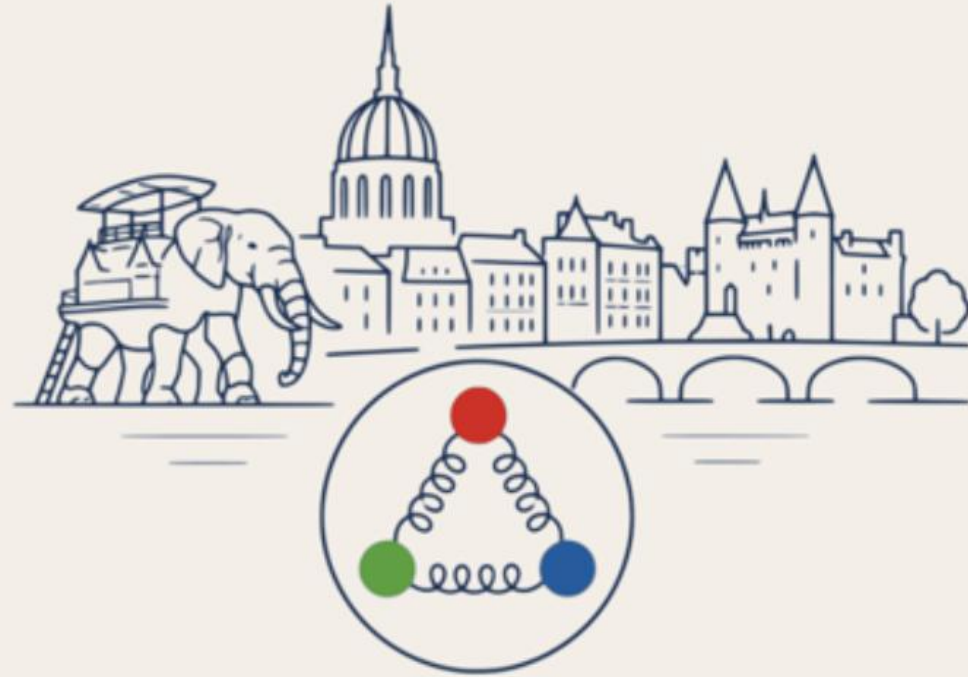


ASSEMBLÉE GÉNÉRALE DU GDR QCD
SUBATECH · 7-9 OCTOBRE 2026



GDR QCD 2026

— NANTES —

The proton charge radius puzzle: observation, bias and scientific rigor.

Or

The proton charge radius measurement
dilemma*

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PARIS-SACLAY

*Having to make a choice between equally unfavorable alternatives



- Some introductory slides: how is the radius defined
- A bit of archeology: how/where was the radius measured
- The puzzle: how it emerged
- Observational bias: how the puzzle got emphasized
- The burden of assumptions: was there really a puzzle?
- Attempts to explain the puzzle
- CODATA2018: the salvation
- New avenues: proposed/upcoming experiments
- Looking to the other infinity: high Q^2 measurements are also a good approach



Proton charge radius from e-p elastic scattering

- Measure the e-p elastic scattering cross section (Lab frame)

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega} \Big|_p \times \left(\frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1 + \tau} + 2\tau \tan^2 \left(\frac{\theta}{2} \right) G_M^2(Q^2) \right)$$

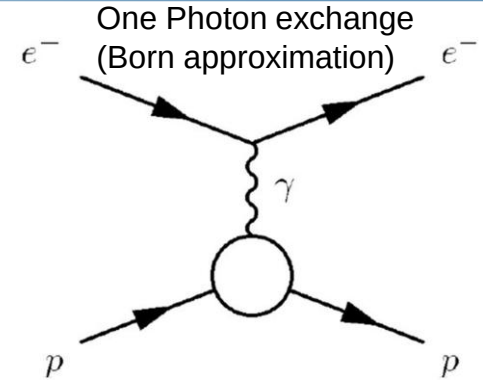
- $G_E(Q^2)$ is the Fourier transform of the spatial density (in the Breit frame)

$$F(Q) = \int \rho(r) e^{iQr} d^3r$$

- Expanding the exponential in multipoles and developing

$$G_E(Q^2) = F(Q^2) = 1 + \sum_{s=1}^{\infty} \frac{(-1)^s}{(2s+1)!} (Q^2)^s \langle r^{2s} \rangle$$

- The radius is defined as $r_p = \sqrt{\langle r^2 \rangle} = \sqrt{-6 \frac{\partial G_E(Q^2)}{\partial Q^2} \Big|_{Q^2=0}}$



comments:

- Breit frame: non-relativistic approach.
- The density is assumed to be radial: we can expand the exponential in multipoles.
- Only positive even-order moments are accessible
- The radius is defined at zero momentum transfer: extrapolation of knowledge is needed.



Proton charge radius from e-p elastic scattering

- The lowest ever reached Q^2 value is $5.51 \times 10^{-3} \text{ fm}^{-2}$: PRad experiment in hall B at Jlab.
- There are no measurement of the proton electric form factor at $Q^2 = 0 \text{ GeV}^2$
- Need to rely on extrapolation of the knowledge on the form factor in one region of four momentum transfer to the point of $Q^2 = 0 \text{ GeV}^2$
 - **Sensitive to the functional form** used in the extrapolation
 - **Sensitive to the interpolation** boundaries of the Form Factor
 - In general, this is not a safe approach.

The principle of induction in logical thinking works fine but is not rational

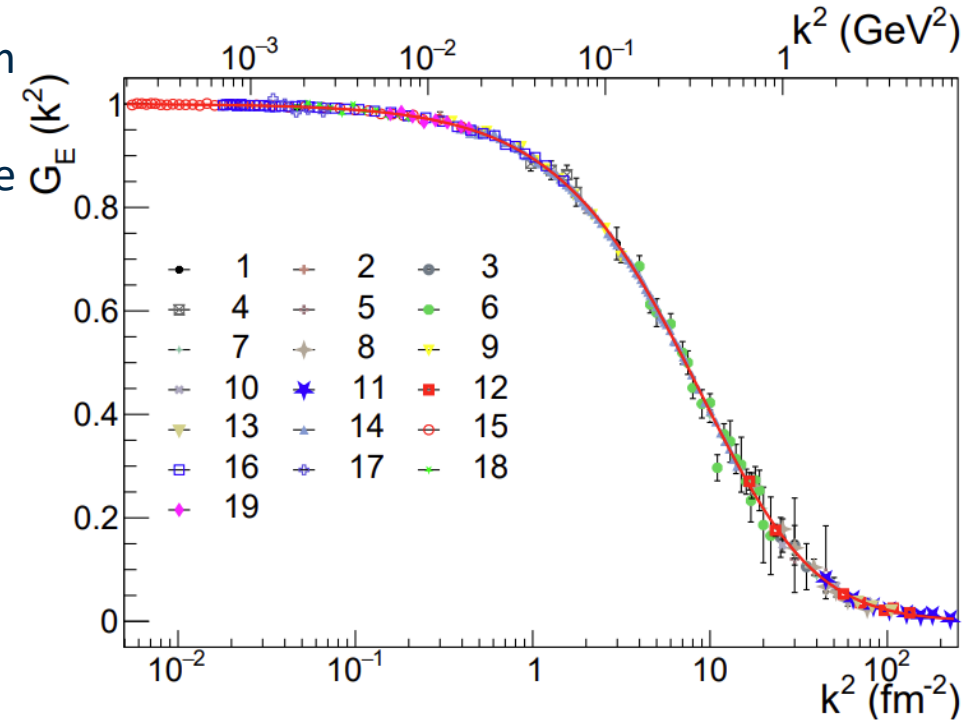


Figure from *Phys. Rev. C* 110 (2024) 015207



Proton radius measurement from hydrogen spectroscopy

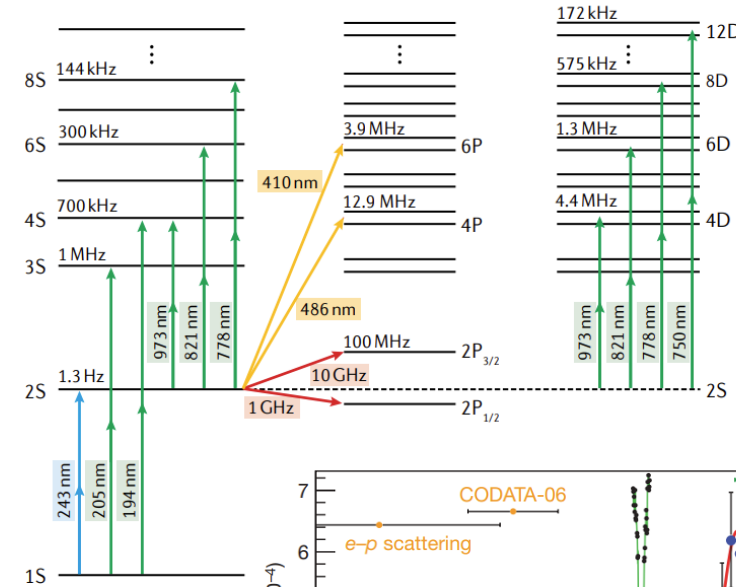
- The energy of an atomic level depends on the proton radius:

$$E_{n,l} \approx -\frac{R_\infty}{n^2} + \delta_{l,0} \frac{L_{1S} + ar_p^2}{n^3}$$

Rydberg constant
Lamb shift

- Many precise measurement with ordinary hydrogen has been performed
- Muonic hydrogen: very high precision in
 - Spectroscopic measurements
 - Theoretical calculations of transition frequencies
- The proton radius is extracted with a sub percent relative uncertainty

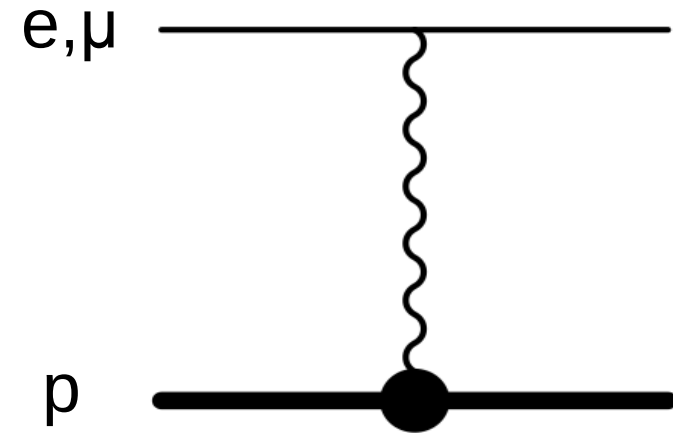
Electronic hydrogen





Atomic spectroscopy vs elastic e-p scattering

- Are atomic spectroscopy and elastic e-p scattering measuring the same thing?
 - Yes, these measurement are accessing the same Feynman diagram
 - This is clearly demonstrated in the nice paper of G. Miller
 - Phys. Rev. C 99, 035202
 - Comparison between the two approaches is fair
 - However, no closure tests can be performed as the analysis and extraction techniques are fundamentally different
 - The comparison stays between high level features, in this case the charge radius value.
- There is a tendency in having more faith in atomic spectroscopy results as they have more control on theory side
 - Although different experiments gave different results for the same studied transition
- The fact that in elastic scattering there is a strong dependence on extrapolation makes the results questionable





