Large N is solvable — and even integrable

correlation functions in $\mathcal{N} = 4$ SYM at strong coupling

D. Serban

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based on:arXiv:2605.03887 w/ B. Basso and E. Kaluç
also arXiv:1903.05038, 1905.1146, w/ I. Kostov and V. Petkova



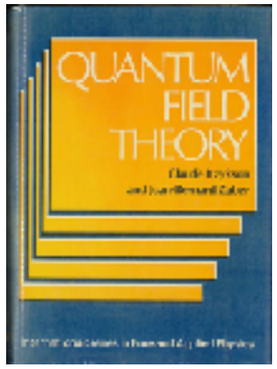


JeanBernardIsEighty
IHP Paris
September 24-25, 2026

How I met Jean-Bernard



IFA București



1992-1993



SPhT Saclay



Jean-Bernard was my official advisor when I started the PhD



Back in time: 1983 Service de Physique Théorique

Jean-Bernard's favourite research themes while he was in Saclay

- Perturbative gauge theory: methods and limits
- Large-N and matrix models
- Lattice (gauge) theories
- Integrable 2d models
- 2d conformal field theories and their algebraic structure

LES CHAMPS DE YANG-MILLS ET LA DIFFUSION DES MESONS PSEUDOSCALAIRES

Soutenue le 28 janvier 1974 devant la Commission d'Examen

INTRODUCTION

La construction de modèles de basse énergie pour les interactions fortes à partir de la théorie des champs se heurte à un sérieux problème : il s'agit de concilier le calcul perturbatif des amplitudes et les importantes contraintes d'unitarité et de croisement. Comme on le verra dans la suite, les approximants de Padé fournissent une solution raisonnable à ce problème et permettent donc d'étudier les implications dynamiques de théories de champs.

Commun. math. Phys. 59, 35—51 (1978)

Planar Diagrams

E. Brézin, C. Itzykson, G. Parisi*, and J. B. Zuber
Service de Physique Theorique, Centre d'Etudes Nucleaires de Saclay, F-91190 Gif-sur-Yvette, France

We present some investigations of the planar approximation to field theory calculated through a limit of a large internal symmetry. Part of the motivation for this work lies in the hope that it might ultimately provide a mean of performing reliable computations in the large coupling phase of non-abelian gauge fields in four dimensions.

→ matrix model program & 2d gravity matrix model approach

REMARKS ABOUT THE EXISTENCE OF NON-LOCAL CHARGES IN TWO-DIMENSIONAL MODELS

E. BRÉZIN, C. ITZYKSON, J. ZINN-JUSTIN and J.-B. ZUBER
CEN, Saclay, 91190 Gif sur Yvette, France

Received 15 January 1979

A simple derivation of the classical non-local conservation laws in two dimensions discovered by Lüscher and Pohlmeyer is given. Several classes of models are shown to possess the same structure.

non-local conserved charges ⟶ Yangian

MODULAR INVARIANT PARTITION FUNCTIONS IN TWO DIMENSIONS

A. CAPPELLI*, C. ITZYKSON and J.-B. ZUBER

Service de Physique Théorique, CEN-Saclay, 91191 Gif-sur-Yvette Cedex, France

Received 3 September 1986

We present a systematic study of modular invariance of partition functions, relevant both for two-dimensional minimal conformal invariant theories and for string propagation on a SU(2) group manifold. We conjecture that all solutions are labelled by simply laced Lie algebras.

ADE-ology

massive perturbations of CFTs ⟶ TBA & Y-systems (different ADE-ology)

1974

solve Yang-Mills at large N

↓

2026

solve planar $\mathcal{N} = 4$ SYM

4d conformal, integrable, supersymmetric,

dual to string theory

spectrum ⟶ correlation functions, amplitudes, Wilson lines, …

$\mathcal{N} = 4$ SYM as an integrable spin chain

- At weak coupling the spin chain is a **supersymmetric**, **long range** generalisation of the **isotropic Heisenberg** spin chain with a continuous parameter ('t Hooft coupling constant)

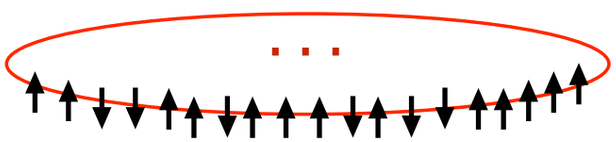
$$g = \sqrt{g_{\rm YM}^2 N_c}/4\pi$$

- The **spin states** correspond to the **fundamental fields** of the gauge theory: SU (N_c) **gauge fields**, six real **bosons** and four complex **fermions**

$$A_\mu,\ \Phi_i \text{ and } \psi_a \qquad\qquad i = 1, ..., 6, \quad a = 1, ..., 4$$

