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Knot enumeration and crossing loop model coupled to gravity

Paul Zinn-Justin

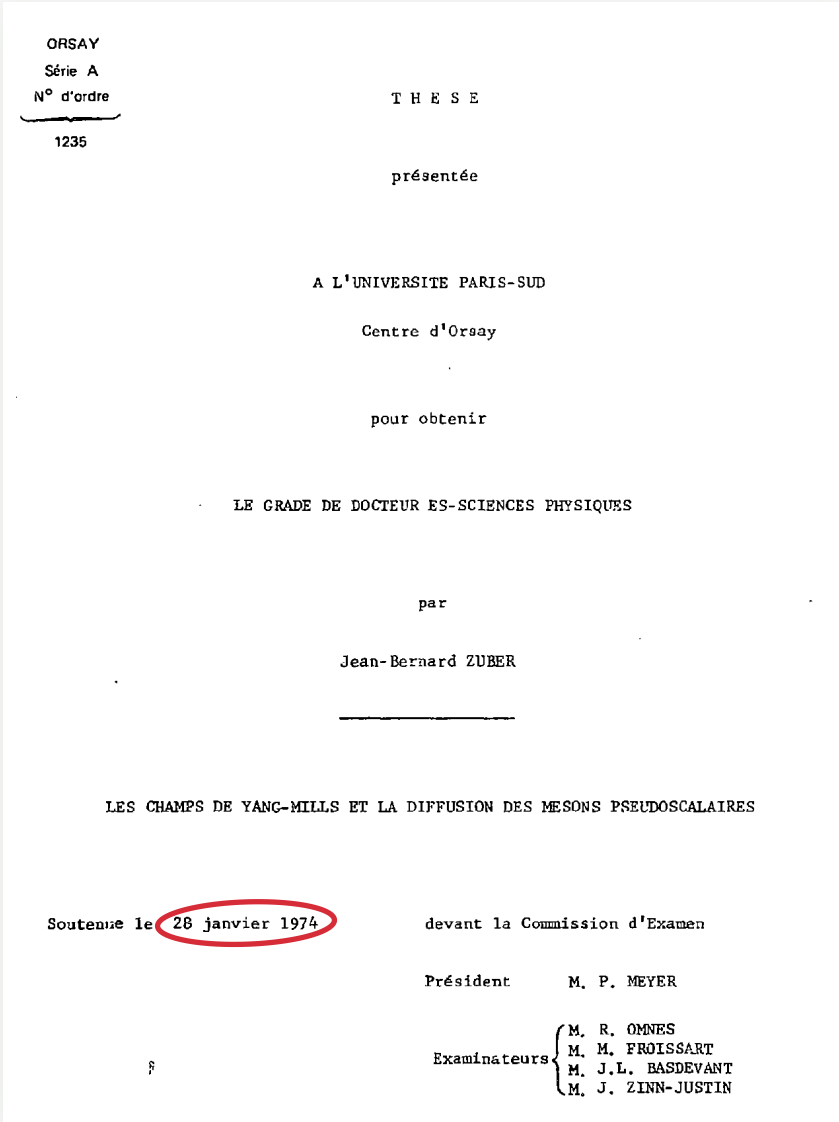
The University of Melbourne

For Jean-Bernard,
with many happy returns.



Our first interaction

Jean-Bernard's PhD defence



28 January 1974

My arrival, five days later

2 February 1974

Le deux février mil neuf cent soixante quatorze à deux heures, est né, 19 rue du Maréchal Gallieni, PAUL,GEORGES, du sexe masculin de Jean,.. Hermann,Paul,Georges, ZINN-JUSTIN, né à Berlin(Allemagne) le dix juillet mil neuf cent quarante trois, Physicien au'C.E.A.' et de Nicole, ... Suzanne, HALLIER, née à Saint-Denis-Hors(Indre et Loire) le vingt neuf juin mil neuf cent quarante et un, Professeur d'Université son épouse domiciliés à Chatenay-Malabry(Hauts de seine) 12 Domaine de Seignelay..

A close call!

THE EARLY YEARS

Together in Les Houches



1975



1982

A question we have been knotting together since 1999

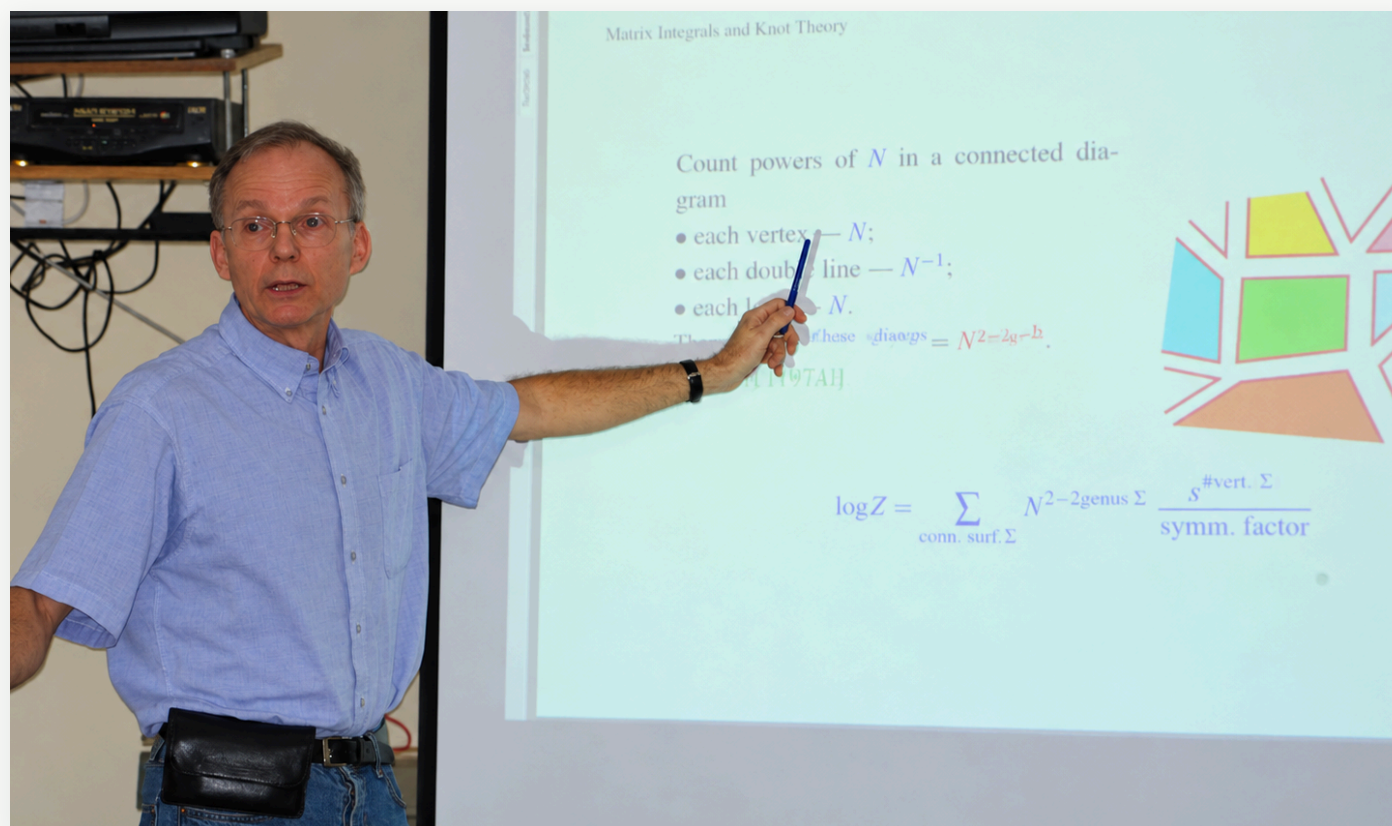
1999 [Matrix integrals and the counting of tangles and links](#)