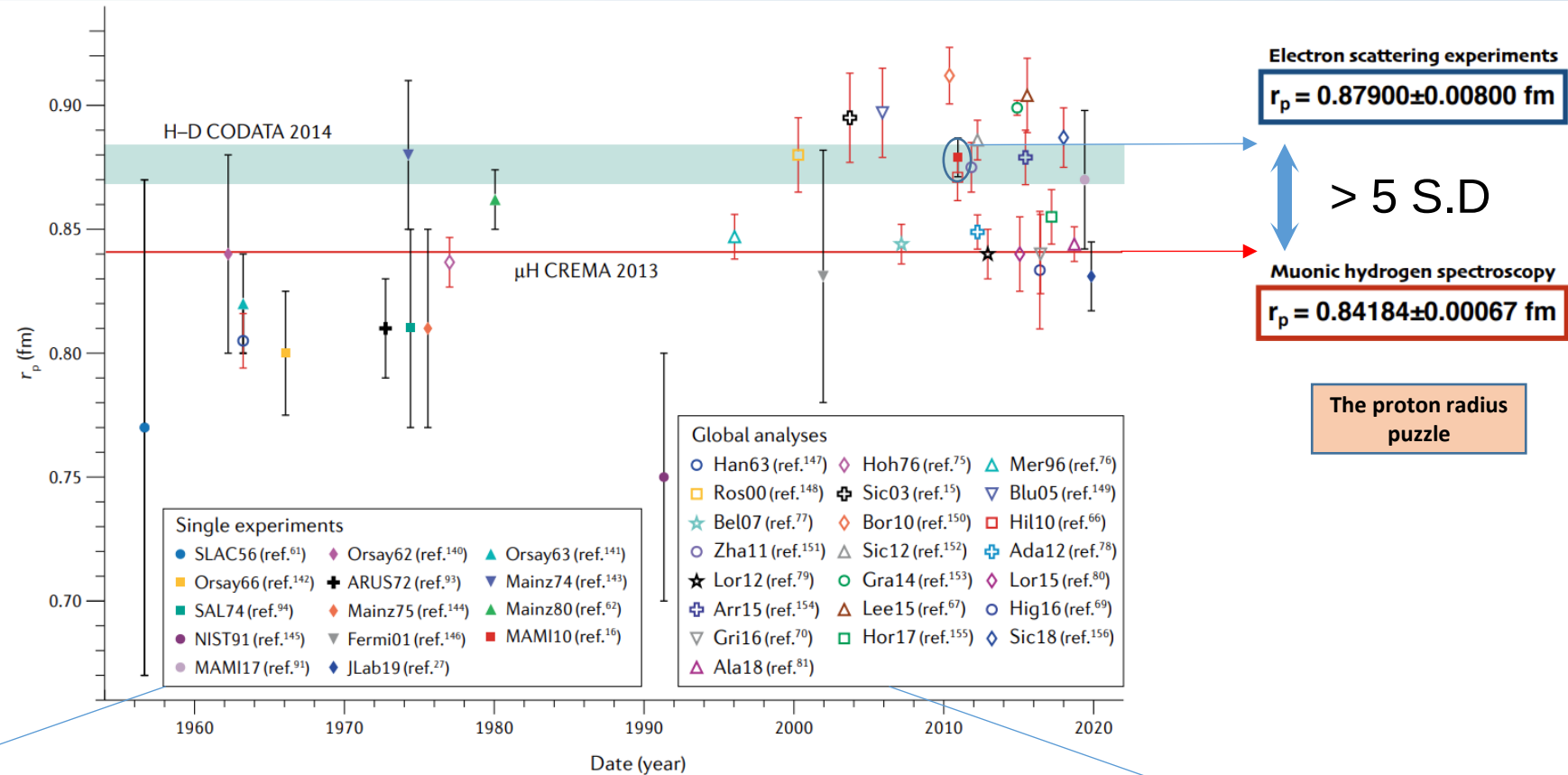
Rational versus irrational beliefs

- Scientific beliefs (all types of beliefs) are usually said to be logical
- However, they can be either
 - Rational: based on direct evidence and examination
 - Irrational: based on testimony
- In the non-scientific community, most beliefs are of the irrational type
- In the scientific community, there is an equilibrium between rational and irrational beliefs
 - This is of extreme importance in the advancement of science
 - But could sometimes lead to dead-ends
- See for reference
 - Testimony and Observation, C. A. J. Coady, American Philosophical Quarterly, Vol. 10, No. 2 (Apr., 1973), pp. 149-155
 - The Place of Testimony in the Fabric of Knowledge and Justification, Robert Audi, American Philosophical Quarterly, Vol. 34, No. 4 (Oct., 1997), pp. 405-422



Status of measurements of the charge radius

This includes mainly e-p scattering experiments

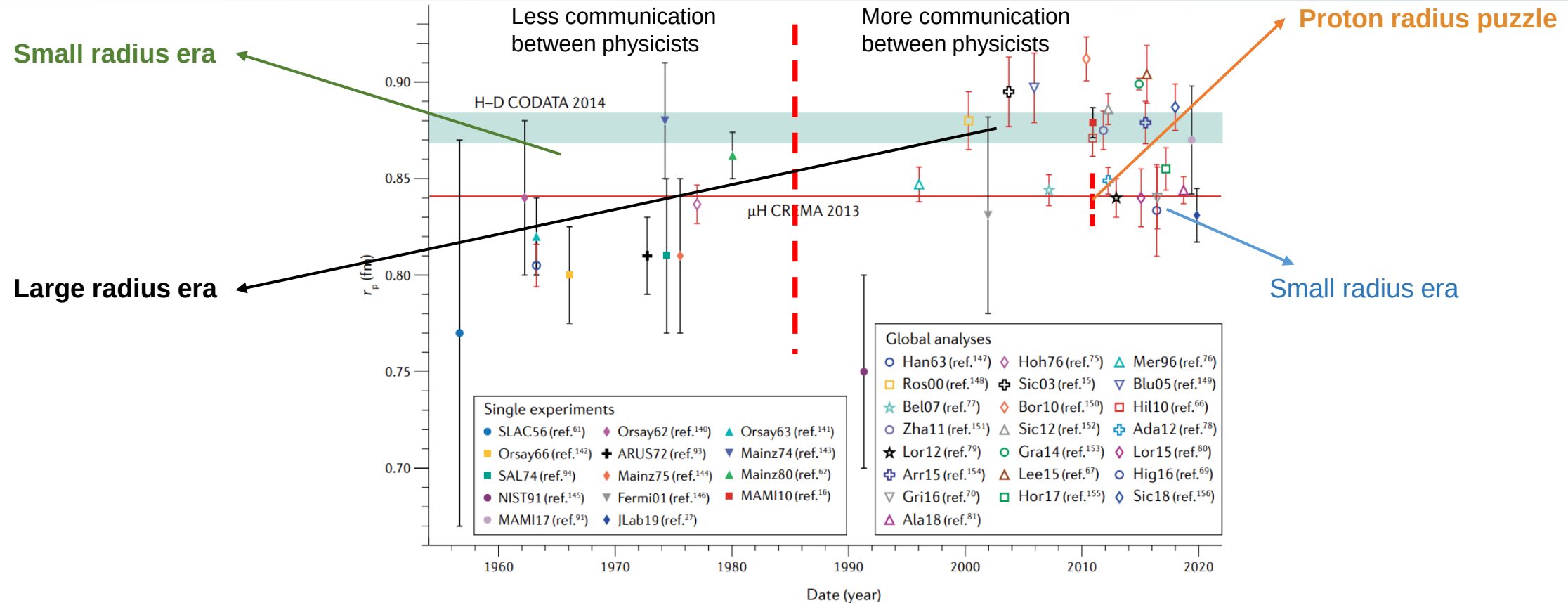


Actual measurements

Reanalysis of existing data



Status of measurements of the charge radius



Researchers are tempted -unconsciously- to find results that validate previous findings
Solution: **blind physical observables** while **optimizing the extraction strategy** so only fit quality and estimated precision guides the analysis

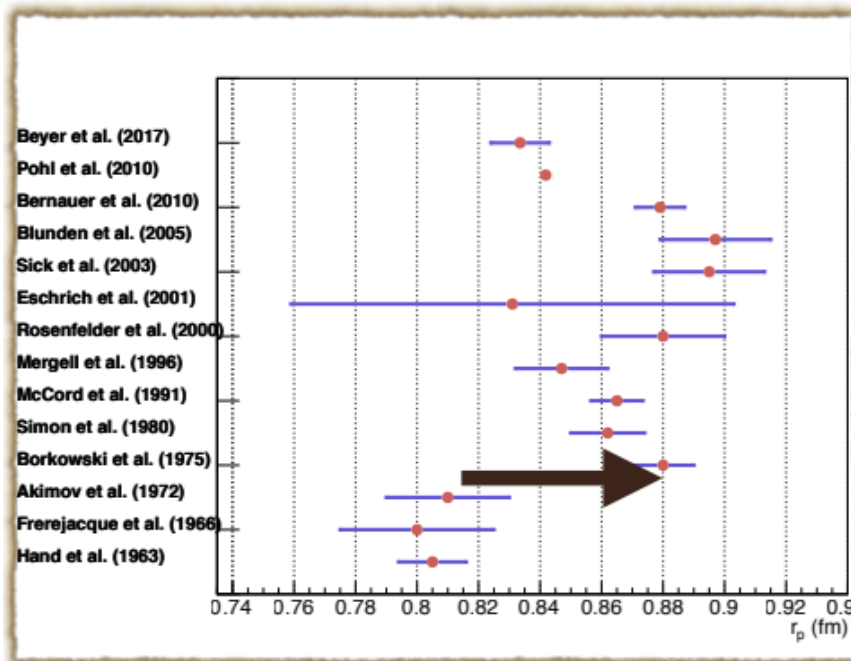
J-P. Karr, D. Marchand, E. Voutier Nature Reviews Physics **2**, 601–614 (2020)



Observational bias

- Who started all of that: Borkowski et al. Nuclear physics B93 (1975) 461-478

the G_E values. The best fit value of the parameter β gives a proton r.m.s. radius of $\langle r_E^2 \rangle^{1/2} = 0.84 \pm 0.02$ fm. This value is higher than the dipole value of 0.81 fm, but within the error limits it is compatible with the result $(0.81 \pm 0.04$ fm) of a similar experiment carried out at Saskatoon [7]. The radius values reported in (I) resulted from the model-dependent extrapolation of the form factors to $q^2 = 0$, where the models demand $G_{EM}(0) = 1$. If we include this demand in the above described fit



F. Borkowski et al. / Form factors of the proton

475

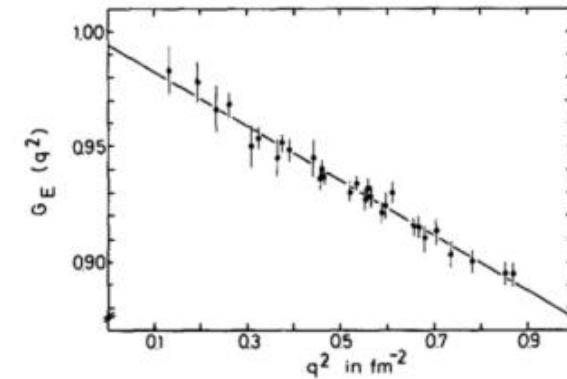


Fig. 5. Form factor values which have been extracted directly from the cross sections of (I) and of the present experiment under the assumption $G_E = G_M$. The contribution of G_M to the cross sections was less than 10%. The solid line is the best fit using the formula $G_E(q^2) = \delta + \beta q^2$.