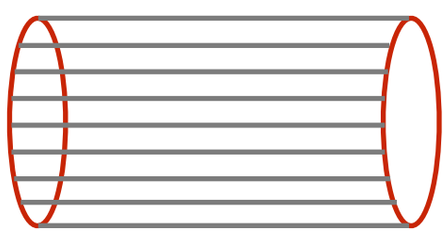
- **Planar limit** $\rightarrow$ single-trace operators $\rightarrow$ **spin chain** with periodic boundary conditions

$$\mathcal{O}_L(x, y) = y_1^{I_1} \dots y_L^{I_L} \ \text{Tr}\ \Phi_{I_1}(x) \dots \Phi_{I_L}(x) \qquad \longleftrightarrow$$



- Two-point correlators at $g \rightarrow 0$:

$$\langle \mathcal{O}_A(x_1) \mathcal{O}_B(x_2) \rangle = \frac{\delta_{AB}}{|x_1 - x_2|^{2L}}$$



$\mathcal{N} = 4$ SYM as an integrable spin chain/2d field theory

- 1/2 BPS state: $\mathcal{O}_L(x,y) = \mathrm{Tr}\ (y \cdot \Phi(x))^L := y^{I_1} \dots y^{I_L}\ \mathrm{Tr}\ \Phi_{I_1}(x) \dots \Phi_{I_L}(x)$

  $\sim$ ferromagnetic **vacuum** of the Heisenberg-like spin chain

- Excitations (spin waves) are called **magnons** and they carry $\mathrm{psu}(2|2)_{\mathrm{L}} \times \mathrm{psu}(2|2)_{\mathrm{R}}$ flavours

- Magnons scatter with **Beisert's scattering matrix**[2]

  $\rightarrow$ factorised scattering $\leftrightarrow$ Yang-Baxter equation

- Two-point correlators at finite coupling : $\langle \mathcal{O}_A(x_1) \mathcal{O}_B(x_2) \rangle = \dfrac{\delta_{AB}}{|x_1 - x_2|^{2\Delta_A(g)}}$

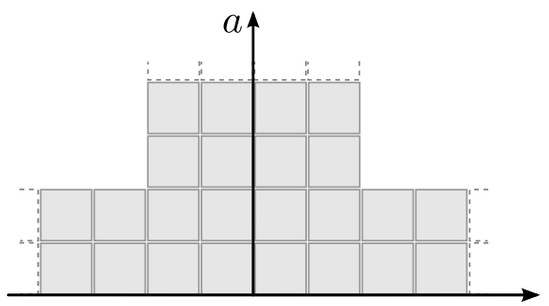


energy of the spin chain

$\mathcal{N} = 4$ SYM as a spin chain/2d field theory

- The spectrum of **conformal dimensions** $\Delta(g)$ is determined by **fusion relations** of **transfer matrices** $T_{a,s}(u)$ + boundary conditions + **analyticity constraints**

$$T_{a,s}(u+\tfrac{i}{2})T_{a,s}(u-\tfrac{i}{2}) = T_{a+1,s}(u)T_{a-1,s}(u) + T_{a,s+1}(u)T_{a,s-1}(u)$$



also written as **Y-systems**:

$$Y_{a,s}(u+i/2)Y_{a,s}(u-i/2) = \frac{(1+Y_{a,s+1}(u))(1+Y_{a,s-1}(u))}{(1+1/Y_{a+1,s}(u))(1+1/Y_{a-1,s}(u))}$$

$$Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$$

and ultimately as **QQ-relations**:

$$\begin{vmatrix} Q^-_{I\mathbf{i}} & Q^+_{I\mathbf{i}} \\ Q^-_{I\mathbf{j}} & Q^+_{I\mathbf{j}} \end{vmatrix} \propto Q_I(u)Q_{I\mathbf{ij}}(u)$$

