

Fluxes near the lightning using WKB

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Context

Consider a test particle of mass m_2 orbiting a black hole of mass $m_1 \gg m_2$. This particle follows a geodesic.

We assume that the motion is planar and choose units where $G = c = M = 1$. The symmetric mass ratio is $\nu = m_1 m_2 / (m_1 + m_2)^2$.

Using this trajectory and the Einstein equations, we obtain the gravitational waveform at leading order in the mass ratio ν :

$$h_+ - i h_\times = \frac{\nu}{r} \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} h_{\ell m}(t) {}_{-2}Y_{\ell m}(\theta, \phi) + \mathcal{O}(\nu^2)$$

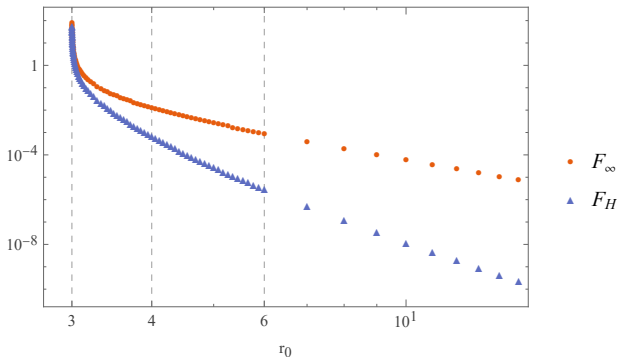
The flux of energy can then be obtained through

$$\mathcal{F} = \nu^2 \sum_{\ell=0}^{\infty} \sum_{m=1}^{\ell} \mathcal{F}_{\ell m} + \mathcal{O}(\nu^3) \quad \text{where} \quad \mathcal{F}_{\ell m} = \frac{1}{8\pi} |\dot{h}_{\ell m}|^2$$

where we note that there are no $m = 0$ contributions at this order.

Circular orbits: stable and unstable

Restrict to **circular orbits** around a Schwarzschild black hole.

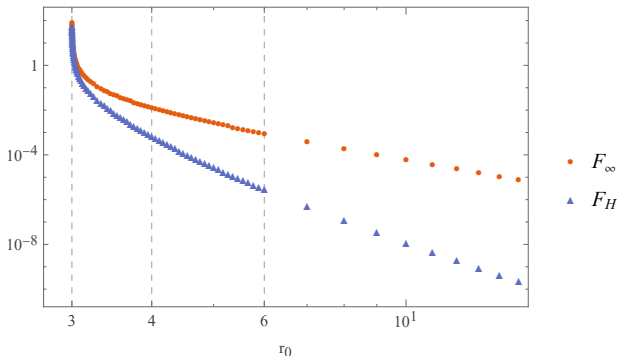


The numerical fluxes of energy through infinity and the horizon are smooth for $r_0 > 3$, but there are three very different regions:

- $r_0 > 6$: stable circular orbits
- $4 < r_0 < 6$: bound unstable circular orbits
- $3 < r_0 < 4$: unbound unstable circular orbits

Circular orbits: stable and unstable

Restrict to **circular orbits** orbits around a Schwarzschild black hole.



For $r_0 \gg 6$, post-Newtonian theory gives us a very good approximation.

If we can have an analytical approximation for the $r_0 \rightarrow 3^+$ divergence ...
can we have an analytical approximant (resummation) for the whole flux function?

Regge-Wheeler equation

The homogeneous radial Regge-Wheeler (RRW) equation governs linear metric perturbations¹ around Schwarzschild:

$$\left[f^2(r) \frac{d^2}{dr^2} + f(r) f'(r) \frac{d}{dr} + \left(\omega^2 - V(r) \right) \right] X_{\ell m}(r; \omega) = 0$$

where the potential is given by $V(r) = f(r) \left(\frac{\ell(\ell+1)}{r^2} - \frac{6}{r^3} \right)$.