P. Zinn-Justin & J.-B. Zuber

2010 [Knot theory and matrix integrals](#)

P. Zinn-Justin & J.-B. Zuber

Along the way: [coloured tangles \(2000\)](#), with JBZ; [plane-curve asymptotics \(2004\)](#), with G. Schaeffer.



February 2005

Moscow

*Combinatorial methods in
Physics and Knot Theory*

Today: logarithmic corrections, an exactly solvable check, and statistics of loop lengths.

BACKGROUND

From knot diagrams to planar maps



An alternating knot diagram



Forget over / under: its shadow

Knot and link types

Diagrams must be identified under Reidemeister moves: a difficult counting problem.

Prime alternating links

For reduced alternating diagrams, flypes suffice—but counting modulo flypes still requires extra combinatorics.

Our simplification

Forget over/under crossings and all move equivalences. Count all connected four-valent planar maps.

Working hypothesis: these simplifications preserve the universality class.

A crossing becomes a degree-four vertex; opposite half-edges trace the loops.

From knot diagrams to planar maps

First count all connected, rooted, four-valent maps on the sphere.

Exact enumeration

$$A_n = \frac{2 \cdot 3^n (2n)!}{n! (n+2)!} \sim \frac{2}{\sqrt{\pi}} 12^n n^{-5/2}.$$

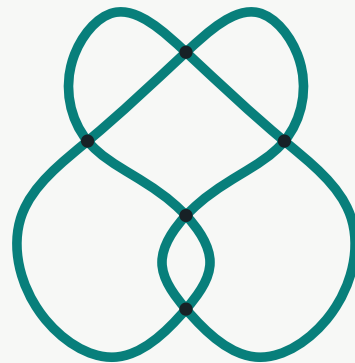
n crossings · $2n$ edges

Root: one distinguished oriented edge.

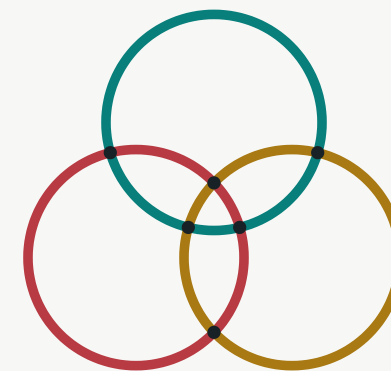
What changes when we prescribe the number of components?

The one-component sector is much harder. In general, introduce a fugacity per component.

The exponential growth depends on the ensemble; the power-law exponent is the proposed universal information.



5_2 knot shadow: **1 component**



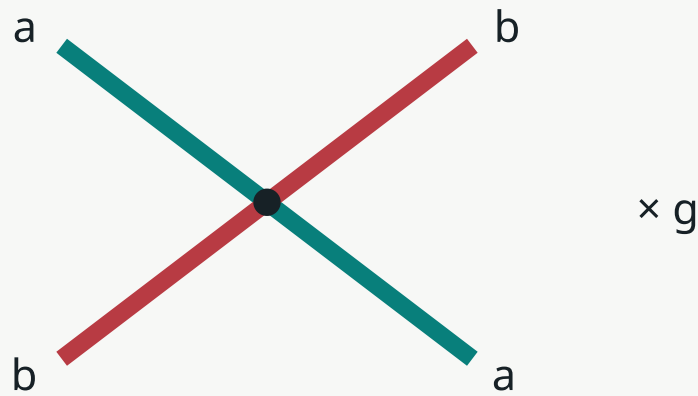
Borromean shadow: **3 components**

Trace each loop through crossings; both planar maps are connected.

The matrix integral for crossing loops

For integer $q > 0$, use q Hermitian matrices of size N :

$$\mathcal{Z}_N(q, g) = \int \prod_{a=1}^q dM_a \exp N \operatorname{tr} \left[-\frac{1}{2} \sum_a M_a^2 + \frac{g}{4} \sum_{a,b} M_a M_b M_a M_b \right].$$



The cyclic order a, b, a, b makes a crossing.

Matrix indices make a ribbon graph.
A surface of genus h contributes N^{2-2h} .

Colour is conserved along a strand.
Each closed loop contributes q .

Large N selects the spherical topology.
At each order, the answer is a polynomial in q .

$$F_0(q, g) = \lim_{N \rightarrow \infty} N^{-2} \log \mathcal{Z}_N(q, g), \quad Z_n(q) = 4n [g^n] F_0(q, g).$$

Formal, Gaussian-normalized integral. Here we keep all maps: no prime/reduced restriction or flype quotient.