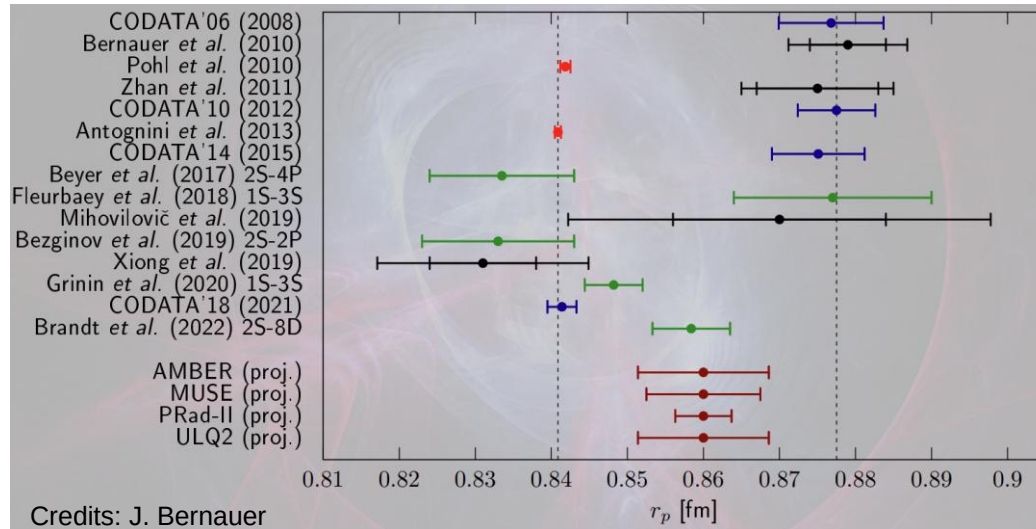
(i.e. $\delta = 1.0$), we obtain a proton r.m.s. radius of 0.88 fm, a value which coincides with the extrapolation results.



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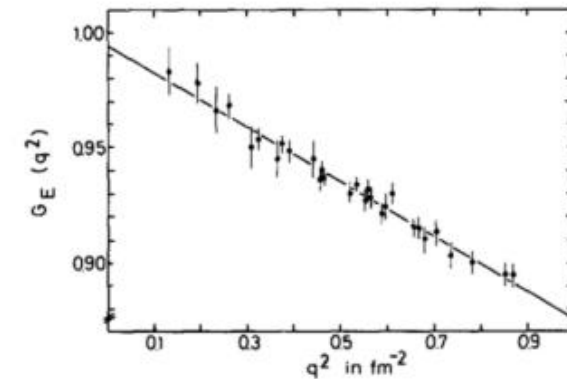


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(i.e. $\delta = 1.0$), we obtain a proton r.m.s. radius of 0.88 fm, a value which coincides with the extrapolation results.

This trend survives up till now



- The muonic hydrogen experiment took longer than expected
 - Biased by the “large radius”, people looked for laser frequency at values corresponding to large radius!
 - Only when scanning other frequency ranges that they saw the peek!

with which the proton radius is known [2]. The radius is also needed for the planned measurements of muonic X-ray transitions [3]; these experiments can only scan a narrow frequency range, which must be chosen according to the best value of the rms-radius presently known.

rms-radius could be calculated. The resulting value of 0.81 fm, which is still quoted in the literature, should have long been superseded by values coming from more precise data at *lower* q which are indeed sensitive to the rms-radius.

0.013 fm. This yields as the final result for the charge radius of the proton $r_{\text{rms}}^e = 0.895 \pm 0.018$ fm. This radius is significantly larger than the values generally cited in the literature. It agrees with the most accurate value derived from atomic transitions [2] 0.890 ± 0.014 fm.

8. Conclusions

From an analysis of the *world*-data on e–p scattering we determine the proton rms-radius and find a value that is significantly larger than previous values. The change is understood as a consequence of treating properly the higher moments $\langle r^n \rangle$.



Focus on PRad

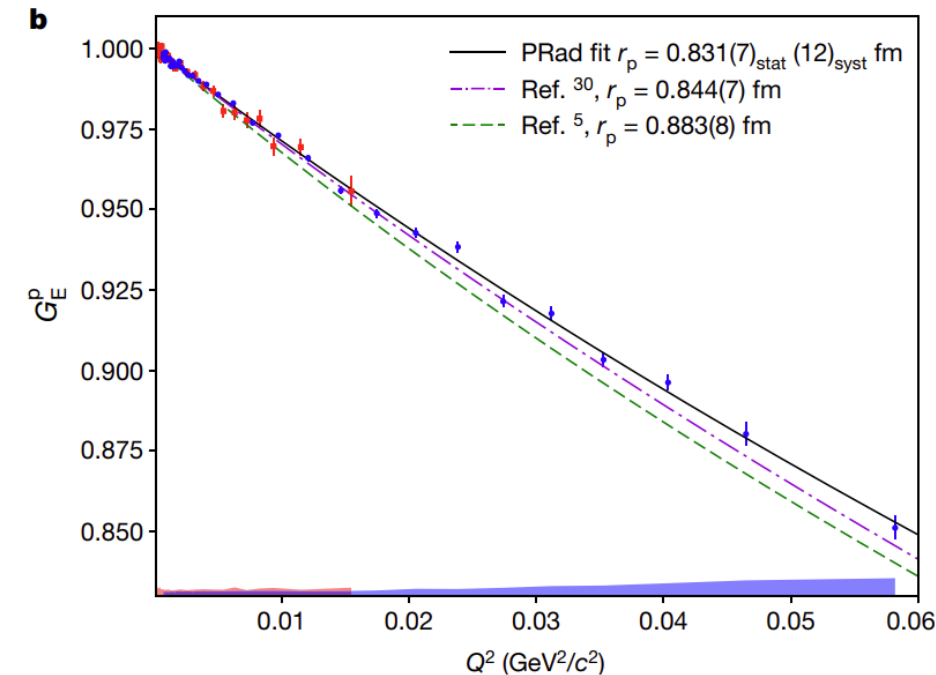
- **PRad experiment in hall B at Jefferson Lab, USA.**
 - Measured e-p scattering cross section at low Q^2
 - Ignored magnetic form factor contribution which is suppressed
 - Two data sets for two different beam energies
 - Absolute normalization wrt Moller scattering data
- **Extract electric form factor**
 - Fit simultaneously the two data sets having the same functional form for the form factor dependence
$$f(Q^2) = nG_E^p(Q^2) = n \frac{1 + p_1 Q^2}{1 + p_2 Q^2}$$
 - Different normalization parameters for each data set
 - $n1 = 1.0002 \pm 0.0002_{\text{stat}} \pm 0.0020_{\text{syst}}$
 - $n2 = 0.9983 \pm 0.0002_{\text{stat}} \pm 0.0013_{\text{syst}}$
 - **A difference of 0.8 S.D**

What motivates the assumption of different normalizations?

Article | Published: 06 November 2019

A small proton charge radius from an electron-proton scattering experiment

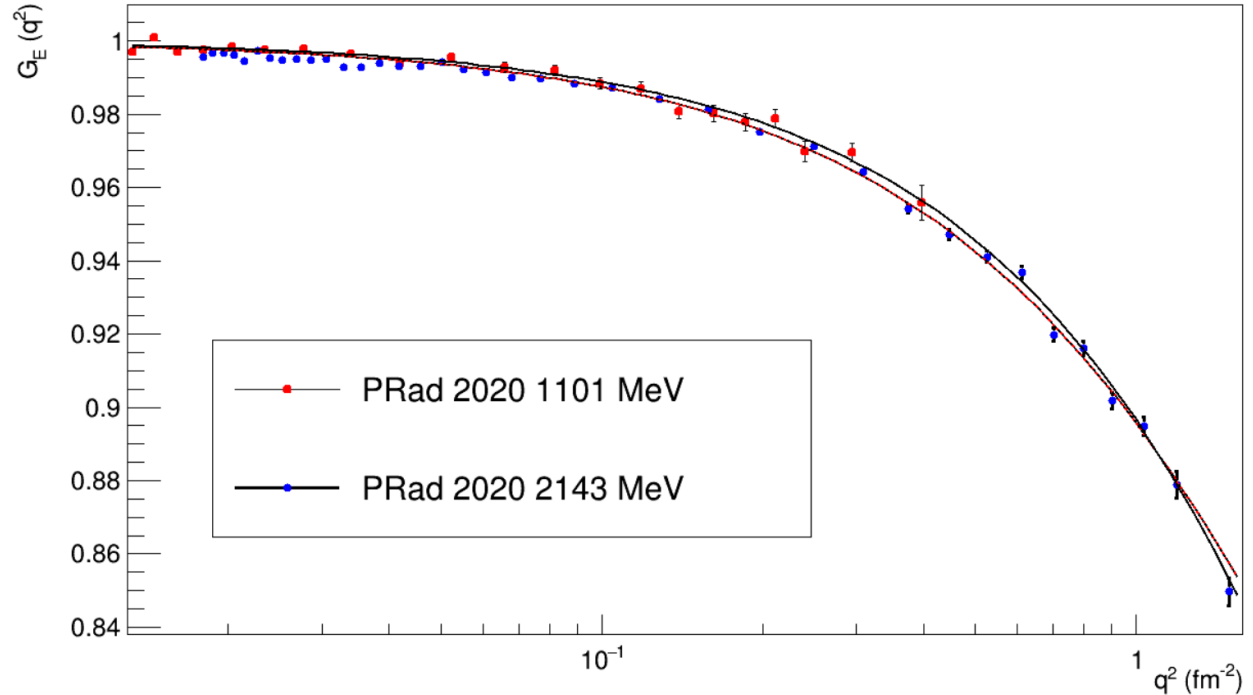
Nature **volume 575**, pages147–150 (2019)





Focus on PRad

- Fixing $G_E(0) = 1$ results in adding a data point with an infinite precision
- It adds a strong constraint from an experimental point of view
- Experiments probably should report this feature as part of their systematic!



