

Next-to-next-to-leading gravitational radiation-reaction corrections to quasi-elliptic and quasi-hyperbolic compact binary orbits

Emeric Seraille



10ème assemblée générale du GDR Ondes Gravitationnelles

October 5th, 2026

Based on

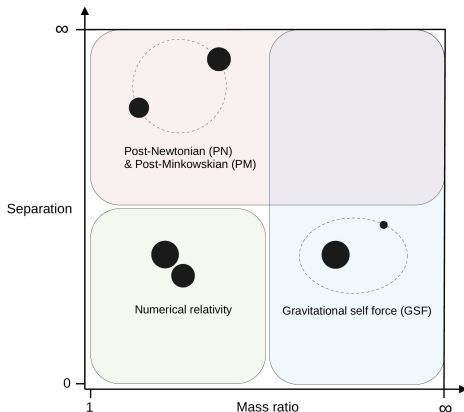
Blanchet, Faye, **Seraille** & Trestini

2601.06743

Bini, Blanchet, Faye, Geralico, Henry, Rufrano Aliberti & **Seraille**

2610.XXXXX

The two-body problem in general relativity



Approximation schemes are necessary

PN: $\frac{v}{c} \ll 1$

PM: $G \ll 1$

GSF: $\frac{m}{M} \ll 1$

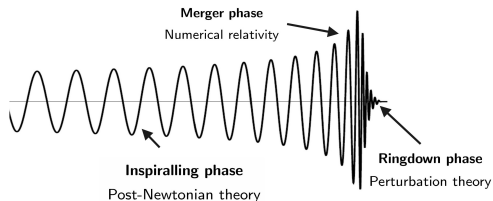
This presentation

Solutions to the Post-Newtonian **dissipative part** up to 4.5PN without spins for

- ▶ bound **quasi-elliptic** orbits
- ▶ unbound **quasi-hyperbolic** orbits

The 4.5 Post-Newtonian radiation reaction

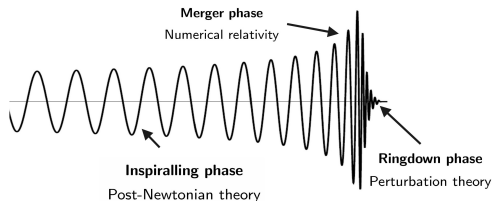
The **radiation reaction** is the part of the equations of motion associated with the **loss of energy** of a gravitational system and the emission of **gravitational waves**.



$$\mathbf{a}_1 = -\frac{Gm_2}{r_{12}^2} \mathbf{n}_{12} + \underbrace{\mathbf{a}_1^{1\text{PN}} + \mathbf{a}_1^{2\text{PN}} + \mathbf{a}_1^{3\text{PN}} + \mathbf{a}_1^{4\text{PN}}}_{\text{conservative part}} + \underbrace{\mathbf{a}_1^{2.5\text{PN}} + \mathbf{a}_1^{3.5\text{PN}} + \mathbf{a}_1^{4\text{PN}} + \mathbf{a}_1^{4.5\text{PN}}}_{\text{dissipative part}}$$

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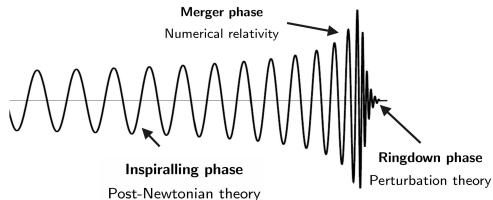
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At the leading (2.5PN) order in harmonic coordinates and in the center-of-mass frame the relative acceleration is [Damour 1982]

$$\mathbf{A}_{2.5\text{PN}}^{\text{RR}} = \frac{G^2 m^2 \nu}{r^3 c^5} \left\{ \dot{r} \mathbf{n} \left(\frac{136}{15} \frac{Gm}{r} + \frac{24}{5} v^2 \right) - \mathbf{v} \left(\frac{24}{5} \frac{Gm}{r} + \frac{8}{5} v^2 \right) \right\}$$