The leading power: matter plus a fluctuating sphere

$$Z_n(q) = \sum_{\substack{M \text{ rooted quartic map} \\ |V(M)|=n}} q^{\mathcal{N}(M)}, \quad \mathcal{N}(M) = \text{number of loops.}$$

Crossing-loop / Goldstone hypothesis

$$c(q) = q - 1.$$

The KPZ susceptibility is

$$\Gamma(q) = \frac{q - 2 - \sqrt{(2 - q)(26 - q)}}{12}.$$

The rooted counting power is

$$Z_n(q) \propto \tau(q)^n n^{\Gamma(q)-2}.$$

Rooting shifts the exponent by one relative to the automorphism-weighted unrooted count.

$\tau(q)$ depends on the microscopic ensemble.

The one-component sector is $[q] Z_n(q)$. Its conjectured power is $n^{-(13+\sqrt{13})/6}$.

The corresponding prime alternating-knot power is $n^{-(19+\sqrt{13})/6}$, with additional universality and quotient assumptions. Neither statement yet specifies the logarithms.

Crossings lead to a Goldstone theory

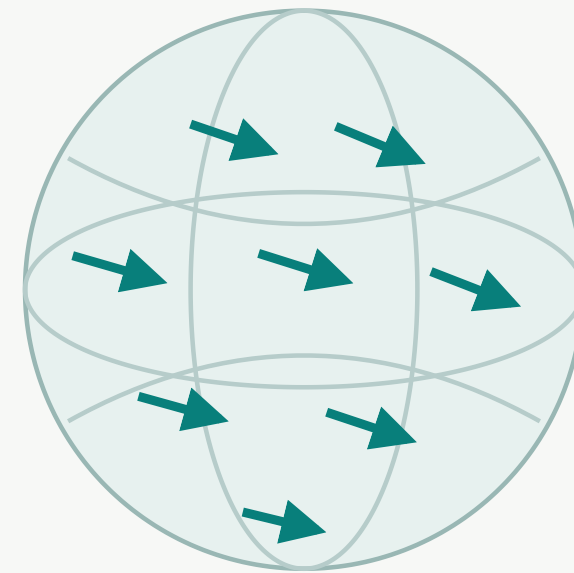
An $O(q)$ sigma model with $q - 1$ Goldstone modes.

$$S = \frac{K}{2} \int (\nabla \mathbf{S})^2, \quad \mathbf{S}^2 = 1.$$

K is the stiffness.

$q - 1$ is continued by replicas or supersymmetry.

This is the vortex-suppressed Goldstone phase, not the ordinary dense non-crossing loop theory.



The sigma model

Marginally irrelevant coupling → **slow running** → **logarithmic corrections**

RG equation (two loops, fixed background): for $q < 2$, with $t = \log(L/L_0)$, where L is the linear size:

$$\frac{dK}{dt} = \frac{2-q}{2\pi} \left[1 + \frac{1}{2\pi K} + O(K^{-2}) \right].$$

$$K(t) = \frac{2-q}{2\pi} t + \frac{1}{2\pi} \log t + O(1).$$

Gravity changes the logarithms, too

Three ingredients must run together.

The spin stiffness

The marginally irrelevant coupling produces $K \sim \log n$, with a two-loop $\log \log n$ correction.

The curvature coupling

The nonlinear dilaton changes the weight of a sphere.

The area operator

Holding physical area fixed is not holding the Liouville field fixed.

$$\log Z_n(q) = n \log \tau(q) + [\Gamma(q) - 2] \log n + p(q) \log \log n + O(1).$$
$$p(q) = p_{\text{collective}}(q) + p_{\text{dilaton}}(q) + p_{\text{area}}(q).$$

The Gaussian collective factor $K^{(q-1)/2}$ is only one part of the answer.

Continuum prediction on the generic Goldstone branch, $-2 < q < 2$, assuming regular microscopic matching and a nonzero leading amplitude. At $q = 0$, extract the one-component sector first.

Gravity changes the logarithms, too

Fixed spherical area · rooted partition function

$$Z_n(q) = C(q)\tau(q)^n n^{\Gamma(q)-2} (\log n)^{p(q)} \times \left[1 + \frac{h(q) \log \log n + c_{\text{match}}(q)}{\log n} + O\left(\frac{(\log \log n)^2}{(\log n)^2}\right) \right].$$

$$u(q) = \sqrt{\frac{26-q}{2-q}}, \quad p(q) = \frac{q-1}{4} [1 + u(q)],$$
$$h(q) = \frac{p(q)[u(q)-1]}{24} \left[4u(q) - 3(q-1) - \frac{3(5-q)}{u(q)} \right].$$

Universal predictions

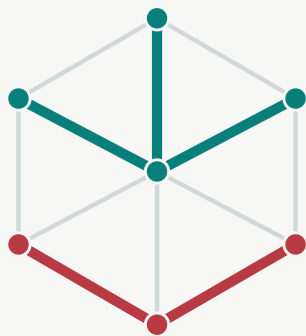
$\Gamma(q), p(q), h(q)$ within the Goldstone universality class.

Microscopic matching

$C(q), \tau(q), c_{\text{match}}(q)$ are non-universal.

Logarithmic origins are absorbed in c_{match} . The formula is not uniform at $q = 2$; a vanishing amplitude at $q = 0$ requires a separate marked/one-component normalization.

At $q = -1$, the two logarithms can be checked



A spanning forest: every vertex is kept, cycles are forbidden; here $k(F) = 2$.

On each graph, negative-fugacity spanning forests are exactly related to an $\text{OSp}(1|2)$ projective-spin model.

$$\sum_{F \text{ spanning}} t^{k(F)}, \quad -1 \leq t < 0, \quad q = 1 - 2 = -1, \quad c = -2.$$

t counts trees, not loops. The map sum is rooted and quartic.

Exact solution on rooted quartic maps $\rightarrow$ singularity expansion

$$Z_n^{\text{forest}}(t) = \frac{C_t \Lambda_t^n}{n^3 (\log n)^2} \left[1 - 2 \frac{\log \log n}{\log n} + \frac{c_t}{\log n} + O\left(\frac{(\log \log n)^2}{(\log n)^2}\right) \right].$$

Independent coupled-gravity prediction $\Gamma(-1) = -1, \quad p(-1) = h(-1) = -2.$ Both logarithms agree.