Same conditions as Prad fit

NO.	NAME	VALUE	ERROR
1	a0	1.00021e+00	2.23683e-04
2	a1	-2.78593e-03	2.15433e-02
3	a2	1.12425e-01	2.33794e-02
4	a3	9.98317e-01	1.87695e-04

Small radius: 0.831 ± 0.007

Normalization fixed to 1

1	a0	1.00000e+00	constant
2	a1	9.97359e-02	2.06941e-02
3	a2	2.27781e-01	2.20774e-02
4	a3	1.00000e+00	constant

Large radius: 0.877 ± 0.006



- 18 data sets: 6 energies and 3 spectrometers
- 1422 measured cross section points
- Redo fits using ratio of polynomials for electric and magnetic form factors

$$\eta_i = \frac{1 + a_1 Q^2}{1 + b_1 Q^2 + b_2 Q^4 + b_3 Q^6}$$

- And different normalization for each sub dataset
- Obtain a good quality of fit
 - Reduced chi2 of 0.8
 - Radius = 0.841 ± 0.008 fm

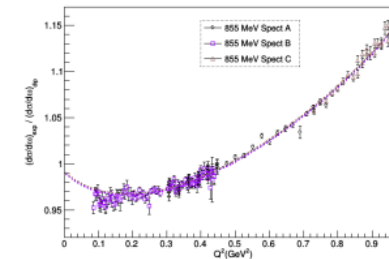
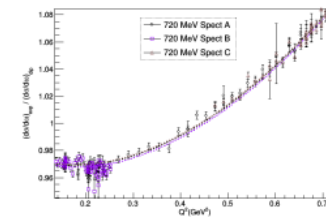
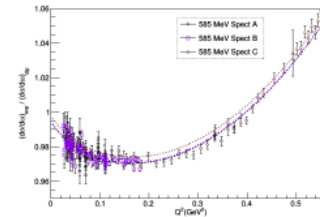
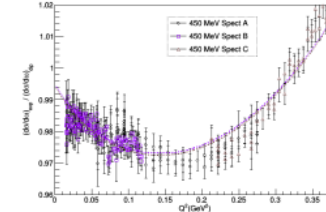
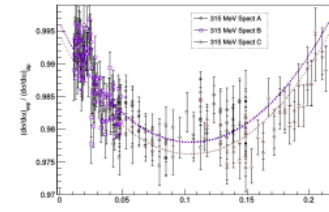
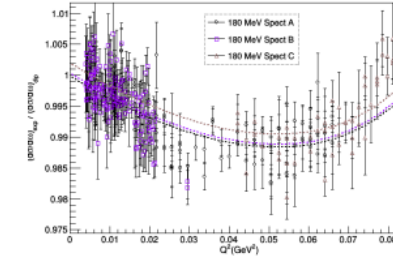
Energy Beam	Specrometer	Normalization Parameter
180 MeV	A	$1.00305 \pm 3.38385 \times 10^{-4}$
	B	$1.00250 \pm 2.15171 \times 10^{-4}$
	C	$1.00486 \pm 5.36807 \times 10^{-4}$
315 MeV	A	$1.00085 \pm 4.28286 \times 10^{-4}$
	B	$1.00114 \pm 3.23118 \times 10^{-4}$
	C	$1.00420 \pm 5.95686 \times 10^{-4}$
450 MeV	A	$1.00068 \pm 5.64634 \times 10^{-4}$
	B	$0.99976 \pm 4.01236 \times 10^{-4}$
	C	$1.00371 \pm 9.57397 \times 10^{-4}$
585 MeV	A	$0.99924 \pm 5.36230 \times 10^{-4}$
	B	$0.99889 \pm 4.29139 \times 10^{-4}$
	C	$1.00448 \pm 1.73328 \times 10^{-3}$
720 MeV	A	$0.99597 \pm 7.93005 \times 10^{-4}$
	B	$0.99734 \pm 4.90752 \times 10^{-4}$
	C	$1.00569 \pm 1.75069 \times 10^{-3}$
855 MeV	A	$0.99141 \pm 1.12938 \times 10^{-3}$
	B	$0.99431 \pm 6.63686 \times 10^{-4}$
	C	$0.93606 \pm 4.72923 \times 10^{-3}$



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$$\eta_i \frac{1 + a_1 Q^2}{1 + b_1 Q^2 + b_2 Q^4 + b_3 Q^6}$$

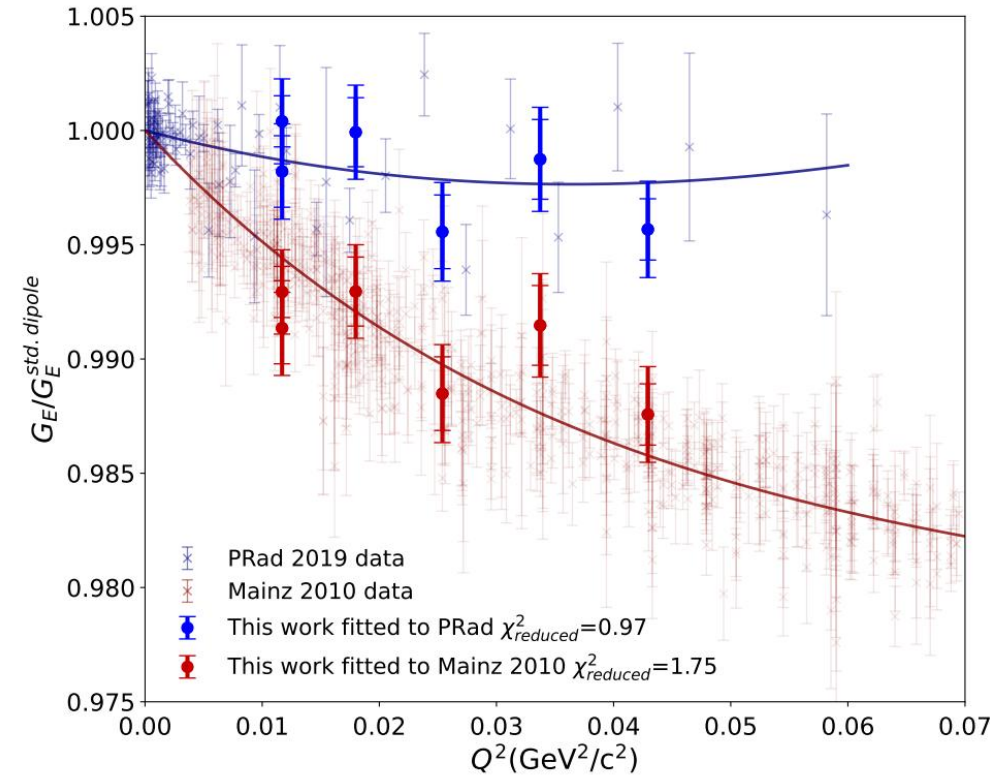
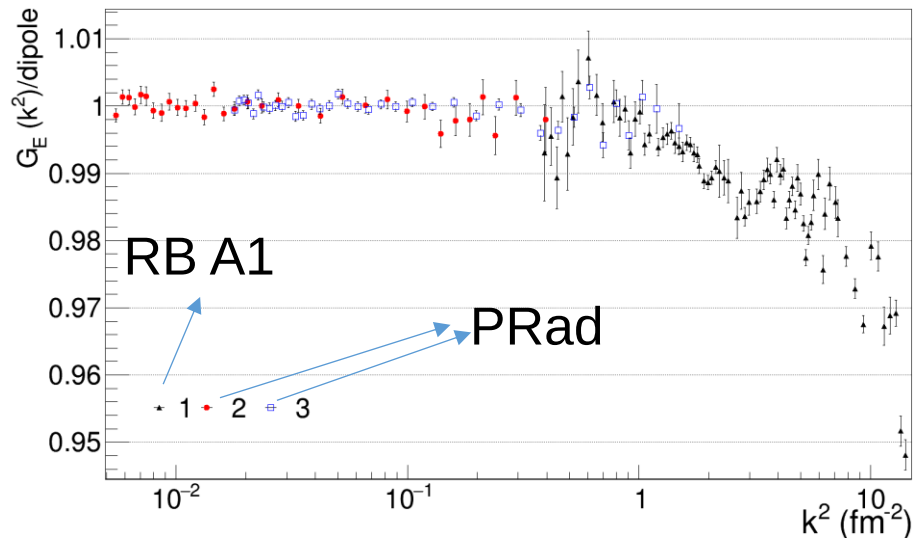
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Disagreement Prad/A1

- Compare both experiments at the level of the form factor
 - PRad has a global normalization
 - Absolute normalization wrt Moller data
 - A1 has a relative normalization
 - 31 normalization parameters of one sub-dataset relative to the other



Phys. Rev. C 106, 044610 (2022)



....Prad II

PRad-I Results on the Form Factor (G_E) at Small Q^2 Range

- There is certain **discrepancy** between two very recent **Form Factor** precision measurements: PRad-I and A1 MAMI at Mainz.
- New **high accuracy ep-scattering measurements** are needed to address these differences.
- PRad-II experimental goals:
 - high accuracy **Ge measurement** in a wide Q^2 range ($Q^2 = 10^{-5} - 10^{-2} \text{ GeV}^2$);
 - reach the $Q^2 = 10^{-5} \text{ GeV}^2$ range
 - a factor of **3 improvement on R_p**

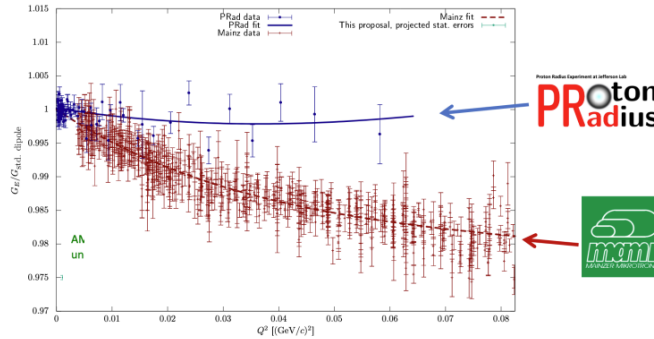


figure: J. Bernauer

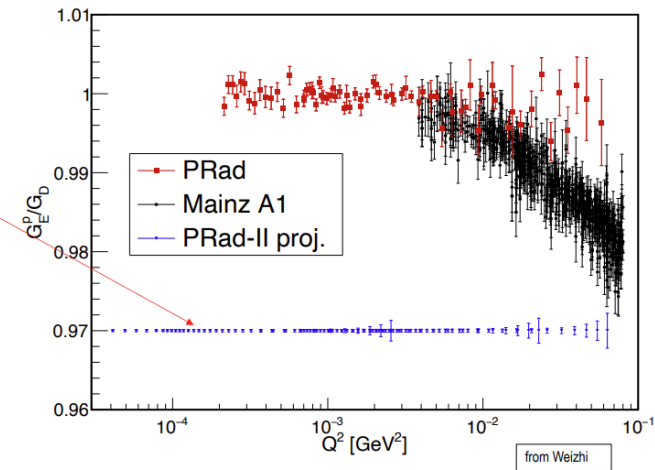
Gasparian

CLAS col. meeting, Nov 21, 2025

15

PRad-II: Improvements on G_E Form Factor Measurement

- To address the **differences** between A1 MAMI, Mainz (2010) and PRad, JLab (2019) results.
- statistical uncertainties only, with:
 - ✓ 6 days with 0.7 GeV, 20nA
 - ✓ 6 days with 2.1 GeV 150nA
 - ✓ 12 days with 3.5 GeV, 150nA
- Total beam time: **40 PAC days**
- Only PbWO₄ part of HyCal used.



Gasparian

CLAS col. meeting, Nov 21, 2025

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After the puzzle: attempts to explain

- The puzzle generated a lot of discussion on what its origin might be

- Electronic hydrogen experiments are not as accurate as stated
- The two-photon exchange correction is under evaluated?
- Pion exchange corrections to the vertex?!!

Or cannot be: Phys Rev D 62 071503

contraction, are no longer negligible. What this means is that an identification of the slope of the form factor with the rms radius of a charge distribution in the rest frame is not unambiguously possible and the best one can achieve is an identification with uncertainties on the order of the Compton wavelength of the target.

~0.2% ?

Cannot account for the difference!?

