+k_{E3}^c} & \alpha_9^i Y_{12,i}^{k_{L3}+k_{E3}^c} \end{pmatrix} v_d$$

$$M_\nu = \begin{pmatrix} g_1^i Y_{12,i}^{2k_{L1}} & g_4^i Y_{12,i}^{k_{L1}+k_{L2}} & g_5^i Y_{10,i}^{k_{L1}+k_{L3}} \\ g_4^i Y_{12,i}^{k_{L1}+k_{L2}} & g_2^i Y_{12,i}^{2k_{L2}} & g_6^i Y_{10,i}^{k_{L2}+k_{L3}} \\ g_5^i Y_{10,i}^{k_{L1}+k_{L3}} & g_6^i Y_{10,i}^{k_{L2}+k_{L3}} & g_3^i Y_{12,i}^{2k_{L3}} \end{pmatrix} \frac{v_d^2}{\Lambda}$$

For example: $k_L = (3, 5, 1) \quad k_{E^c} = (1, 1, 1)$

$$\tau = 0,21 + 2,58i$$

$$\alpha_2 = -0,68$$

$$\alpha_3 = 4,39$$

$$\alpha_4 = -0,76$$

$$\alpha_5 = -14,58$$

$$g_2 = 1,43$$

$$g_3 = 0,19$$

$$g_4 = 1,94$$

$$\chi^2 = 0,55$$

$$P_{MG} = 2,44$$

$$d_{FT} = 1,78$$



$$m_e/m_\mu = 0.00474$$

$$m_\mu/m_\tau = 0.0588$$

$$\Delta m_{21}^2/\Delta m_{31}^2 = 0.0296$$

$$\sin^2(\theta_{12}) = 0.306$$

$$\sin^2(\theta_{13}) = 0.022$$

$$\sin^2(\theta_{23}) = 0,449$$

$$\delta_{CP} = 1.15\pi$$