This model is *not* the crossing-loop model at $q = -1$; it is only a test of the QFT predictions.

$q = -2$ · exact fermionic matrix model; exceptional point.

$q = 1$ · pure gravity; $p(1) = h(1) = 0$.

$q = 2$ · exact six-vertex model; separate $c = 1$ endpoint.

At $q = 1$, differentiate instead of counting knots

At $q = 1$, all rooted maps have equal weight.

$$\partial_q \log Z_n(q)|_{q=1} = \mathbb{E}\mathcal{N}_n, \quad \kappa = \partial_q \log \tau(q)|_{q=1}.$$

The logarithms disappear from $Z_n(1)$, but not from its derivative:

$$\Gamma'(1) = \frac{3}{10}, \quad p'(1) = \frac{3}{2}, \quad h'(1) = \frac{22}{5}.$$

Crossing / Goldstone model (II)

$$\mathbb{E}\mathcal{N}_n = \kappa n + \frac{3}{10} \log n + \dots$$

Dense non-crossing model (I)

$$\text{Instead, } \Gamma'(1) = \frac{3\sqrt{3}}{4\pi} \simeq 0.4135.$$

Idea: easy to sample $q = 1$ planar maps, and e.g., distinguish two families that both have $c = 0$ at $q = 1$.

At $q = 1$, differentiate instead of counting knots

Continuum prediction under the stated matching assumptions

The mean number of loops, for n crossings:

$$\mathbb{E}\mathcal{N}_n = \kappa n + \frac{3}{10} \log n + \frac{3}{2} \log \log n + C \\ + \frac{22}{5} \frac{\log \log n}{\log n} + \frac{c}{\log n} + O\left(\frac{(\log \log n)^2}{(\log n)^2}\right).$$

Universal prediction

$3/10$, $3/2$, $22/5$, for this fixed-area observable.

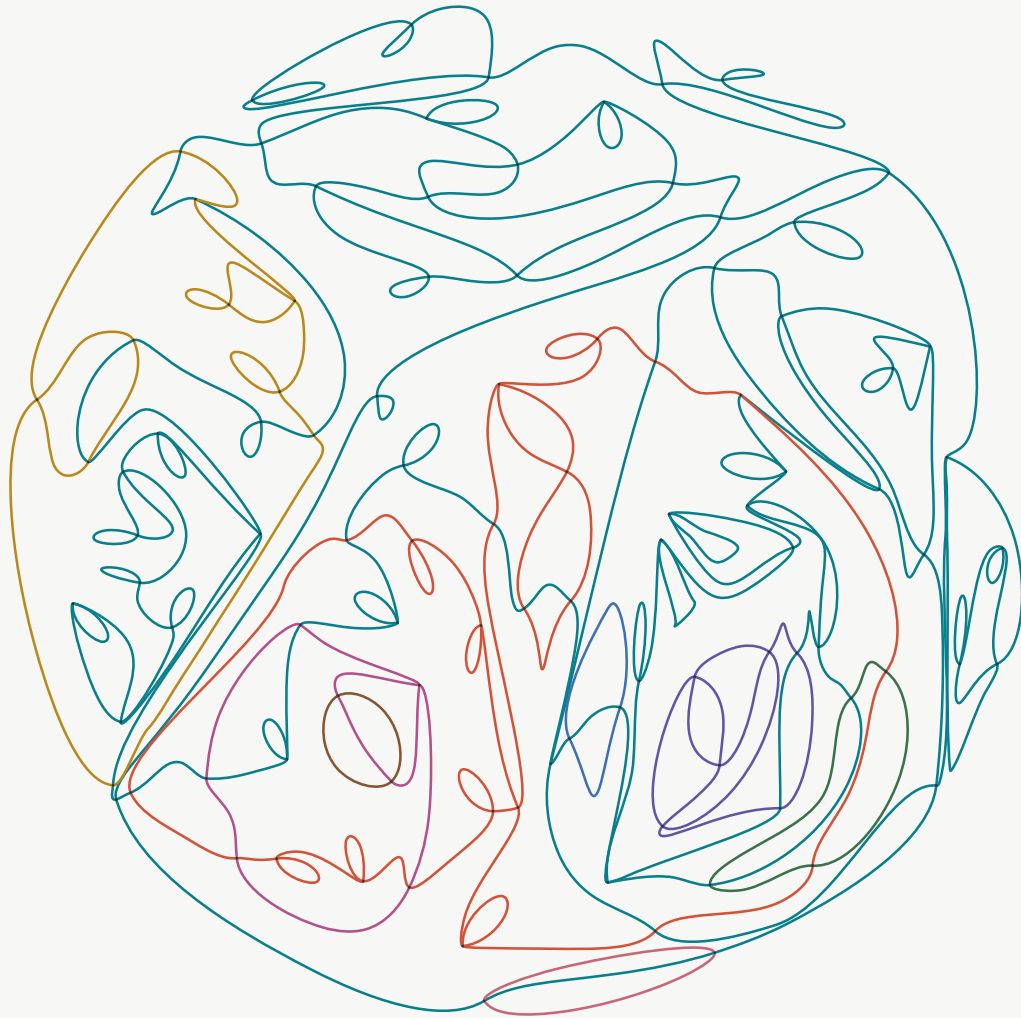
Microscopic information

κ , C , c : non-universal.

This is the original numerical test. Today, we go further: which loop lengths contribute?

Closed loops include the root loop. The original open-root convention differs by one component, hence only by a constant.

Many short loops—or a few very long ones?



100 crossings · 200 edges · 9 loops

Loop length 2ℓ is the number of edges in the loop.

$$a_{\ell,n} = \mathbb{E}[\text{number of loops of length } 2\ell].$$

Every edge lies on exactly one loop.

$$\sum_{\ell \geq 1} \ell a_{\ell,n} = n.$$

All loop lengths are even. We use the half-length ℓ throughout.

$$\text{Fixed } \ell: a_{\ell,n} = n\alpha_{\ell} + \beta_{\ell} + O(n^{-1}).$$

α_{ℓ} : extensive density · β_{ℓ} : finite-size intercept.