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In [2601.06743] we have extended this result up to **4.5PN** in harmonic coordinates:

$$\mathbf{A}^{RR} = \mathbf{A}_{2.5PN}^{RR} + \mathbf{A}_{3.5PN}^{RR} + \mathbf{A}_{4PN, \text{ tail}}^{RR} + \mathbf{A}_{4.5PN, \text{ loc}}^{RR} + \mathbf{A}_{4.5PN, \text{ recoil}}^{RR}$$

local (instantaneous) vs. non-local in time (depends on the whole history)

Conservative quasi-Keplerian parametrization

Quasi-elliptic parametrization for the 2PN conservative dynamics [Damour & Deruelle 1985; Damour & Schäfer 1988]:

$$r = a_r (1 - e_r \cos u)$$

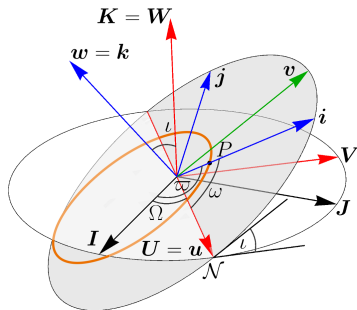
$$\ell = u - e_t \sin u + \underbrace{f_t \sin V_\phi + g_t (V_\phi - u)}_{2\text{PN}}$$

$$\phi = \omega + K[V_\phi + \underbrace{f_\phi \sin 2V_\phi + g_\phi \sin 3V_\phi}_{2\text{PN}}]$$

$u \rightarrow$ eccentric anomaly

$$V_\phi = u + 2 \arctan \left[\frac{\beta_\phi \sin u}{1 - \beta_\phi \cos u} \right] \rightarrow \text{true anomaly}$$

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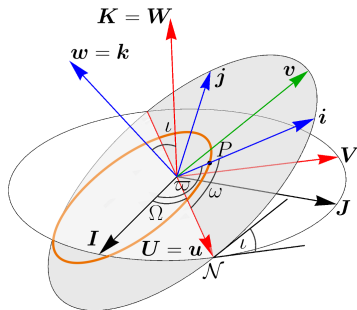
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- ▶ All coefficients are functions of the conserved **energy** E and **angular momentum** J (equivalently of a semi-major axis a and an eccentricity e)
- ▶ They are gauge dependent, except the **radial frequency** $n = 2\pi/P$ and the **orbital precession** $\Delta\phi = 2\pi K$



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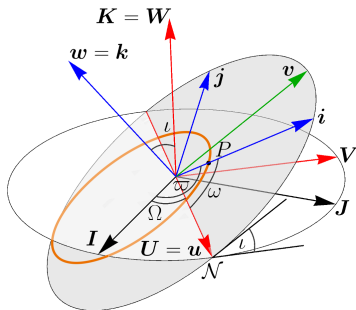
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Six constants of motion

$$c_\alpha = \left(\underbrace{a, e}_{\text{shape}}, \underbrace{\bar{e}, \varpi}_{\text{origin choice}}, \underbrace{l, \Omega}_{\text{orbital plane}} \right)$$

Dissipative quasi-Keplerian parametrization

To include dissipation, we keep the **same functional form** for the position and velocity (the phase space equations including $\dot{r}$ and $\dot{\phi}$) as in the conservative case, but promote the constants to functions of time [\[Damour , Gopakumar & Iyer 2004\]](#)

