Non Perturbative aspects of Supersymmetric Gauge Theories

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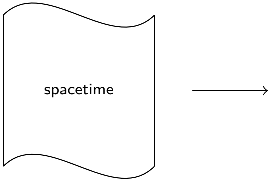
What is a QFT

- **Fields** : Scalars, Fermions, Gauge bosons **particles** are represented by **functions** from spacetime to some representation spaces of the symmetry groups of the theory.



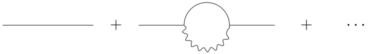
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \psi(x) \text{ for a weyl spinor}$$

$$a = h(x) \text{ for a spin } 0 \text{ boson} = \text{scalar}$$

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} = A(x) \text{ for a spin } 1 \text{ boson}$$

- Perturbative interaction with coupling g :

$$\langle \phi(x_1)\phi(x_2) \rangle = G_{\text{free}} + g^2 \cdot G_{\text{1loop}} + g^4 \cdots$$



Perturbative vs Non-perturbative contributions

- The perturbative expansion fails to see certain effects like

$$e^{-1/g^2}$$

- These effects can already be seen for the path integral Partition Function of gauge theories. These come from what we call **Instantons**, gauge configurations such that the gauge field strength is (anti)-self dual

$$F_{\mu\nu} = \pm\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} = \pm(\star F)_{\mu\nu}$$

What is a SUSY QFT ?

- Idea : Symmetry group become **supersymmetry** group, we add $\mathcal{N}$ transformation generators

$$\mathcal{Q}_I : \text{fermions} \longleftrightarrow \text{bosons}$$

In 4d, **the number of supercharges** $= 4\mathcal{N}$

- Our fields organize in representation called **multiplets**

$$(\phi, \psi_\alpha, \overline{\psi}^{\dot{\alpha}}, A_\mu) \text{ our building block } \mathcal{N} = 2 \text{ vector multiplet}$$

More symmetry = More structure on our theory

Why a SUSY QFT?

- Cancellation between fermionic and bosonic contributions
- Finite perturbative expansion

For example, 4d theories with **8 supercharges**

$$\mathcal{Z}^{\mathcal{N}=2} = \exp \mathcal{F} = \exp\left(\mathcal{F}_{\text{classical}} + \mathcal{F}_{1-\text{loop}} + \mathcal{F}_{\text{instantons}}\right)$$

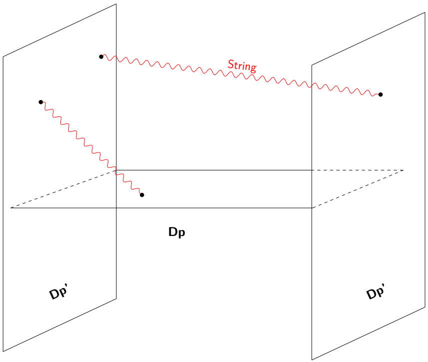
Even better, 16 supercharge theories in 4d have quasi trivial partition function

$$\mathcal{Z}^{\mathcal{N}=4} = \exp(\mathcal{F}_{\text{classical}})$$

[Seiberg,Witten,1994]

Stringy constructions tools to non perturbative results

- **D-branes** in string theories : D+1 dimensional objects living in 10d
- Restrict to the worldvolume of the branes $\xrightarrow[\text{Low energy}]{}$ SUSY gauge theories :



Example :

- Brane construction of a $SU(2)$ 8 supercharges SYM
- Vertical distance $\sim$ gauge coupling

[Very active field since the 80's]

Instanton counting via equivariant localization I

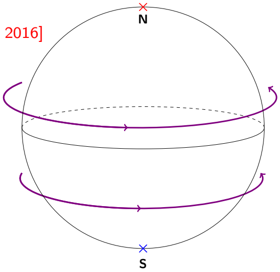
Topological Twist: the content of a multiplet is reorganized. [Witten,1991]

- new field χ that **forces the gauge field to be an instanton configuration**

$$\int [\mathcal{D}\chi][\mathcal{D}A]e^{-S} = \int [\mathcal{D}A]\delta(F + \star F = 0)e^{-S}$$