A length penalty is a mass

One ordinary colour and r auxiliary marked colours.

Each marked edge carries $e^{-\mu/2}$, so a 2ℓ -edge loop carries $e^{-\mu\ell}$.

$$Z_n(r, \mu) = \sum_{M: |V(M)|=n} \prod_{j \in \text{loops}(M)} (1 + r e^{-\mu\ell_j}),$$
$$\mathcal{M}_n(\mu) = \partial_r \log Z_n(r, \mu)|_{r=0} = \left\langle \sum_j e^{-\mu\ell_j} \right\rangle_{q=1} = \sum_{\ell \geq 1} a_{\ell,n} e^{-\mu\ell}.$$

Rooted maps M ; linear in r selects one marked loop. At $\mu = 0, q = 1 + r$.

Marked colours: transverse components.

$$\frac{K}{2} \gamma \mathbf{S}_{\perp}^2 = \frac{h}{2} \mathbf{S}_{\perp}^2, \quad h = K\gamma.$$

Bare mass squared: $\gamma \propto \mu$.

R denotes running couplings; $\gamma_R = h_R/K$.

Dress the operator; divide by K .

$$h_R \propto \frac{\mu n}{(\log n)^5}, \quad \gamma_R \propto \frac{\mu n}{(\log n)^6}.$$

Fixed background: powers one and two, respectively.

A length penalty is a mass

Recall: $a_{\ell,n} = n\alpha_\ell + \beta_\ell + O(n^{-1})$ as $n \rightarrow \infty$ at fixed ℓ .

$a_{\ell,n}$: mean number of 2ℓ -edge loops; α_ℓ : density; β_ℓ : finite-size intercept.

Tail predictions

Now take $\ell \rightarrow \infty$ in the fixed-length coefficients.

$$\alpha_\ell \sim \frac{A}{\ell^2 (\log \ell)^6}.$$

$$\beta_\ell = \frac{3}{10\ell} \left[1 + \frac{28}{3 \log \ell} + \cdots \right].$$

The coefficient $28/3$ also uses massive curvature/area matching and ordinary massive decoupling—not just the source flow.

Here A is a non-universal tail amplitude. Logarithmic origins are absorbed into the omitted subleading corrections.

Local tail regularity is assumed.

A length penalty is a mass

The length penalty reveals two crossover scales.

At fixed map size n , how strong must μ be to affect the long-distance marked sector?

Bulk: nonconstant modes

The mass competes with gradient energy.

$$\gamma_R \asymp 1 \implies \mu_{\text{bulk}} \asymp \frac{(\log n)^6}{n}.$$

Collective: uniform orientation

No gradient cost; $h_R = K\gamma_R$ controls pinning.

$$h_R \asymp 1 \implies \mu_{\text{coll}} \asymp \frac{(\log n)^5}{n}.$$

Because $e^{-\mu\ell}$ suppresses loops with $\ell \gtrsim 1/\mu$, define the corresponding inverse-source half-lengths:

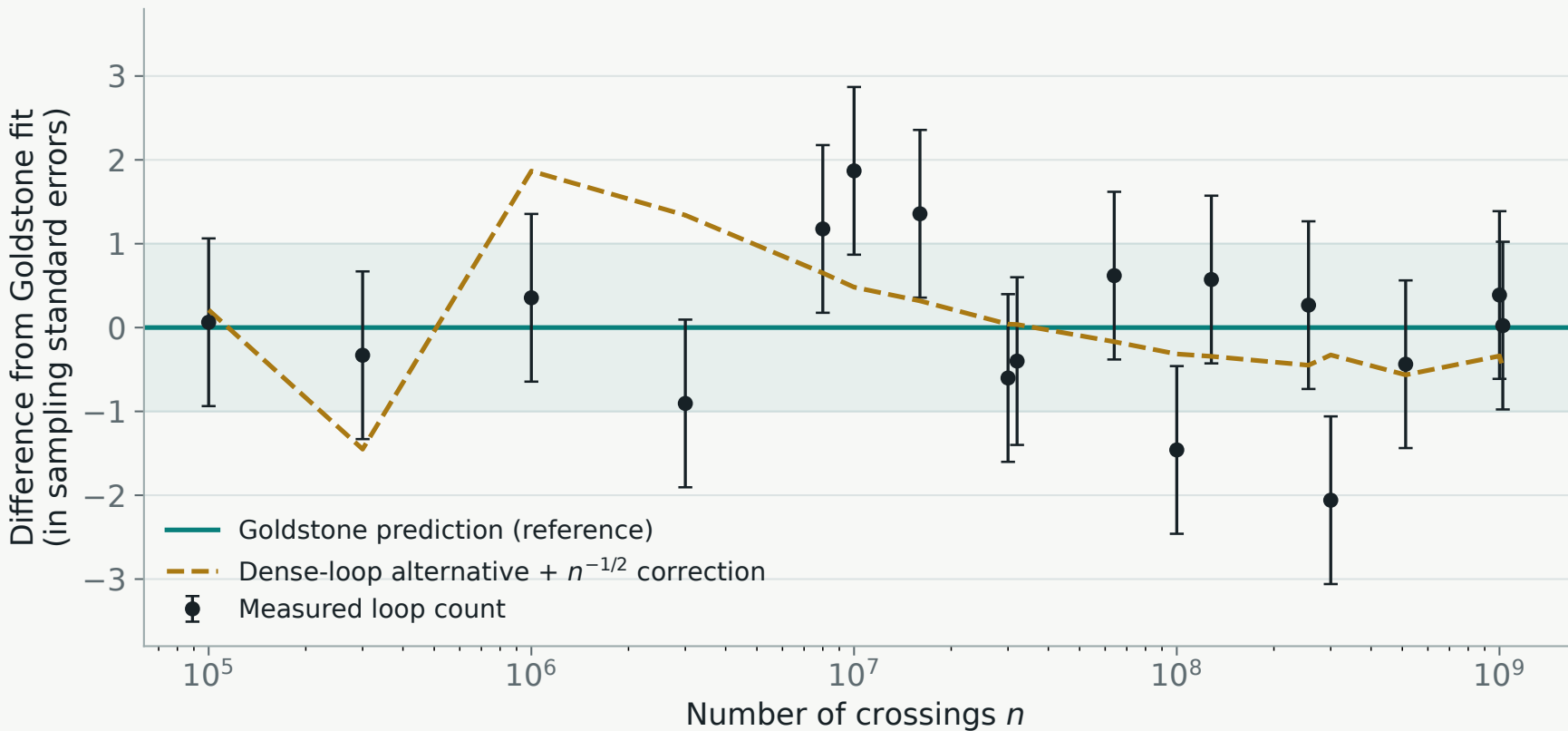
$$L_n := \frac{1}{\mu_{\text{bulk}}} \asymp \frac{n}{(\log n)^6}, \quad \frac{1}{\mu_{\text{coll}}} \asymp K L_n \asymp \frac{n}{(\log n)^5}.$$

The collective threshold is weaker by $K \asymp \log n$.