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The constants obey **Lagrange's equations**, which are inverted with Poisson brackets:

$$\sum_\beta [c_\alpha, c_\beta] \frac{dc_\beta}{dt} = \frac{\partial \mathbf{x}}{\partial c_\alpha} \cdot \mathbf{A}^{\text{RR}} \Leftrightarrow \frac{dc_\alpha}{dt} = \sum_\beta \{c_\alpha, c_\beta\} \frac{\partial \mathbf{x}}{\partial c_\beta} \cdot \mathbf{A}^{\text{RR}} \equiv \mathcal{A}_\alpha^{\text{RR}}[t, c_\beta(t)]$$

$$[c_\alpha, c_\beta] \equiv \frac{\partial \mathbf{x}}{\partial c_\alpha} \cdot \frac{\partial \mathbf{v}}{\partial c_\beta} - \frac{\partial \mathbf{x}}{\partial c_\beta} \cdot \frac{\partial \mathbf{v}}{\partial c_\alpha}, \quad \{c_\alpha, c_\beta\} \equiv \frac{\partial c_\alpha}{\partial \mathbf{v}} \cdot \frac{\partial c_\beta}{\partial \mathbf{x}} - \frac{\partial c_\beta}{\partial \mathbf{v}} \cdot \frac{\partial c_\alpha}{\partial \mathbf{x}}$$

We recover $\frac{d\iota}{dt} = \frac{d\Omega}{dt} = 0$, i.e. the orbital plane stays fixed for non-spinning bodies.

Parameters Secular evolution and oscillating part

Radiation reaction is treated as a **perturbation** of the conservative orbit:

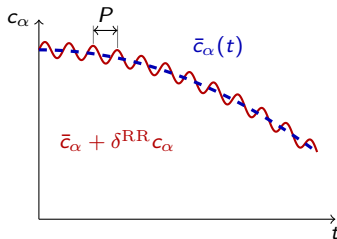
$$c_\alpha(t) = \underbrace{\bar{c}_\alpha(t)}_{\text{conservative part up to 2PN}} + \underbrace{\delta^{\text{RR}} c_\alpha(t)}_{\text{RR perturbation 2.5PN to 4.5PN}}$$

- **Slow secular evolution** (adiabatic) :

$$\frac{d\bar{c}_\alpha}{dt} = \bar{\mathcal{A}}_\alpha^{\text{RR}}[\bar{c}_\beta] \equiv \frac{1}{P} \int_0^P dt \mathcal{A}_\alpha^{\text{RR}}[t, \bar{c}_\beta]$$

- **Fast oscillations** at the orbital period, with zero average:

$$\frac{d\delta^{\text{RR}} c_\alpha}{dt} = \mathcal{A}_\alpha^{\text{RR}}[t, \bar{c}_\beta] - \bar{\mathcal{A}}_\alpha^{\text{RR}}[\bar{c}_\beta]$$



- Neglected terms are quadratic, i.e. $\mathcal{O}(1/c^{10})$: beyond 4.5PN
- Closed-form solutions, built from $\chi \equiv 1 - e \cos u$ with

$$\frac{1}{\chi^n}, \quad \frac{\sin u}{\chi^n}, \quad \frac{V_\phi - u}{\chi^n}, \quad \frac{(V_\phi - u)^2}{\chi^n}, \quad \frac{\ln \chi}{\chi^n}, \quad \frac{\text{Li}_2(\beta_\phi e^{iu})}{\chi^n}$$

Quasi-hyperbolic orbits and analytic continuation

Unbound orbits: hyperbolic quasi-Keplerian parametrization

$$r = \hat{a}_r (\hat{e}_r \cosh v - 1), \quad \hat{\ell} = \hat{e}_t \sinh v - v + \hat{f}_t \sin \hat{V}_\phi + \hat{g}_t \hat{V}_\phi$$

$$\phi = \hat{c}_\phi + \hat{K} [\hat{V}_\phi + \hat{f}_\phi \sin 2\hat{V}_\phi + \hat{g}_\phi \sin 3\hat{V}_\phi]$$