Or more exotically:

- Relativistic corrections?
- New Physics?
 - Lepton universality violation
 - Heavy/Dark photons

What we measure is the Sach radius and not the real proton radius

Licht & Pagnamenta PRD 2, 6, 1970 (I and II)

J. J. Kelly PRC 66, 065203 (2002)

arxiv 1505.04723v2 Donnelly, Hasell and Milner

Y. Chen, C. Lorcé, Phys. Rev. D 107, 096003 (2023)

C. Lorcé, Phys. Rev. Lett. 125, 232002 (2020)

G.A. Miller, Phys. Rev. C 99, 035202 (2019)

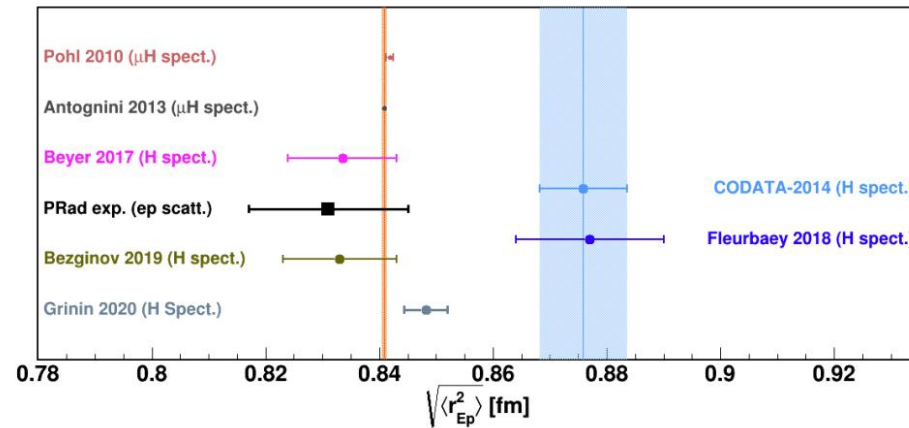
Dark/heavy photon to solve the puzzle

Liu, McKeen and Miller et al. Phys. Rev. Lett. 117, 101801 (2016)



The puzzle: how it actually lost popularity

- CODATA estimated that there is no weakness to be found in the muonic experiments (~8 years after it got published in nature) and the related theory (more than a decade old theoretical effort)
 - Included the muonic results in the world average in 2018 recommended values
CODATA-recommended value of $r_p = 0.8414(19)$ fm
- Nevertheless, experiments coming after, mainly atomic spectroscopy ones, reported high radius values
 - Tensions between Fleurbaey et al. (Paris) and Grinin et al. (Garching) results for 1S-3S H : Grinin et al.: factor 2 more precise, $\sim 2\sigma$ smaller than Fleurbaey et al., $\sim 2\sigma$ larger than μ H result



Rev. Mod. Phys. 94 (2022) 015002



Next on proton size: perspective at low Q²

- Muonic atom spectroscopy: shift from puzzle to precision !

- μH Lamb shift: CREMA coll.: factor 5 improvement planned
- μH 1S Hyper fine splitting: next frontier 1ppm precision !

Theory accuracy is at the same level as experimental accuracy for muonic hydrogen measurements. Should cope with expected improvements on the experimental side.

- Lepton scattering

- Muon scattering:
 - MUSE@PSI : production runs in 2023/2024, under analysis
 - AMBER@CERN: pilot run fall 2024, beam time was foreseen in 2025?
- Electron/positron:
 - ULQ2@Sendai: e- scattering @ 10 - 60 MeV, Q² down to $3 \times 10^{-5} \text{ GeV}^2$ (**analysis ongoing waiting for publication**)
 - PRad-II@hall B JLab: **ran this year**, improvement of statistical uncertainty by factor 3 vs Prad
 - Systematic errors should be controlled at better level!!!
 - MAGIX@MESA: **under construction**, should start in 2026??!
 - Further plans/opportunities: positron upgrade at Jlab (if it happens), ARIEL e- LINAC



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- Lepton scattering

Credits: J. Bernauer

Exp	Regime	Settings	e/μ	Antiparticle
PRAD2@JLAB	small angle	few	e^-	no
ULQ2@TOHOKU	large angle	many	e^-	no
MAGIX@MESA	large angle	many	e^-	no
AMBER@CERN	small angle	few	μ	yes
MUSE@PSI	large angle	few	e/μ	yes



GPDs as a tool to extract Charge radius

- Determine GPDs at zero skewness through the analysis of ep scattering data **Phys. Rev. D 72, 054013 (2005)** and **Eur. Phys. J. C73, no.4, 2397 (2013)** and most recently **Phys. Lett. B 864 (2025) 139423**

- Extract functional form for GPD evolution using a χ^2 with

$$F_1^p(t) = \sum_q e_q F_1^q(t) = \sum_q e_q \int_0^1 dx H_v^q(x, \mu^2, t) \quad F_2^p(t) = \sum_q e_q F_2^q(t) = \sum_q e_q \int_0^1 dx E_v^q(x, \mu^2, t)$$

- Parametrize GPDs as

$$\begin{aligned} H_v^q(x, \mu^2, t) &= q_v(x, \mu^2) \exp[tf_v^q(x)] \\ E_v^q(x, \mu^2, t) &= e_v^q(x, \mu^2) \exp[tg_v^q(x)] \end{aligned} \quad \begin{aligned} &\rightarrow \mathcal{F}(x) = \alpha'(1-x)^3 \log \frac{1}{x} + B(1-x)^3 + Ax(1-x)^2 \end{aligned}$$

- Ignore Strange quark contribution
- Take PDFs from the NNPDF analysis at NLO
- Satisfy forward limit constraint: GPDs reduce to PDFs
- Densities of quarks and anti-quarks differ for same x and at a given transverse distance (strong constraint on the functional form)

- Determine a functional form for the Dirac and Pauli form factors as moments of the H and E GPDs

Observable	Base Fit	PRad	PRad_NoMainz
r_{pE}	0.8564 ± 0.0136	0.8558 ± 0.0135	0.8542 ± 0.0137
r_{pM}	0.8242 ± 0.0596	0.8268 ± 0.0533	0.8243 ± 0.0605
$< r_{nE}^2 >$	-0.1132 ± 0.0272	-0.1181 ± 0.0270	-0.1244 ± 0.0274
r_{nM}	0.8330 ± 0.1136	0.8367 ± 0.0845	0.8426 ± 0.1230



GPDs as a tool to extract Charge radius

- The method seems to be robust: there is some systematic from the choice of the ansatz for the GPDs but still more analytically constrained than that of the FF
- Tension in Mainz data:

so that including them in the analysis does not change significantly the χ^2 of the other data sets. Moreover, removing the Mainz data from the analysis leads to a significant decrease in the value of total χ^2 and hence $\chi^2/\text{d.o.f.}$

the central values of some parameters exhibit notable changes. On the other hand, removing the Mainz data from the analysis does not substantially alter the final results, particularly for the GPDs H_v^q . However,

- The question remains: How did Mainz analysis paper get good chi2?
 - Probably include the systematic errors in the chi2 minimization?

Observable	$\chi^2/N_{\text{pts.}}$		
	Base Fit	PRad	PRad_NoMainz
Mainz G_E^p [58]	275.1/77	281.6/77	—
AMT07 G_E^p/G_D [57]	37.2/47	37.1/47	40.3/47
AMT07 R^p [50]	115.6/69	115.7/69	111.7/69
AMT07 G_E^n [50]	30.1/38	27.1/38	25.2/38
AMT07 $G_M^n/\mu_n G_D$ [50]	32.5/33	34.4/33	33.8/33
GMp12 σ_R [51]	16.2/13	18.4/13	13.7/13
PRad σ_R [15] 1.1 GeV	—	10.5/33	10.2/33
PRad σ_R [15] 2.2 GeV	—	45.1/38	44.2/38
Total $\chi^2/\text{d.o.f.}$	506.7/262	569.9/333	279.1/256

Recommendation

analysis of GPDs. Actually, since the forward limit e_v^q and the profile functions $f_v^q(x)$ and $g_v^q(x)$ are obtained from whole $-t$ regions, the medium and large $-t$ regions are also important and can affect the value of radii. Mathematically, this means that one should first obtain the whole $-t$ dependency of FFs and then calculate their derivations at $t = 0$.

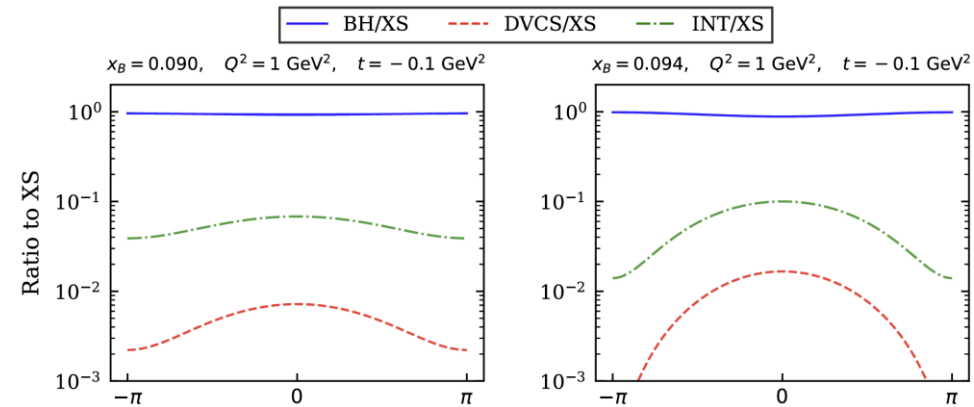
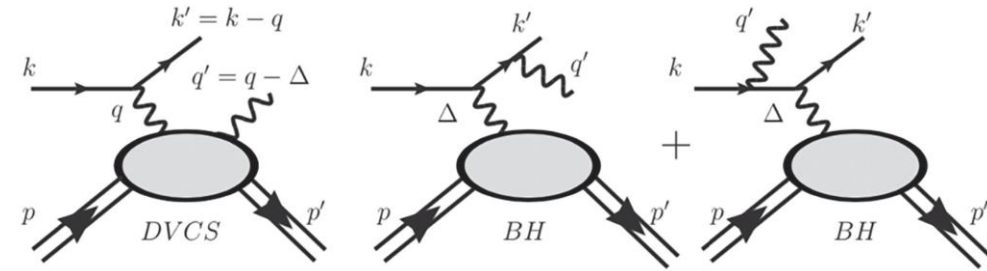
We already did that at Orsay!



DVCS/BH as a tool to extract Charge radius

- Exclusive $ep \rightarrow e\gamma$ measurements: BH dominated phase space
 - BH is exactly calculable in QED: extract Dirac and Pauli FF
 - Major assumption of F2 functional form that could significantly alter the radius extraction
 - Why not take elastic data in consideration via a simultaneous fit?

Fit	r_E (fm)	r_M (fm)
Fit 5	0.779 ± 0.013	0.842 ± 0.027
Fit 6	0.804 ± 0.011	0.860 ± 0.022
Fit 7	0.792 ± 0.012	0.889 ± 0.030
Fit 8	0.815 ± 0.011	0.831 ± 0.004



PHYSICAL REVIEW D 113, 054034 (2026)

Side note: clas2018 cross section data is in tension with other datasets; although the clas12 authors insist they are compatible with previous measurements!!