These are loop half-length response scales, not spatial distances or a derived longest-loop law.

The total count: agreement, not a measurement of every coefficient

Uniform maps from the blossom-tree bijection · two completed campaigns · sizes $10^5 \leq n \leq 1.024 \times 10^9$.



Compatible

Goldstone prediction

$\chi^2 = 15.8$ for 14 degrees of freedom.

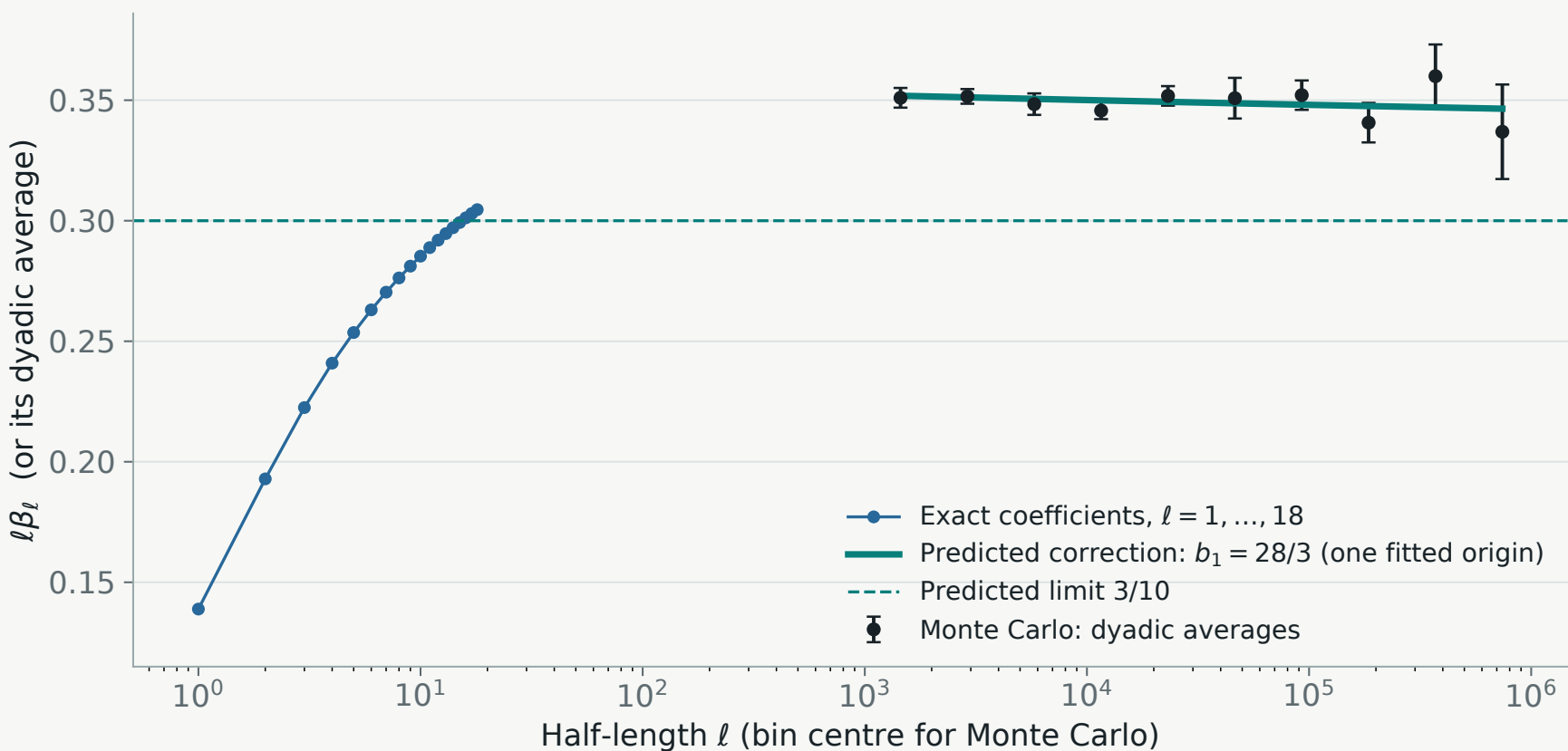
Only κ, C, c fitted, with $n_0 = 1$.

The power-corrected dense-loop alternative also fits: $\chi^2 = 19.6$ for 14 degrees of freedom.

One common subtraction for both models. Error bars: one sampling standard error.

The uncorrected alternative fails. Once allowed corrections are included, these data do not uniquely select the theory.

Why an observed value near 0.35 can still tend to 0.30



Compatible

$$\ell\beta_\ell \simeq \frac{3}{10} \left[1 + \frac{28/3}{\log(\ell/\ell_{0,\beta})} \right].$$