For the **local parts** both representations are equivalent up to **analytic continuation** with $u \rightarrow iv$

$$\begin{aligned} \hat{n} &= i n (1 + g_t), & \hat{K} &= K, & \hat{a}_r &= -a_r \\ \hat{e}_r &= e_r, & \hat{e}_t &= e_t (1 + g_t), & \hat{e}_\phi &= e_\phi \\ \hat{f}_t &= i f_t, & \hat{g}_t &= i g_t, & \hat{f}_\phi &= f_\phi, & \hat{g}_\phi &= g_\phi \end{aligned}$$

In the hyperbolic case the evolution equations follow from the elliptic ones:

$$\frac{d\hat{a}}{dt} = -\frac{da}{dt}, \quad \frac{d\hat{e}}{dt} = \frac{de}{dt}, \quad \frac{d\hat{c}_\ell}{dt} = i \left[(1 + g_t) \frac{dc_\ell}{dt} + \ell \frac{dg_t}{dt} \right], \quad \frac{d\hat{c}_\phi}{dt} = \frac{d\omega}{dt} - \ell \frac{dK}{dt}$$

We have obtained the **radiation reaction contribution to the scattering angle**

$\chi^{\text{RR}} = [\delta^{\text{RR}}\phi]_{-\infty}^{+\infty}$ up to **4.5PN** and found complete agreement with the linear-response formula [Bini, Damour & Geralico 2023]

Non-local contributions up to 4.5PN : tail and recoil

Two contributions to $\mathbf{A}^{\text{RR}}$ depend on the **past history** of the binary, so inserting them in Lagrange's equations requires integrating over the past orbit.

4PN tail

GW tails acting back on the binary:

$$A_{\text{tail}}^i = -\frac{4G^2M}{5c^8} x^j \int_0^{+\infty} d\tau \ln\left(\frac{\tau}{\tau_0}\right) \left[I_{ij}^{(7)}(t - \tau) + I_{ij}^{(7)}(t + \tau) \right]$$

Contribution computed as a Fourier series in the elliptic case and a Fourier transform in the hyperbolic case

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In the center-of-mass frame, the linear momentum radiated by the binary induces a non-local term:

$$\mathbf{A}_{\text{recoil}} = \frac{G(m_1 - m_2)}{r^2 m c^2} \left[2\mathbf{n} (\mathbf{v} \cdot \mathbf{\Pi}) + \mathbf{v} (\mathbf{n} \cdot \mathbf{\Pi}) \right], \quad \mathbf{\Pi}(t) = \int_{-\infty}^t dt' \mathcal{F}_{\mathbf{P}}(t')$$

$\mathbf{\Pi}$ is built from the flux $\mathcal{F}_{\mathbf{P}} = \mathcal{O}(1/c^7)$, hence a 4.5PN contribution.

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The analytic continuation **breaks down** for the **non-local contributions**.

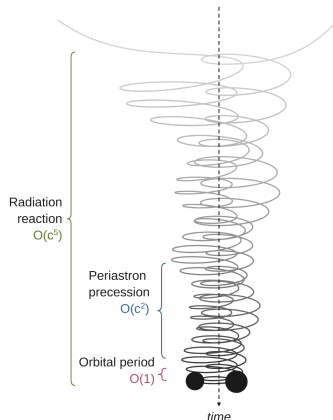
PN order reduction due to the recoil in the elliptic case

We need to integrate the flux over the past in the quasi-elliptic parametrization

$$\mathbf{\Pi} \simeq \sum_{p \in \mathbb{Z}, q = \pm 1} \int_0^{\infty} dV_{\phi} f_{p,q}(a, e) e^{i(p+q\mathbf{k})V_{\phi}}$$

Relativistic precession: $\mathbf{k} = \frac{3Gm}{ac^2(1-e^2)} + \dots$

- ▶ $p \neq 0$: **orbital** frequency
 $\Rightarrow \mathbf{\Pi} = \mathcal{O}(1/c^7)$, as expected
- ▶ $p = 0$: **precession** frequency
 $\Rightarrow$ integration over a time $\Delta t \propto k$
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However, the computation remains valid since precession time $\ll$ RR time.
The 2.5PN contribution vanishes in the circular limit.