Standard basis of “in” and “up” solutions defined by normalization:

$$X_{\ell m}^{\text{in}}(r; \omega) \sim \begin{cases} e^{-i\omega r^*} & \text{as } r^* \rightarrow -\infty \\ \frac{1}{\mathcal{T}_{\ell m}} e^{-i\omega r^*} - \frac{\mathcal{T}_{\ell m}^*}{\mathcal{T}_{\ell m}^*} e^{+i\omega r^*} & \text{as } r^* \rightarrow +\infty \end{cases}$$
$$X_{\ell m}^{\text{up}}(r; \omega) \sim \begin{cases} \frac{\mathcal{R}_{\ell m}}{\mathcal{T}_{\ell m}} e^{-i\omega r^*} + \frac{1}{\mathcal{T}_{\ell m}} e^{+i\omega r^*} & \text{as } r^* \rightarrow -\infty \\ e^{+i\omega r^*} & \text{as } r^* \rightarrow +\infty \end{cases}$$

where $r^* = r + 2 \ln(r/2 - 1)$.

¹Using Chandrasekhar transformation, RW works for both axial and polar cases

Fluxes

The fluxes generated by a circular orbit at radius $r_0 = \omega_0^{-2/3} = 3$ is given explicitly in terms of the standard solutions to the RW equation:

$$\mathcal{F}_{\ell m}^{\infty/\mathcal{H}} = \frac{\nu^2}{2\pi(m\omega_0)^2} |\chi_{\ell m}|^2 |\mathcal{T}_{\ell m}|^2 \left\{ |\kappa_{\ell m}|^2 \mathcal{X}_{\ell m}^{\text{in/up}} + |\lambda_{\ell m}|^2 \mathcal{Y}_{\ell m}^{\text{in/up}} + \kappa_{\ell m}^* \lambda_{\ell m} \mathcal{Z}_{\ell m}^{\text{in/up}} + \kappa_{\ell m} \lambda_{\ell m}^* (\mathcal{Z}_{\ell m}^{\text{in/up}})^* \right\}$$

where

$$\begin{aligned} \chi_{\ell m} &= -\frac{2^6(m\omega_0)^2(1-2im\omega_0)(1-4im\omega_0)}{(\ell-1)\ell(\ell+1)(\ell+2)-12im\omega_0} \\ \mathcal{X}_{\ell m}^{\text{in/up}}(r_0) &= |X_{\ell m}^{\text{in/up}}(r_0; m\omega_0)|^2 \\ \mathcal{Y}_{\ell m}^{\text{in/up}}(r_0) &= \left| \frac{dX_{\ell m}^{\text{in/up}}}{dr^*}(r_0; m\omega_0) \right|^2 \\ \mathcal{Z}_{\ell m}^{\text{in/up}}(r_0) &= X_{\ell m}^{\text{in/up}*}(r_0; m\omega_0) \frac{dX_{\ell m}^{\text{in/up}}}{dr^*}(r_0; m\omega_0) \end{aligned}$$

Note that this is the same expression for ∞ and $\mathcal{H}$!

Analytical fluxes

Usually, the RRW equation is solve numerically and injected into the flux formula.

Here we solve the RRW equation *analytically* using WKB methods in the $k = \ell - m \gg 1$ limit. This is because the RRW equation is actually the stationary Schrödinger equation:

$$-\frac{\hbar}{2m_p} \frac{d^2\psi}{dx^2}(x) + [U(x) - E]\psi(x) = 0$$

with $\psi = X_{\ell m}$, $x = r^* = r + 2 \ln(r/2 - 1)$ and

$$\frac{\hbar}{2m_p} = \frac{1}{m^2} \quad U(x) = \frac{V(r(x))}{m^2} \quad E = \frac{\omega^2}{m^2} = \omega_0$$

The WKB limit is thus $m \gg 1$, i.e. $\hbar \ll 1$. We will be interested in this approximation near the light ring ($\delta = r_0 - 3 \ll 1$), because that is where high- m modes are non-negligeable.

See paper ...