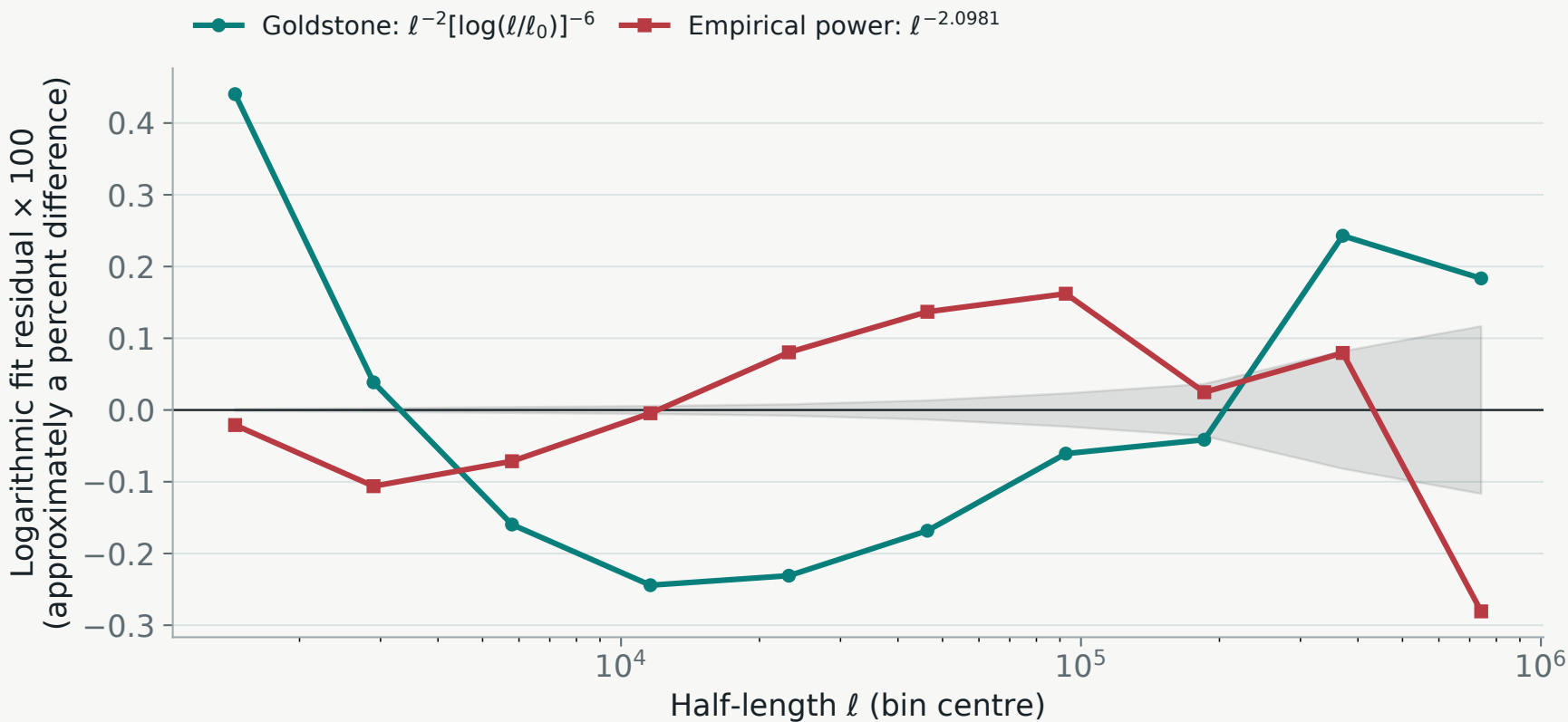
$\chi^2 = 5.22$ for 9 degrees of freedom, fitting one logarithmic origin.

But a free constant also fits. Even the downward drift is not resolved.

Dots at small ℓ : exact. Large-length points: finite-size intercepts, averaged over dyadic bins.

Compatibility does not determine $28/3$: the coefficient and logarithmic origin compensate each other over this range.

The microscopic tail is more demanding



Finite-range tension

The leading six-log curve is not the best fit here.

RMS fractional discrepancy:
Goldstone: about 0.215%.
Free power: about 0.124%.

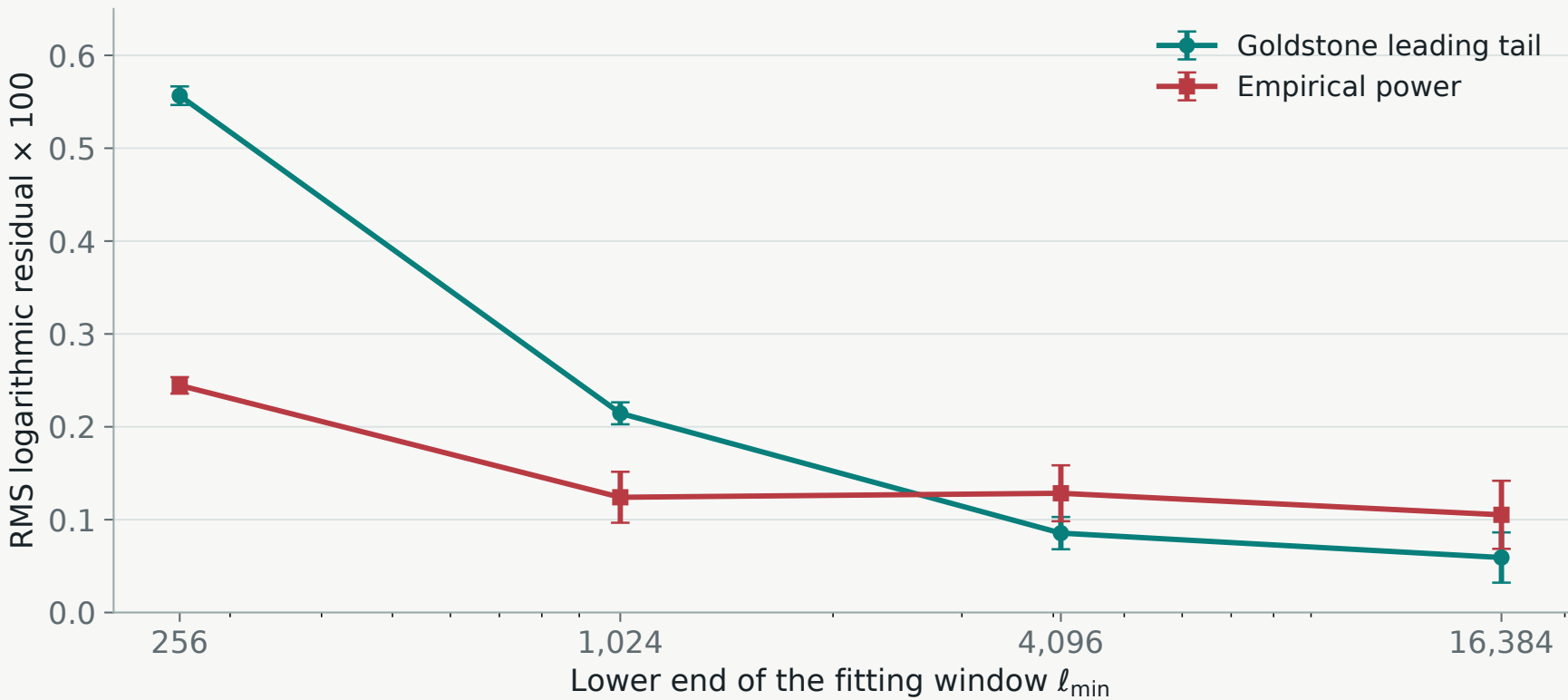
The mismatch is larger than sampling noise.