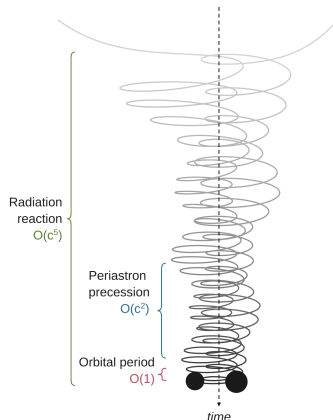
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The 4.5PN non-local acceleration contributes already at **3.5PN** order!

Summary and prospects

Using the 4.5PN radiation reaction in harmonic coordinates and Lagrange's method of variation of constants, we obtained the RR corrections (secular + oscillatory) to **quasi-elliptic** and **quasi-hyperbolic** orbits at **4.5PN** order.

- ▶ The analytic continuation provides a **one-to-one correspondence** for the local contributions to the equations of motion for the parameters between elliptic and hyperbolic orbits.
- ▶ The analytic continuation **breaks down** for non local contributions.
- ▶ In the elliptic case the non-local terms in the acceleration enter the corrections to the quasi Keplerian parameters at **3.5PN order**.

Combined with the radiative multipole moments for eccentric orbits, this work allows for instance to compute the full eccentric waveform at the same PN order.

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Thank you for your attention !

Additional slides

Newtonian Lagrange brackets

For $c_\alpha = (a, e, \bar{e}, \varpi, \iota, \Omega)$, with $\varpi = \omega + \Omega$ and $\bar{e} = \varpi + c_\ell$:

$$\begin{aligned} [a, \bar{e}] &= -\frac{an}{2}, & [a, \varpi] &= \frac{an}{2} \left(1 - \sqrt{1 - e^2}\right), \\ [a, \Omega] &= \frac{an}{2} \sqrt{1 - e^2} (1 - \cos \iota), & [e, \varpi] &= \frac{ea^2 n}{\sqrt{1 - e^2}}, \\ [e, \Omega] &= -\frac{ea^2 n}{\sqrt{1 - e^2}} (1 - \cos \iota), & [\iota, \Omega] &= a^2 n \sqrt{1 - e^2} \sin \iota, \end{aligned}$$

all the other brackets being zero.

Newtonian-like semi-major axis and eccentricity, defined from the conserved quantities:

$$a \equiv -\frac{Gm}{2\tilde{E}}, \quad e \equiv \sqrt{1 + \frac{2\tilde{E}\tilde{J}^2}{G^2 m^2}}, \quad \text{with} \quad \tilde{E} \equiv \frac{E - mc^2}{m\nu}, \quad \tilde{J} \equiv \frac{J}{m\nu}$$

Recoil on elliptic orbits

Constant part (along the direction $\mathbf{l}$ of the periastron), starting at 2.5PN:

$$\mathbf{\Pi}_{\text{const}} = -\frac{G^3 m^4 \delta \nu^2}{a_r^3 c^5 (1 - e_r^2)^3} \frac{e_r}{90} (312 + 456 e_r^2 + 37 e_r^4) \mathbf{l} + \mathcal{O}\left(\frac{1}{c^7}\right)$$

Circular-orbit limit ($e_r \rightarrow 0$), directly from $\dot{\mathbf{n}} = \dot{\phi} \boldsymbol{\lambda}$ and $\dot{\boldsymbol{\lambda}} = -\dot{\phi} \mathbf{n}$:

$$\mathbf{\Pi}_{\text{circ}} = -\frac{464}{105} \frac{G^4 m^5 \delta \nu^2}{a_r^4 c^7} \mathbf{n} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