Localization : On any space $\mathscr{F}$ that admits a group action [Pestun et al., 2016]

$$\int_{\mathscr{F}} [\mathcal{D}A]e^{-S[A]} = \sum_{\substack{\text{fixed points} \\ p\in\mathscr{F}}} C(p) \times e^{-S[p]}$$



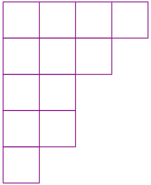
Instanton counting via equivariant localization II

Partition function on $\mathbb{R}^4$ [Nekrasov,2003]

$$\mathcal{Z} = \mathcal{Z}_{\text{pert}} \times \mathcal{Z}_{\text{inst}} \; ; \; \mathcal{Z}_{\text{inst}} = \sum_{\substack{\text{fixed points} \\ \lambda}} e^{-8\pi^2/g^2 \times k} Z_\lambda$$

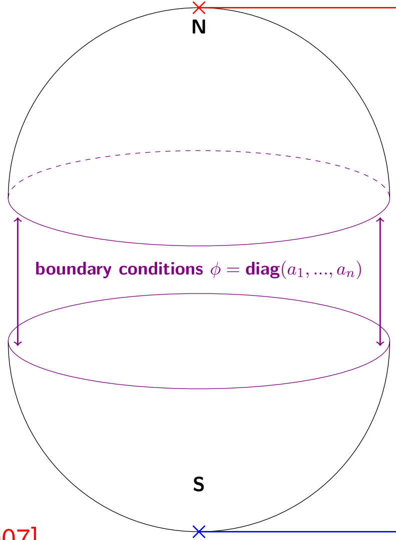
- k : number of instanton
- $\tau = 4i\pi/g^2$: gauge coupling
- $\mathfrak{q} = e^{2i\pi\tau}$: instanton counting parameter $\propto e^{-8\pi^2/g^2}$
- λ : A partition of the integer k

$\lambda =$



4
3
2
2
1
= k = 12

The setup : 4 dimensional sphere



$$Z(\mathbb{S}^4) = \int d^n a \; Z^N[a] Z^S[a]$$

[Pestun,2007]

The Partition function, anatomy I

$U(2)$ $\mathcal{N}=2^*$ theory : vector multiplet + adjoint hypermultiplet of mass m.

- On $\mathbb{R}^4_{\varepsilon_1,\varepsilon_2}$:

$$Z_{pert}[a] \sim e^{\frac{-8\pi^2}{g^2}a^2} \frac{H(a)}{H(a+m)} \,,$$

$$Z_\lambda[a] \sim \prod_{s\in\lambda} \wp(a+x(s)) - \wp(m)$$

With special functions composed of limits of Gamma functions and Weierstrass elliptic functions

$$H(x) := e^{(1+\gamma)x^2} G(1+ix)G(1-ix) = \exp\left[-\sum_{k\geq 1} \frac{\zeta_{2k+1}}{k+1} x^{2k+2}\right]$$

$$\wp(z) = \lim_{\tau\to\mathfrak{Im}\infty} \wp(z,\tau) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1)\times 2\zeta_{2k+2} z^{2k}$$

The Partition function, anatomy II

- On the 4-sphere $\mathbb{S}^4$, we integrate :

$$Z[\mathbb{S}^4] \sim \sum_{\lambda,\mu} \int_{\mathbb{R}^n} d^n a \quad \underbrace{Z^{pert}(a,m)}_{\text{independent of } \lambda,\mu} \quad \times \quad \underbrace{Z^N_{\text{inst}}[a,m,\lambda] Z^S_{\text{inst}}[a,m,\mu]}_{\text{depends on the partitions}}$$

$$\sim \sum_{\lambda,\mu} \int \frac{H(a)}{H(a+m)} \prod_{s\in\lambda} \wp(a+x(s)) - \wp(m) \quad da$$

- Very hard, almost impossible to compute analytically
- But $m \to 0$ limit is $\mathcal{N}=4$ limit which we saw to be trivial !