Motivation: developing another method to evaluate the moments of the charge density from experimental data

- **New approach** to extract the **moments of a probability density function** through integral forms of its Fourier transform [Phys . Let. B 808 135669 \(2020\)](#)
- With specific attention to negative order moments [Eur. Phys. J. A. 61, 29 \(2025\)](#)
- Application to **proton electric form factor data** taking into account all sources of uncertainties [Phys . Rev. C 110 015207 \(2024\)](#)

With major contributions from
M. Atoui, M.B. Barbaro, C. Keyrouz, R. Kunne, M. Lassaut, D. Marchand, G. Quéméner,
E.Voutier, J. van de Wiele



The integral Method

- Spatial density $\rho_E(\mathbf{r}) = \frac{1}{(2\pi)^3} \int_{R^3} d^3\mathbf{k} e^{i\mathbf{k}\mathbf{r}} \mathbf{G}_E(\mathbf{k})$
- Moments $(\mathbf{r}^\lambda, \rho_E) = \int d^3\mathbf{r} r^\lambda \rho_E(\mathbf{r})$

$$(\mathbf{r}^\lambda, \rho_E) = \frac{1}{2\pi^3} \int_{R^3} d^3\mathbf{k} \mathbf{G}_E(\mathbf{k}) \int_{R^3} d^3\mathbf{r} e^{i\mathbf{k}\mathbf{r}} r^\lambda$$

Finite term

Divergent term needs to be regularized as by definition the moment r^λ is finite

- The integral: $g_\lambda(k) = \int_{R^3} d^3r e^{i\mathbf{k}\mathbf{r}} r^\lambda$ needs to be regularized
- Two methods to regularize:
 - The principle value regularization: integral method 1 (IM1)
 - The exponential regularization: integral method 2 (IM2)
- Both give the same results with differences originating in the rapidity with which each approach converges



The principle value regularization IM1

- **The integral** $g_\lambda(\mathbf{k}) \equiv g_\lambda(k) = \int_{\mathbb{R}^3} d^3\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}} r^\lambda$ satisfies the relation $g_\lambda(t\mathbf{k}) = \frac{1}{t^{\lambda+3}} g_\lambda(\mathbf{k})$

- The integral is to be considered as a distribution and counter terms $(\tilde{f}_{2j})$ **need to be subtracted to insure convergence**

$$(r^\lambda, f) = \mathcal{N}_\lambda \int_0^\infty dk \left\{ \frac{\tilde{f}(k)}{k^{\lambda+1}} \right\} \quad \text{with} \quad \mathcal{N}_\lambda = \frac{2^{\lambda+2}}{\sqrt{\pi}} \frac{\Gamma(\frac{\lambda+3}{2})}{\Gamma(-\frac{\lambda}{2})} \quad \text{with the regularized moment} \quad (r^{m-\eta}, f) = \mathcal{N}_{m-\eta} \int_0^\infty dk \frac{\tilde{f}(k) - \sum_{j=0}^{\lfloor \frac{\lambda}{2} \rfloor} \tilde{f}_{2j} k^{2j}}{k^{m-\eta+1}}$$

$$\tilde{f}_{2j} = \frac{1}{j!} \left. \frac{d^j \tilde{f}(k)}{d(k^2)^j} \right|_{k=0}$$

- **The divergence appearing in the normalization term is canceled by the divergence in the integral**

$$(r^m, f) = \lim_{\eta \rightarrow 0^+} (r^{m-\eta}, f) \quad m \text{ even}$$

$$(r^m, f) = (r^{m-\eta}, f)|_{\eta=0} \quad m \text{ odd.}$$

Condition: $\lambda > -3$



The exponential regularization IM2

- The integral $g_\lambda(\mathbf{k}) \equiv g_\lambda(k) = \int_{\mathbb{R}^3} d^3\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}} r^\lambda$ can be taken as the weak limit of the convergent integral

$$g_\lambda(\mathbf{k}) = \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^3} d^3\mathbf{r} r^\lambda e^{-\epsilon r} e^{i\mathbf{k}\cdot\mathbf{r}} = \lim_{\epsilon \rightarrow 0^+} \mathcal{I}_\lambda(k, \epsilon) \quad \text{with} \quad \mathcal{I}_\lambda(k, \epsilon) = \frac{4\pi \Gamma(\lambda + 2) \sin[(\lambda + 2)\text{Arctan}(k/\epsilon)]}{k(k^2 + \epsilon^2)^{\frac{\lambda}{2} + 1}}$$

Standard technique to regularize the Fourier transform of the Coulomb potential

- The moment r^λ can then be written as

$$(r^\lambda, f) = \frac{2}{\pi} \Gamma(\lambda + 2) \lim_{\epsilon \rightarrow 0^+} \int_0^\infty dk \tilde{f}(k) \frac{k \sin[(\lambda + 2)\text{Arctan}(k/\epsilon)]}{(k^2 + \epsilon^2)^{\lambda/2 + 1}}$$

- For integer values of λ

$$(r^m, f) = \frac{2}{\pi} (m + 1)! \lim_{\epsilon \rightarrow 0^+} \epsilon^{m+2} \int_0^\infty dk \tilde{f}(k) \frac{k}{(k^2 + \epsilon^2)^{m+2}} \Phi_m(k/\epsilon)$$

$$\text{with} \quad \Phi_m(k/\epsilon) = \sum_{j=0}^{m+2} \sin\left(\frac{j\pi}{2}\right) \frac{(m+2)!}{j!(m+2-j)!} \left(\frac{k}{\epsilon}\right)^j$$

Condition: $\lambda > -3$



- The integral method advocates the necessity of **fitting all available data** for the determination of the functional form
- The model needs only to be **integrable over the domain where data exists**. The convergence of the integrand is insured by a **cut-off**.
- Can **access moments with orders $\lambda > -3$** , integer and non – integer moments
 - Allows to experimentally determine all moments, even if the inverse Fourier transform of the FF doesn't exist
- The extraction of even moments are unchanged w.r.t. previous determinations
- **The extraction of odd moments represents a first experimental determination**



Application to real data

- **Select data extracted from electron scattering experiments**

- via a Rosenbluth separation
- for kinematical conditions where the magnetic form factor contribution to the cross section is strongly suppressed

- **Use the functional form**

$$\tilde{f}_K(\mathbf{k}) \equiv \tilde{f}_K(k) = \frac{1 + a_1 k^2}{1 + b_1 k^2 + b_2 k^4 + b_3 k^6}$$

- **Fit simultaneously the different datasets**

- The **same functional behavior** is assumed for each dataset
- **A separate normalization** parameter is considered for each dataset

Data Set Number	Year	Authors	Ref.	Number of data points	k^2 -range k_{min}^2 [fm ⁻²]	k_{max}^2 [fm ⁻²]
1	1962	Lehmann <i>et al.</i>	[19]	1	2.98	2.98
2	1963	Dudelzak <i>et al.</i>	[20]	4	0.30	2.00
3	1963	Berkelman <i>et al.</i>	[21]	3	25.0	35.0
4	1966	Frèrejacque <i>et al.</i>	[22]	4	0.98	1.76
5	1966	Chen <i>et al.</i>	[23]	2	30.0	45.0
6	1966	Janssens <i>et al.</i>	[24]	20	4.00	22.0
7	1971	Berger <i>et al.</i>	[25]	9	1.00	50.0
8	1973	Bartel <i>et al.</i>	[26]	8	17.2	77.0
9	1975	Borkowski <i>et al.</i>	[27]	10	0.35	3.15
10	1994	Walker <i>et al.</i>	[28]	4	25.7	77.0
11	1994	Andivahis <i>et al.</i>	[29]	8	44.9	226.
12	2004	Christy <i>et al.</i>	[30]	7	16.7	133.
13	2005	Qattan <i>et al.</i>	[31]	3	67.8	105.
* 14	2014	Bernauer <i>et al.</i>	[10]	77	0.39	14.2
# 15	2019	Xiong <i>et al.</i> - 1.1 GeV	[11]	33	5.51×10^{-3}	3.96×10^{-1}
# 16	2019	Xiong <i>et al.</i> - 2.1 GeV	[11]	38	1.79×10^{-2}	1.49
17	2021	Mihovilović <i>et al.</i> - 195 MeV	[32]	6	3.43×10^{-2}	6.99×10^{-2}
% 18	2021	Mihovilović <i>et al.</i> - 330 MeV	[32]	11	4.69×10^{-2}	2.00×10^{-1}
19	2021	Mihovilović <i>et al.</i> - 495 MeV	[32]	8	1.57×10^{-1}	4.37×10^{-1}

#PRad

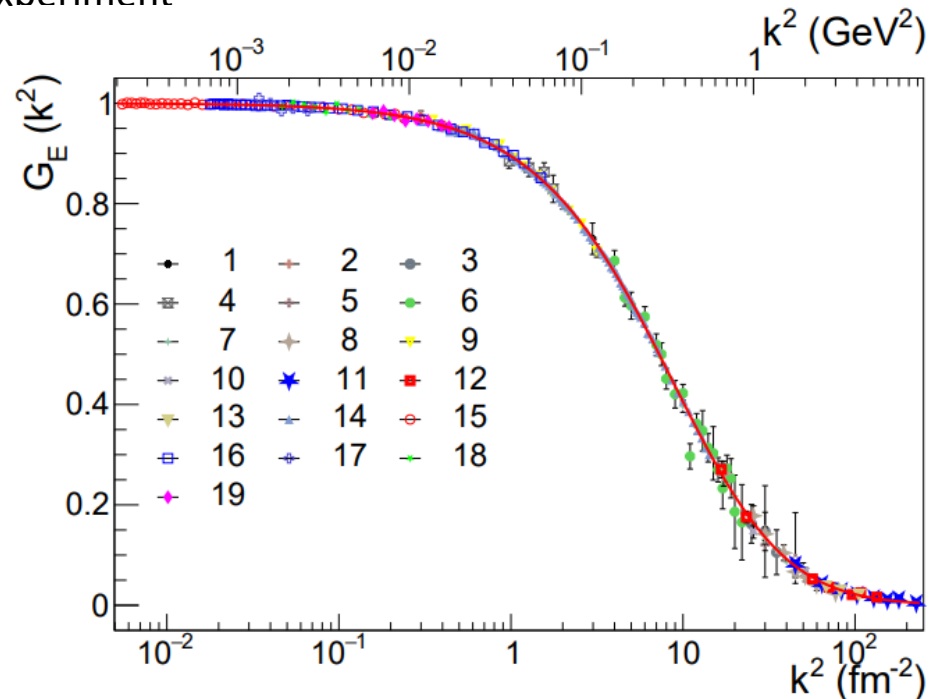
*A1 Rosenbluth

%ISR

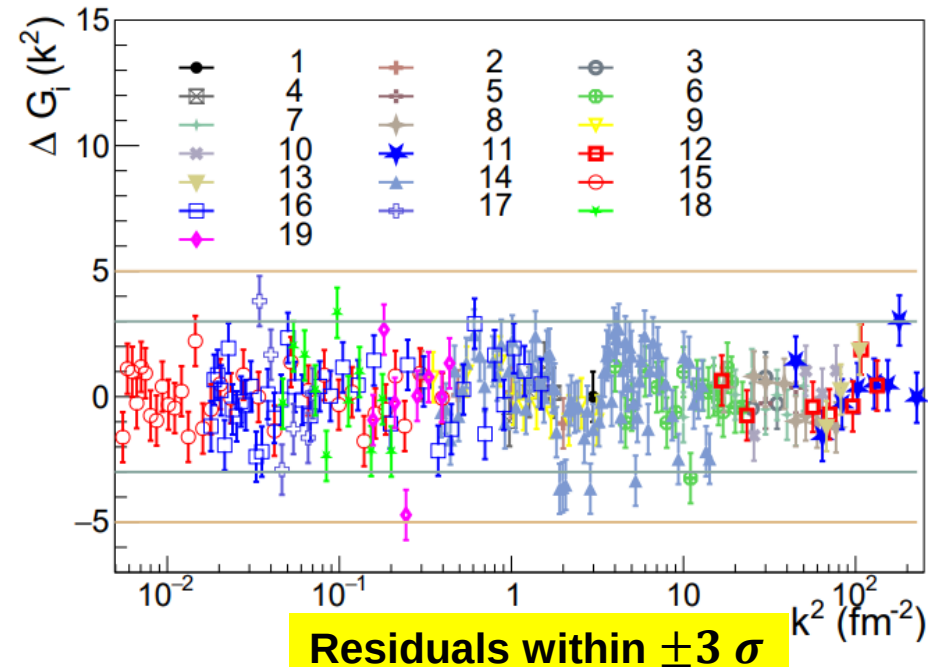


Application to real data: fit results

- **Acceptable fit quality** with a reduced χ^2 of ~ 1.97
- The evaluated **normalization parameters (all compatible with 1 with deviations up to 17%)** reflect **the imprecision** on the determination of the absolute normalization for each experiment