Finite-size-corrected densities, not raw counts divided by the largest n . Each curve has two fitted parameters. Grey band: one sampling standard error.

A good power-law fit is not yet evidence for a different asymptotic exponent.

PZJ, unpublished. RMS logarithmic residual is a relative-shape score, not a χ^2 or model probability.

The microscopic tail is more demanding



The preference changes when the fitting window moves.

Later windows can favour the six-log form.

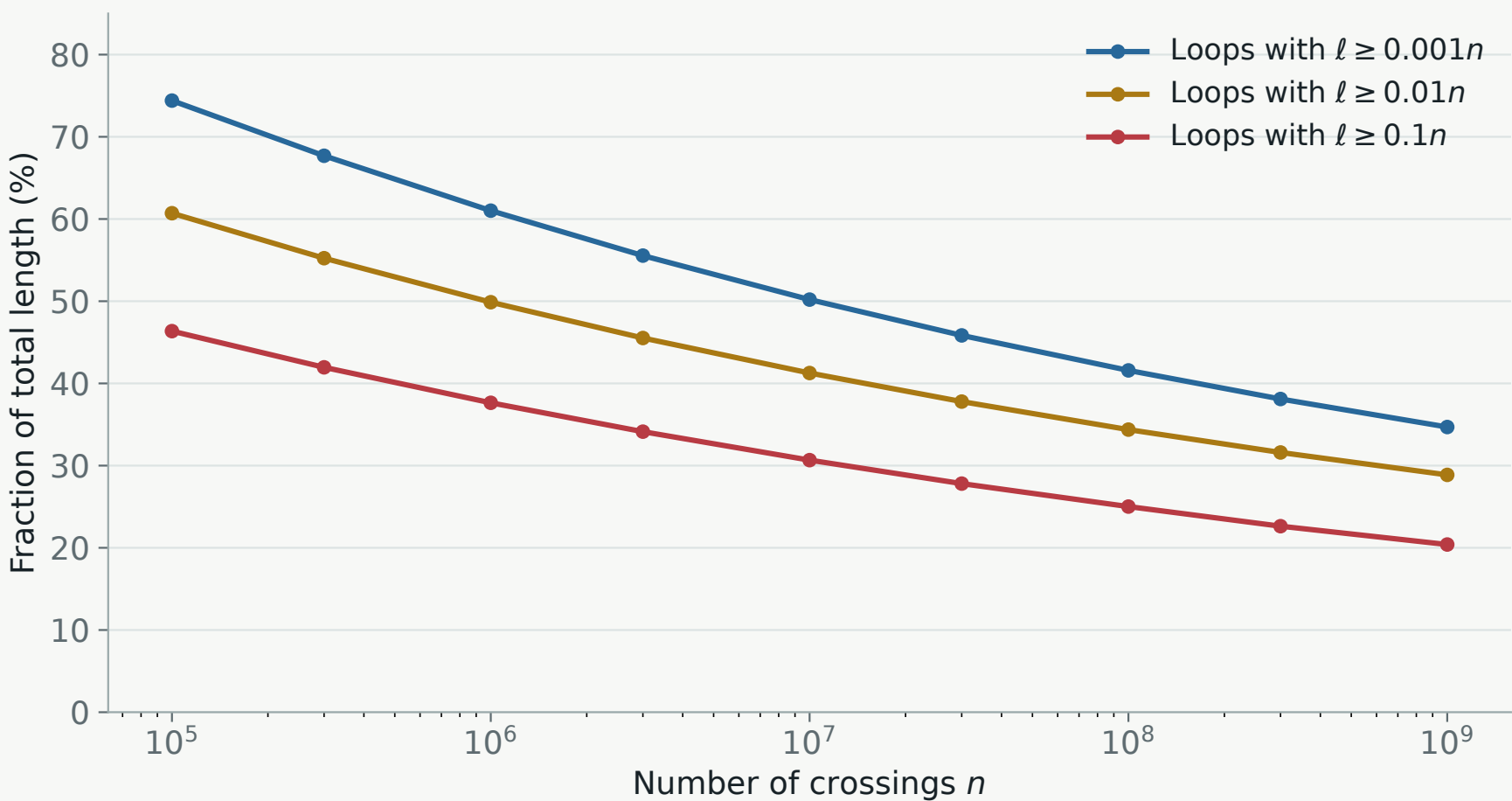
Those wins are not statistically resolved.

Subleading tail corrections are not controlled well enough to identify the asymptotic law from these leading curves.

Same upper endpoint 2^{20} ; move only the lower endpoint. Sampling errors shown. Nested windows are correlated.

The data constrain pre-asymptotic corrections; they do not eliminate the Goldstone asymptote.

A billion crossings is not the asymptotic limit



Very large loops still carry substantial length.

At $n = 10^9$, loops with $\ell \geq 0.1n$ carry about 20%.

The matched-collapse tests also do not distinguish the five-log and six-log scales.

Fraction of all edges belonging to loops whose half-length exceeds a fixed fraction of n .

Vanishing macroscopic mass would still not rule out length escaping to growing sublinear scales.

A billion crossings is not the asymptotic limit

Exact

Fixed-length series.

Exact series up to ~ 20 ; knot expansion; sum rules.

Physics

Universal logarithmic corrections.

Gravity dresses the observable as well as the theory.

Numerics

Current Montecarlo simulations don't reach the asymptotic regime.

The data are compatible with the conjectured universality, but don't single it out convincingly.