The first mass correction as an infinite series

We take only the first mass correction

$$Z_{\mathbb{S}^4}(m) = 1 + \cancel{m\partial_m Z_{\mathbb{S}^4}} + \frac{m^2}{2}\partial_m^2 Z_{\mathbb{S}^4} + \cdots \tag{1}$$

(Related to some observable in $\mathcal{N} = 4$ SYM)
We can actually compute explicitely in function of the gauge coupling g

$$\partial_m^2 Z_{\mathbb{S}^4} \sim \text{perturbative} + \text{instantons}$$

$$\downarrow$$

$$\sim \sum_{k\geq 1} \frac{(2k)!}{k!}\zeta_{2k+1}\left(\frac{g}{4\pi}\right)^{2k} = \mathcal{G}$$

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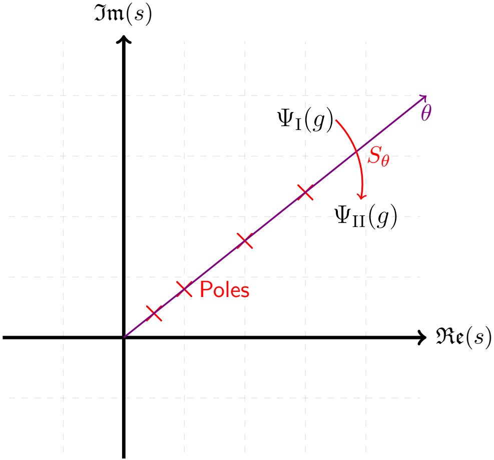
$\Rightarrow$ Divergence !

Resurgence analysis : new perturbative contributions



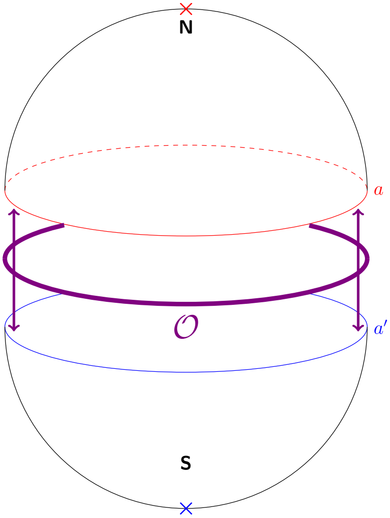
- Consider a new object, the borel transform $\mathcal{B}[\mathcal{G}](s)$
- Different regions on this s-plane $=$ finite solutions $\Psi(g)$
- Different states are counted, discontinuity between solutions
- Framework called **Resurgence**. Modern point of view for new non perturbative contributions.

$$\Rightarrow \text{Discontinuity} \sim \sum_{n\geq 1} \exp(-\frac{4\pi^2}{g^2} n^2) = \sum_{n\geq 1} \left(\mathfrak{q}^{1/2}\right)^{n^2}$$

Generalizations

- We have made sense of new kinds of non-perturbative states counted by the partition function, from the perturbative 1-loop part
- This analysis can be performed in the same way with instantons
- Work in progress : Generalization to higher rank, higher mass correction.

Extension :Gluing conditions, operator insertions

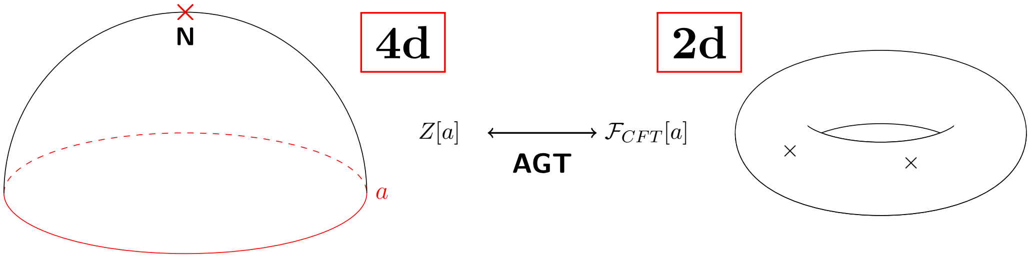


Operator insertion $\mathcal{O}$ = boundary conditions

Gluing $\rightarrow \int da \int da' \; Z^N[a]\mathcal{O}[a,a']Z^S[a']$

[Hosomichi,Lee,Park,2010]

The AGT correspondance



- Boundary conditions on the 4d side = " Choice of basis" of the 2d theory
- Different techniques should lead to the same result

[Alday,Gaiotto,Tachikawa,2009]

Conclusion

- Framework to get exact quantum results. Better understanding of QFT in general.
- Overarching goal to apply to more realistic models
- Can be connected to many modern areas of research.