- A1 & ISR @ Mainz
 - Fit without them has χ^2 of ~ 1.2
 - A1 data very precise?!
 - ISR fixed higher orders in their fit based on A1 $\rightarrow$ bias





Application to real data: evaluation of errors

- **Statistical error** coming from **experimental data**
- Sources of **systematic errors**:
 1. Dat.: Originating from the systematic error that is reported by each considered **experiment on the EFF**
 2. Nor.: errors originating from the assumption of a global normalisation
 3. Fun.: Bias that could be generated on the fit parameters **from the fitting model itself**
 4. Mod. Error coming from the **choice of the fitting model**.
- **Errors are propagated to the evaluated moments using MC methods**
 - Take into account correlations between parameters to all orders

λ				Statistical Error		Systematic Error				
	$\langle r^\lambda \rangle_Q$ [fm $^\lambda$]	$\langle r^\lambda \rangle$ [fm $^\lambda$]	$\langle r^{2p} \rangle_d$ [fm $^\lambda$]	$\delta [\langle r^\lambda \rangle_Q]$ [fm $^\lambda$]	$\delta [\langle r^{2p} \rangle_d]$ [fm $^\lambda$]	Dat. [fm $^\lambda$]	Int. [fm $^\lambda$]	Fun. [fm $^\lambda$]	Nor. [fm $^\lambda$]	Mod. [fm $^\lambda$]
-2	6.5245	8.7226	—	0.0172	—	0.0106	2.1981	0.0001	0.0095	0.3731
-1	1.9681	2.0906	—	0.0024	—	0.0019	0.1225	0.0001	0.0029	0.0278
1	0.7206	0.7179	—	0.0020	—	0.0025	0.0027	0.0001	0.0011	0.0029
2	0.6937	0.6937	0.6937	0.0094	0.0105	0.0111	0	0.0001	0.0010	0.0116
3	0.8697	0.8701	—	0.0457	—	0.0494	0.0004	0.0007	0.0013	0.0633
4	1.4728	1.4728	1.4728	0.2461	0.2365	0.2474	0	0.0065	0.0022	0.3805
5	3.8139	3.8137	—	1.4822	—	1.4297	0.0002	0.0343	0.0056	2.6276
6	16.405	16.405	16.405	10.058	10.839	9.4985	0	0.1871	0.0240	21.531
7	104.64	104.64	—	76.676	—	71.727	0.0001	2.8169	0.1528	212.48

Larger statistical errors for high order positive moments (probing the large distance behavior of the charge density):
lack of measurements at ultra low k^2



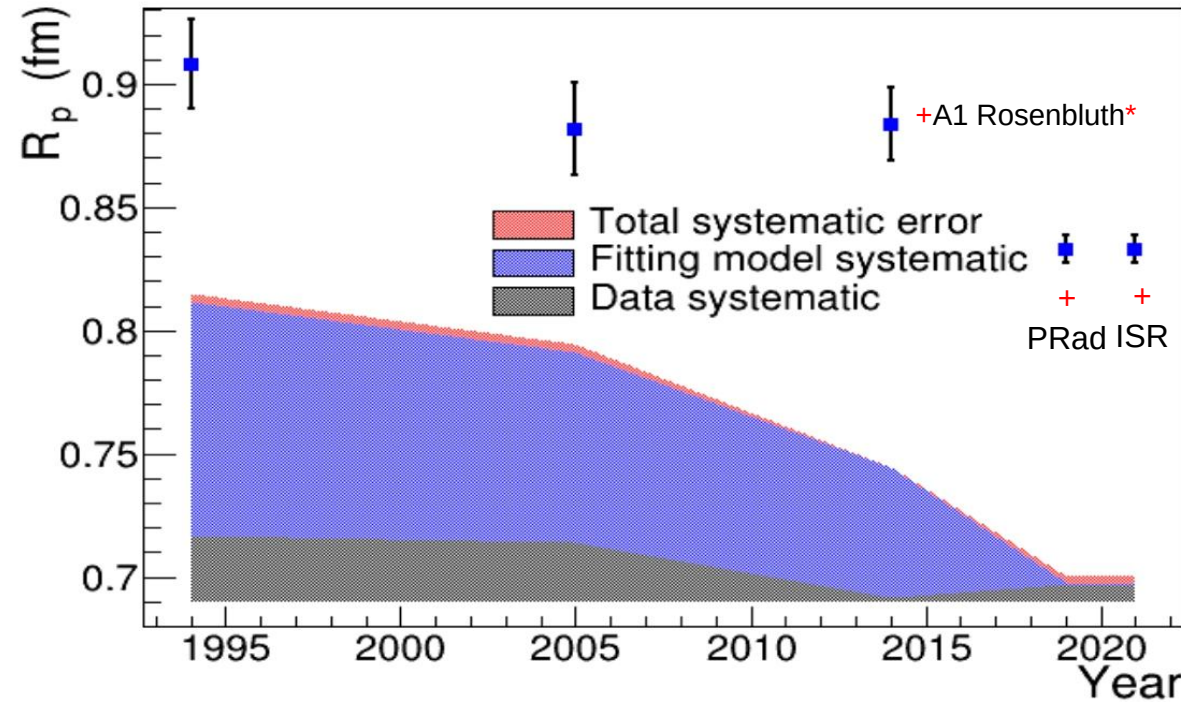
Proton charge radius

- Data are grouped into **5 time periods**, from 1963 to 2021
- For each period we have performed the same analysis procedure estimating the **radius** and the **systematic errors**

Time period	Data set range	R_p [fm]	$(\delta R_p)_{Sta.}$ [fm]	$(\delta R_p)_{Sys.}$ [fm]
1962-1994	1 - 11	0.9081	0.0178	0.1249
1962-2005	1 - 13	0.8813	0.0191	0.1044
1962-2014	1 - 14	0.8837	0.0148	0.0544
1962-2019	1 - 16	0.8329	0.0057	0.0102
1962-2021	1 - 19	0.8329	0.0056	0.0097

Up to 2014 the **dominant** source of **systematic** uncertainty is the choice of the **fitting model** (blue band)

With data at very low Q^2 (Mainz A1, PRAD, ISR) constraints on the fit model are reinforced and this systematics is strongly reduced: the **data systematics** becomes **dominant** (grey band)





Why large Q^2 is good?

- The spatial density cannot be reconstructed from the knowledge of its moments
- But, using negative order moments, one can access information about the shape of the density near $r=0$ fm.
 - We derived a method (extending the IM) to extract the Maclaurin expansion of the spatial density
 - This requires knowledge of the form factor at large Q^2 values.

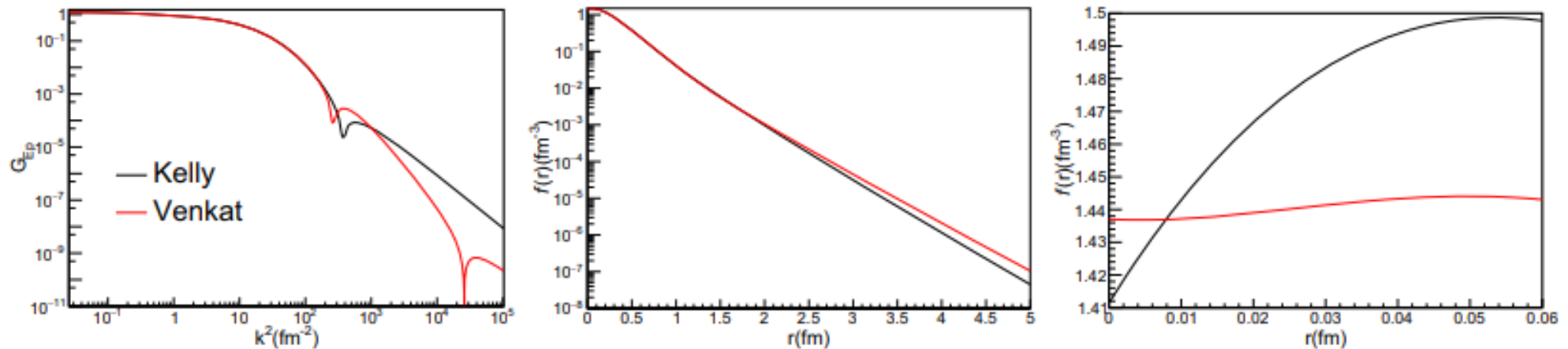


Fig. 4 (Left) Electric form factor of proton from Refs. [11] and [12] drawn in natural units of fm^{-2} ; $1\text{GeV}^2 = 1/(\hbar c)^2 \text{fm}^{-2}$. (Middle) Radial densities corresponding to the two considered parametrizations of G_{Ep} . (Right) Radial densities at small distances (for $r \ll 1$ fm)



Summary and conclusions

- Puzzle lost popularity but still survives
- Too many attempts to explain, too many hypothesis to make
 - Most of which are not valid
- Mixed messages from spectroscopy
- Mixed messages from scattering
 - Assumptions made without clear discription?
- More measurements from both approaches to come
 - Should be relieved from the burden of assumption by considering all possible assumptions!

Our contribution in Orsay

- A **novel method** for the **determination of the moments** of the charge density via **integral forms of the electric FF**.
- A conceptual implication on **the consideration of Q^2 domain** for the fit
- The experimental extraction of **odd, even, fractional and negative** ($\lambda > -3$) **moments** of the charge density
- Emphasis on negative order moments to probe the spatial density near the center
- Perspective: **Zemach radius and the HFS: FAMU experiment 10^{-5} accuracy on HFS $\rightarrow$ 0.3% on zemach radius**
 - Fit simultaneously GE and GM data to get a convoluted density and extract the Zemach moments



- Observational bias: should be avoided with a blind analysis
- The assumptions: should be clear and results should be presented with the different possible assumptions



- The concept of an observable quantity in quantum mechanics is ill-defined
- The principle of induction in logical thinking works fine but is not rational



spare



Comments on some newer fitting results

2010: **>0.870** Hill, Paz: old data, z expansion with disp. bounds

» Bounds on infinite exp. → bounds for truncated exp.?