Finally expression for the fluxes

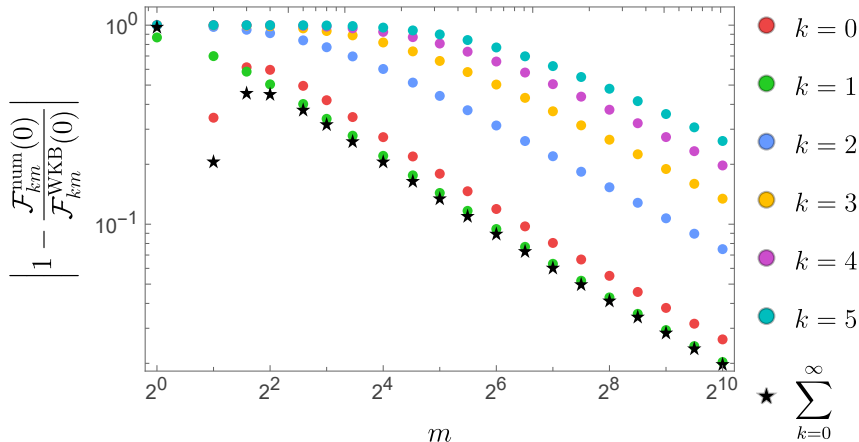
At fixed $k = \ell - m$ and for $m \gg 1$, we find the asymptotic statement:

$$\mathcal{F}_{m+k,m}^{\infty/\mathcal{H}} \sim \frac{1}{m\delta} \frac{2^{4\mathfrak{p}(k)-k} k!}{27\pi^{3/2} (\lfloor k/2 \rfloor!)^2} \left(1 + 2k + \frac{m\delta}{2}\right)^{2[1-\mathfrak{p}(k)]} \\ \times e^{-\frac{\pi}{4}(1+2k+m\delta)} \left| \Gamma\left(\frac{1 + 2\mathfrak{p}(k) + i(1 + 2k + m\delta)}{4}\right) \right|^2$$

where $\delta = r_0 - 3 \ll 1$; and $\mathfrak{p}(k) = k \bmod 2$.

We see that each (m, k) mode diverges like $1/\delta$, so we can compute the finite quantity $\delta \mathcal{F}_{m+k,m}^{\infty/\mathcal{H}}$ as $\delta \rightarrow 0$ (i.e., on the lightring) and compare it against a numerical prediction.

Numerical comparison of $\delta \mathcal{F}_{m+k,m}^\infty$ (through infinity)



Summing over m -modes

We know the analytical flux for every (m, k) mode. We now wish to study the $\delta \rightarrow 0$ behavior of

$$\delta \mathcal{F}_k^{\infty/\mathcal{H}} = \sum_{m=1}^{\infty} \delta \mathcal{F}_{mk}$$

This sum converges for $\delta > 0$, but diverges logarithmically for $\delta = 0$ (we are summing $\sum_m m^{-1}$!).

To analyze the $\delta \rightarrow 0$ behavior, we choose an arbitrary switch m^* :

- for $m < m^*$, sum the numerical fluxes at $\delta = 0$
- for $m > m^*$, use the WKB result; notice that it is a *harmonic sum*
- From [hal:inria-00074307], we know that the $\delta \rightarrow 0$ behavior follows from taking the Mellin transform: straightforward for harmonic sums
- Take $m^* \rightarrow \infty$ (it converges)

Then take the sum over k (converges exponentially!)

Final asymptotics

As $\delta = r_0 - 3 \rightarrow 0$, the fluxes behave like

$$\frac{\mathcal{F}^\infty}{\nu^2} \sim \frac{\alpha \ln(1/\delta) + \beta^\infty}{\delta} \quad \text{and} \quad \frac{\mathcal{F}^{\mathcal{H}}}{\nu^2} \sim \frac{\alpha \ln(1/\delta) + \beta^{\mathcal{H}}}{\delta}$$