Non-local acceleration up to 5.5PN, with $(n\Pi) = \mathbf{n} \cdot \mathbf{\Pi}$, $(\nu\Pi) = \mathbf{\nu} \cdot \mathbf{\Pi}$, $(n\nu) = \mathbf{n} \cdot \mathbf{\nu}$:

$$\begin{aligned} A_{\text{recoil}}^i = & \frac{G\delta}{r^2 c^2} \left[2n^i (\nu\Pi) + \nu^i (n\Pi) \right] \\ & + \frac{G\delta}{r^2 c^4} \left\{ n^i \left[\frac{3\nu}{2} (n\nu)(n\Pi)(3\nu^2 - 5(n\nu)^2) + (\nu\Pi) \left(3\nu(n\nu)^2 - 4\nu\nu^2 - \frac{Gm}{r}(8 - \nu) \right) \right] \right. \\ & \left. + \nu^i \left[(n\Pi) \left(\frac{9\nu}{2} (n\nu)^2 + (2 - \nu) \left(\frac{\nu^2}{2} - \frac{2Gm}{r} \right) \right) - (4 + \nu)(n\nu)(\nu\Pi) \right] \right\} \end{aligned}$$

Tail on elliptic orbits

Fourier series of the Newtonian quadrupole using the Hansen coefficients (ψ : true anomaly):

$$\left(\frac{r}{a}\right)^n e^{im\psi} = \sum_{p=-\infty}^{+\infty} X_p^{n,m}(e) e^{ip\ell}$$

where the $X_p^{n,m}(e)$ are finite sums of Bessel functions $J_q(pe)$. The tail integral is then computed mode by mode, and the arbitrary scale τ_0 drops out:

$$\int_0^{+\infty} d\tau \ln\left(\frac{\tau}{\tau_0}\right) \left(e^{-ipn\tau} + e^{ipn\tau}\right) = -\frac{\pi}{pn} \text{sign}(p)$$

$$\int_0^{+\infty} d\tau \ln\left(\frac{\tau}{\tau_0}\right) \left[I_{ij}^{(7)}(t-\tau) + I_{ij}^{(7)}(t+\tau) \right] = i\pi \sum_p (pn)^6 \text{sign}(p) I_{ij}^p e^{ip\ell(t)},$$

$$\text{with } I_{ij}(t) = \sum_p I_{ij}^p e^{ip\ell(t)}$$

- ▶ Only a and e have a secular tail contribution: $\frac{d\bar{t}}{dt} = \frac{d\bar{\varepsilon}}{dt} = \frac{d\bar{\varpi}}{dt} = \frac{d\bar{\Omega}}{dt} = 0$
- ▶ Small-eccentricity expansions are straightforward, since $J_q(pe) = \mathcal{O}(e^q)$

Computation of the non-local 4.5PN contributions

To be consistent at 4.5PN in the evolution of the orbital elements, we need to extend beyond the leading order

- ▶ Π at **next-to-leading order**: linear-momentum flux at 4.5PN and k at 2PN
- ▶ the **5.5PN non-local acceleration** in the center-of-mass frame

$$\mathbf{A}_{\text{recoil}} = \frac{G(m_1 - m_2)}{r^2 mc^2} \left[2\mathbf{n}(\mathbf{v} \cdot \Pi) + \mathbf{v}(\mathbf{n} \cdot \Pi) \right] + \underbrace{\frac{G(m_1 - m_2)}{r^2 mc^4} \left[\mathbf{n}(\dots) + \mathbf{v}(\dots) \right]}_{\text{new: 5.5PN}}$$

We have checked that the secular evolution from Lagrange's method matches the flux-balance equations

$$\frac{d\bar{c}_\alpha}{dt} = -\frac{\partial c_\alpha}{\partial E} \langle \mathcal{F}_E \rangle - \frac{\partial c_\alpha}{\partial J} \langle \mathcal{F}_J \rangle$$

only if the computation is fully consistent at 4.5PN order.

The analytic continuation **breaks down** for the non-local contributions.