2012: **0.840(10)** Lorenz, Hammer, Meissner: Disp. relation fit.

» Same value but a lot more data. Probably model dominated.

2014: **0.84** Lorenz, Meissner: z expansion without bounds

» Fit did not converge. In real minimum, large radius is found.

2014: **0.8989(1)** Gracyk/Juszczak: Bayesian estimation

» Interesting technique, unbelievable? small errors

2016: **0.84?** Higinbotham: F-Test to select max. order

» Misunderstood F-test. Absence of proof $\neq$ proof of absence.

2016: **0.84?** Horbatsch/Hessels/Griffioen/Carlson/Maddox... Low-Q

» Low-Q fits with low order don't work.

2018: **XXX** Yan/Higinbotham/...

» Small radius fraction finally does bias testing

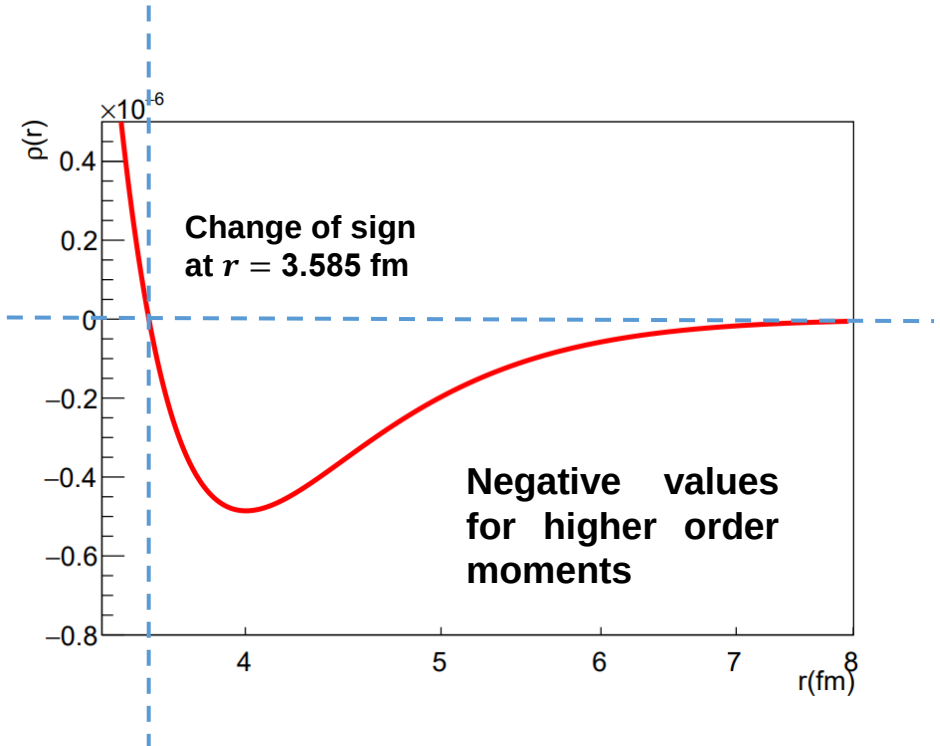
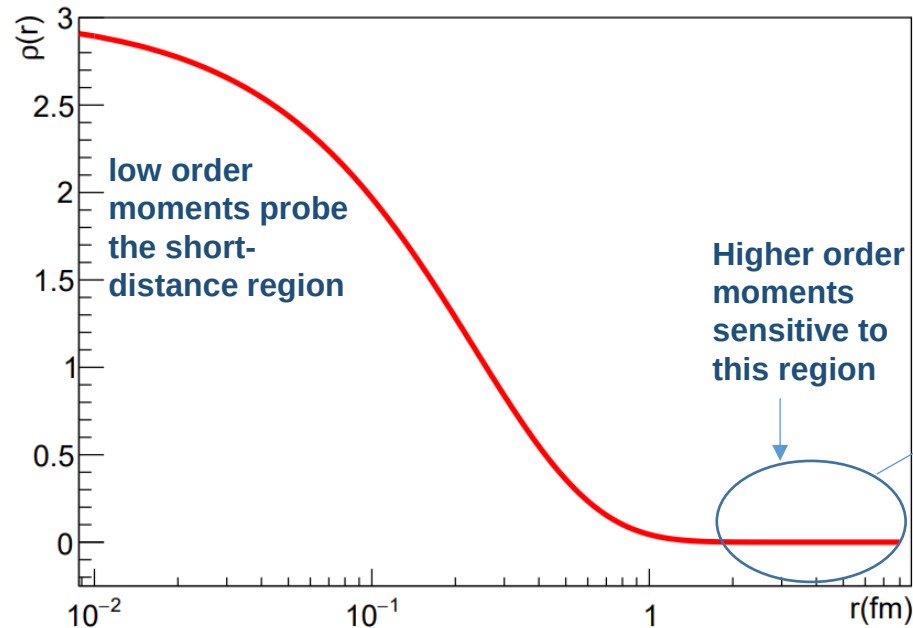
Credits: J. Bernauer



Distribution of the charge density function

Spatial density $\rho_E(r) = \frac{1}{2\pi^3} \int_{R^3} d^3k e^{ikr} G_E(k)$

Fourier transform of our functional form $G_E(k)$



high positive order moments describe the tail of the charge distribution



The moments

Moments of the charge density: Refer to how charge is distributed inside the nucleon

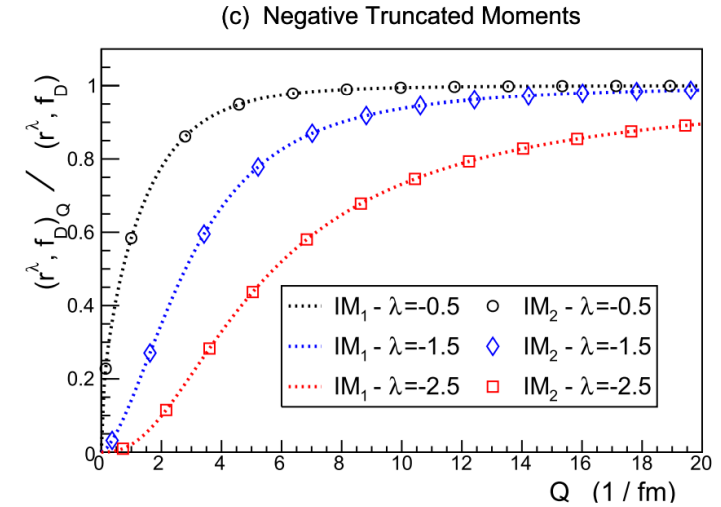
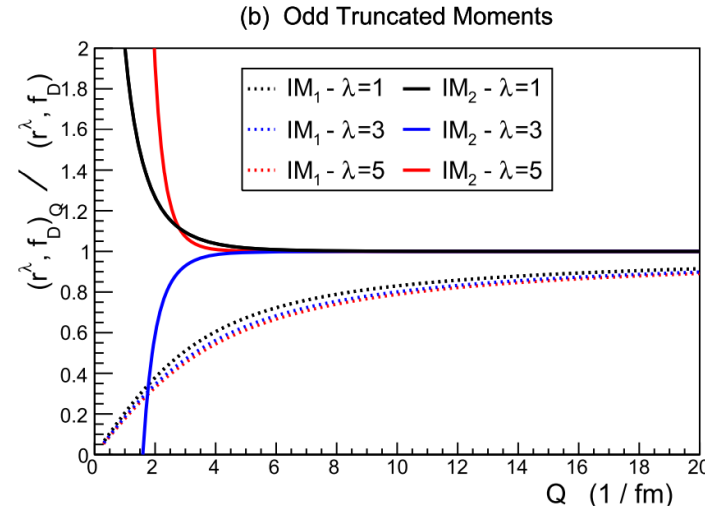
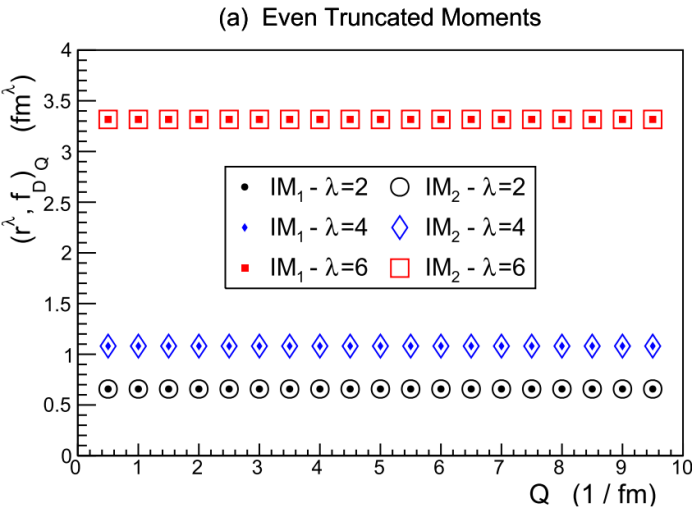
- Moments **beyond the second order** : **Complementary information** on the charge distribution **inside the nucleon**.
- **Negative orders:**
 - Relevant for the **study of the high-momentum dependence** of the form factor
 - Essential to understand **short range effects** near the nucleon's center
- **High positive order** moments
 - Probe the **low-momentum behavior** of the form factor
 - Essential to understand **long range effects**



Validation of the integral method

Use the polynomial ratio parametrization $\tilde{f}_K(\mathbf{k}) \equiv \tilde{f}_K(k) = \frac{1 + a_1 k^2}{1 + b_1 k^2 + b_2 k^4 + b_3 k^6}$

J.J. Kelly, Phys. Rev. C 70 (2004) 068202.



Q-independence is reproduced by each prescription: the **IM recovers the same quantities as the derivative method**. The practical constraint is to obtain an **appropriate description of the data over a large k^2 -domain**.

Different saturation behaviours of the two regularization methods: IM1 asks for large Q -values, IM2 rapidly saturates about 6 fm^{-1} . Sensitive to the high-momentum behavior of the FF.

Same behavior for both methods IM1 and IM2

Convergence is not guaranteed within the domain covered by experimental data **Negative moments are of particular importance:** behavior of the charge density near the nucleon's center



The approach

- The Maclaurin expansion of a radial density in D dimensions writes:

$$f_D(r) = \frac{1}{r^\beta} \left(\sum_{j=0}^m g_j r^j + \mathcal{O}(r^{m+1}) \right), \quad m \geq 0 \quad \lim_{r \rightarrow 0} r^\beta f_D(r) = g_0$$

Determine these unknowns -> Get the density

- Rewrite the moment as: $(r^\lambda, f_D) = \omega_D \int_0^\infty dr r^{\lambda-\lambda_c-1} g(r)$. with $g(r) = f_D(r) r^\beta$ and $\lambda_c \equiv -D + \beta$
- The moment can directly be related to the coefficients of the Maclaurin expansion

$$\frac{(r^\lambda, f_D)}{\omega_D} \approx \frac{g_{j_0}}{\lambda - (\lambda_c - j_0)}$$