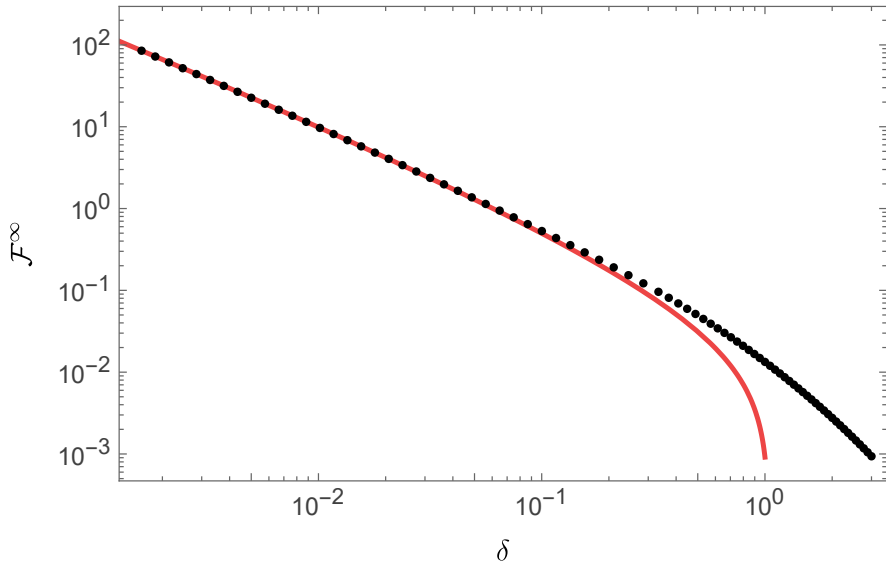
where

$$\alpha = \sum_{k=0}^{\infty} \frac{2^{4\mathfrak{p}(k)-k} (1+2k)^{2[1-\mathfrak{p}(k)]} k! e^{-\frac{\pi(1+2k)}{4}}}{27\pi^{3/2}(\lfloor k/2 \rfloor!)^2} \left| \Gamma\left(\frac{1+2\mathfrak{p}(k)+i(1+2k)}{4}\right) \right|^2$$

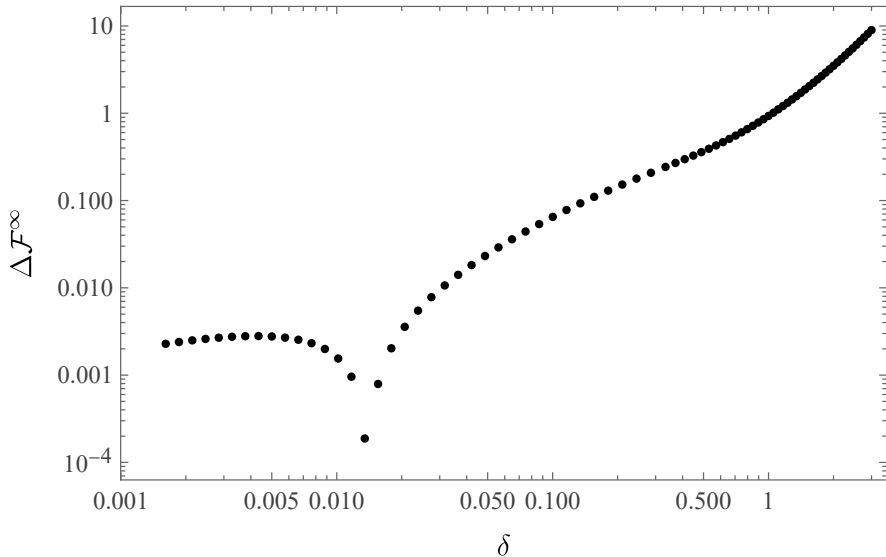
Numerically:

$$\begin{aligned}\alpha &\approx +0.021218 \\ \beta^\infty &\approx +0.000914 \\ \beta^{\mathcal{H}} &\approx -0.035283\end{aligned}$$

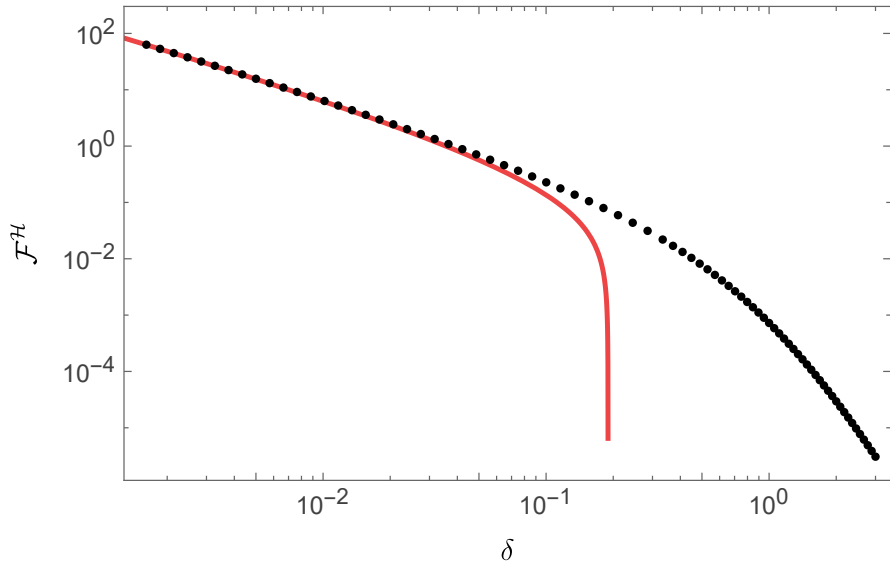
Final comparison : infinity flux (overlapped)



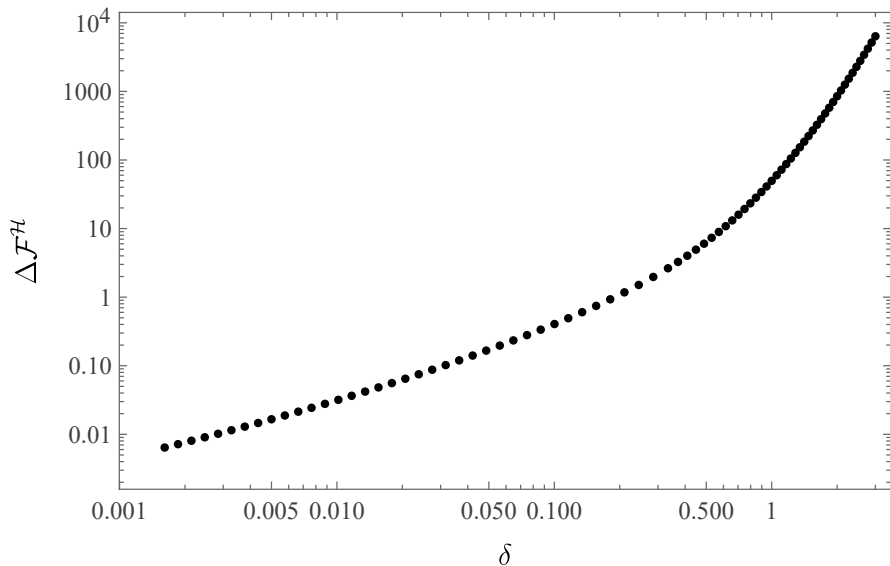
Final comparison : infinity flux (relative difference)



Final comparison : horizon flux (overlapped)



Final comparison : horizon flux (relative difference)

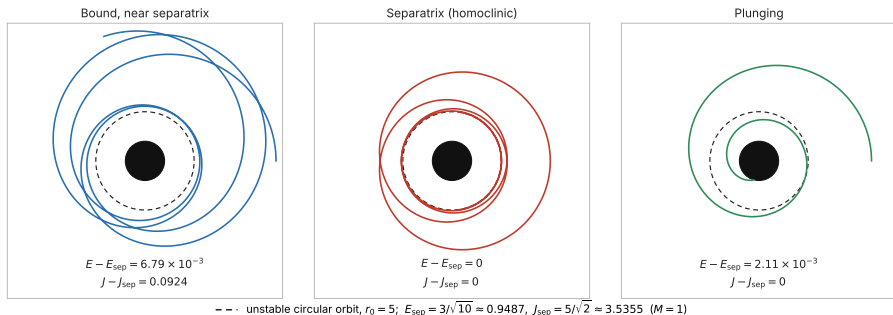


As $\delta = r_0 - 3 \rightarrow 0$, the fluxes behave like

$$\frac{\mathcal{F}^\infty}{\nu^2} \sim \frac{\alpha \ln(1/\delta) + \beta^\infty}{\delta} \quad \text{and} \quad \frac{\mathcal{F}^\mathcal{H}}{\nu^2} \sim \frac{\alpha \ln(1/\delta) + \beta^\mathcal{H}}{\delta}$$

Homoclinic orbits

Bound unstable circular orbits ($4 < r_0 < 6$) have same energy and angular momentum as marginally stable “homoclinic” orbits.



Heteroclinic orbits

Unbound unstable circular orbits ($3 < r_0 < 4$) have same energy and angular momentum as marginally stable “heteroclinic” orbits.