u rapidity variable

$\frac{u}{g} = x + \frac{1}{x}$ Zhukovsky variable

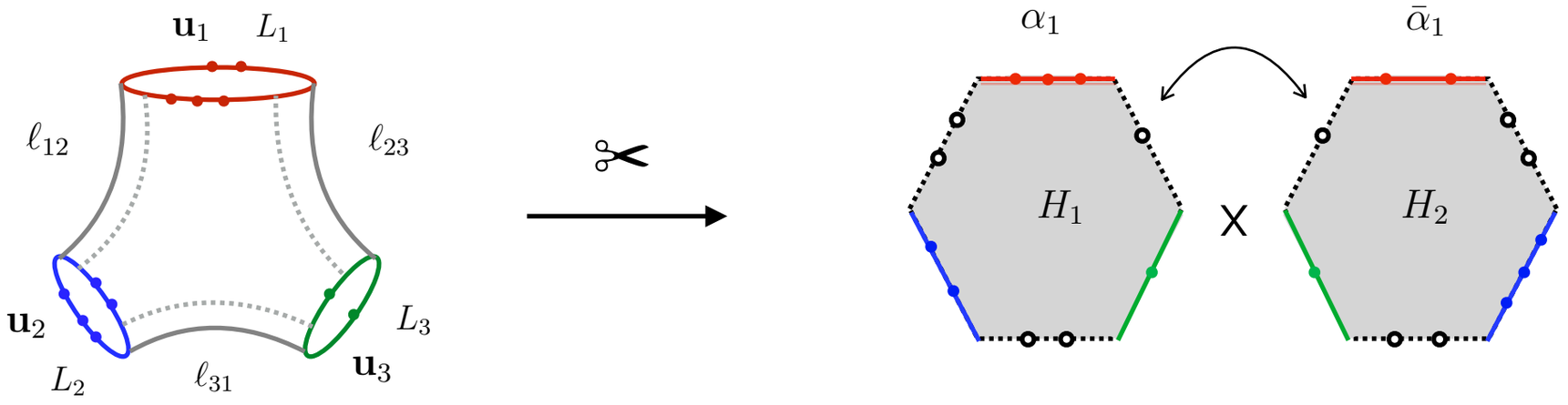
$f^{\pm} \equiv f(u \pm i/2)$

→ **Quantum Spectral Curve** (QSC): **[Cavaglia, Fioravanti, Gromov, Kazakov, Leurent, Sorin, Tateo, Tsuboi, Vieira, Volin, Zabrodin…]** based on the **TBA approach** for integrable 2d field theories

Correlation functions and the hexagon expansion

- Three point function: $$\langle \mathcal{O}_{\mathbf{u}_1}(x_1)\,\mathcal{O}_{\mathbf{u}_2}(x_2)\,\mathcal{O}_{\mathbf{u}_3}(x_3)\rangle = \frac{\mathcal{C}_{123}(g)}{x_{12}^{\Delta_{12}}\,x_{23}^{\Delta_{23}}\,x_{31}^{\Delta_{31}}}$$

- pants decomposition: **[Basso, Komatsu, Vieira, 15]**



$$\mathcal{C}_{123} = \sum_{\alpha_i \cup \bar{\alpha}_i = \mathbf{u}_i} \sum_{\text{mirror}} w_{\ell_{12}} w_{\ell_{23}} w_{\ell_{31}}\; H_1 \times H_2$$

- sum over partitions for the three groups of **physical magnons**

$$\mathbf{u}_1 = \alpha_1 \cup \bar{\alpha}_1, \mathbf{u}_2 = \alpha_2 \cup \bar{\alpha}_2, \mathbf{u}_3 = \alpha_3 \cup \bar{\alpha}_3$$

- insertion of an arbitrary number of virtual **(mirror) magnons** on the bridges of lengths

$$\ell_{ij} = \frac{1}{2}(L_i + L_j - L_k)$$

From sphere to disk geometry

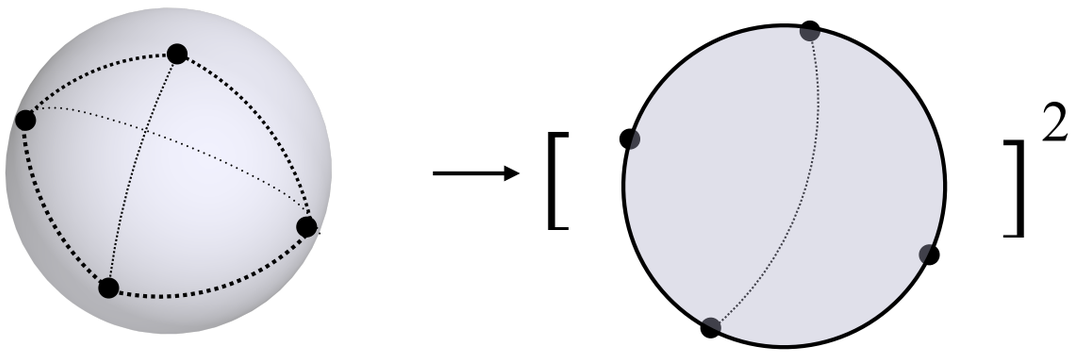
- Further simplification: cut open the sphere into two disks:

  - set $L = 2$ and send a collection of points to **light-like distances** $x^2_{i\,i+1} \to 0$; connection to **amplitudes** **[Alday, Eden, Korchemsky, Maldacena, Sokatchev, 10]**

or

  - send $L \to \infty$ on some edges to **decouple two hemispheres** **[Coronado, 18];** duality with scattering amplitudes on the Coulomb branch **[Caron-Huot, Coronado, 21; Bargheer, Bekov, Bercini, Coronado, 25]**

$n = 4$:



- For $n = 4$ points (**octagon**) this allows to find the result **in closed form for any coupling**
- For $n > 4$ results in **particular limits** (strong coupling, null polygonal limit, origins)

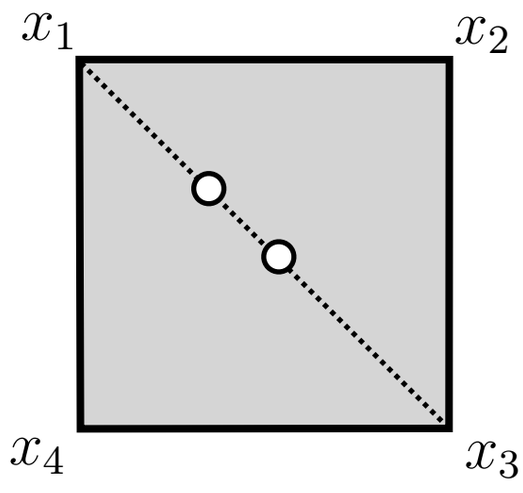
The octagon kernel

- the octagon depends on **two space-time cross ratios**

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{32}^2} \qquad (1-z)(1-\bar{z}) = \frac{x_{13}^2 x_{24}^2}{x_{14}^2 x_{32}^2}$$

$$z = e^{-\sigma+i\phi}\,, \qquad \bar{z} = e^{-\sigma-i\phi}$$

and two R-charge cross ratios: $\alpha, \bar{\alpha}$



- **analytical computation of the octagon** by summing up the mirror particle contribution

→ Fredholm determinant **[Kostov, Petkova, D.S., 19]**

$$\mathbb{O}_\ell(z,\bar{z}) = \det(1-\mathbf{K}_\ell^{\rm oct})$$

- **octagon (Bessel) kernel**: simplified by **[Belitsky, Korchemsky, 19]**

$$(\mathrm{K}_\ell^{\rm oct})_{mn} = -2\sqrt{(2m+\ell+1)(2n+\ell+1)}\int_0^\infty \frac{dt}{t}\,\chi(t)\,J_{2m+\ell+1}(2gt)\,J_{2n+\ell+1}(2gt) \qquad m,n\geq 0$$

$$\chi(t) = \frac{\cos\phi-\cosh\sigma}{\cos\phi-\cosh\sqrt{t^2+\sigma^2}}$$

contains all the information about the specifics of the correlation function (cross ratios, coupling constant,…)

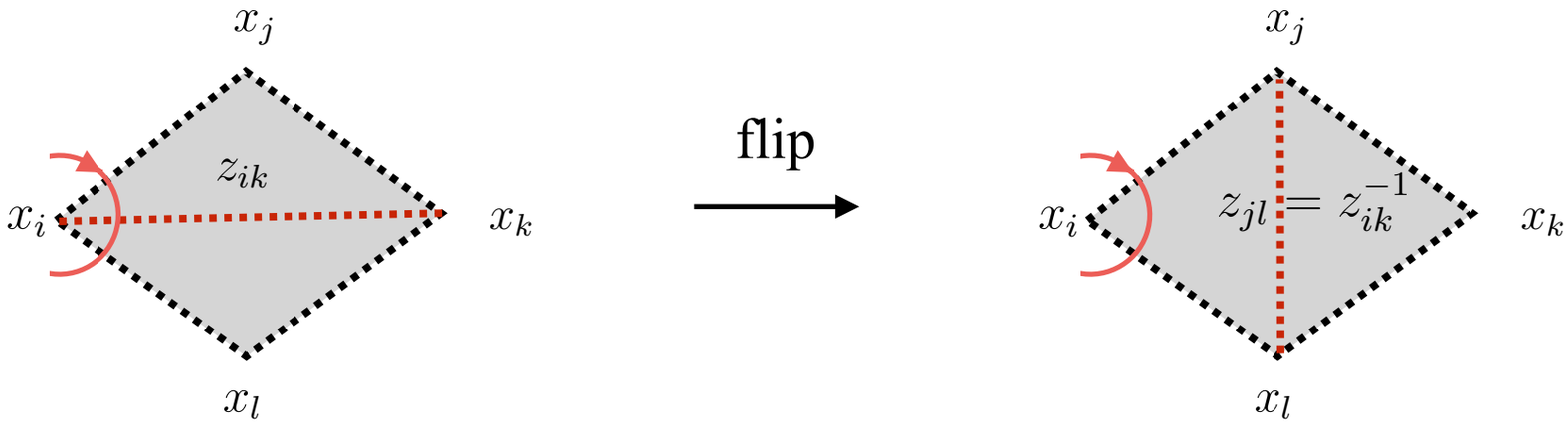
Polygonal correlation functions

- **2d kinematics** (easier to handle) ⟶ x_i become complex coordinates
- from hexagon tilings → triangulation → Fock-Goncharov coordinates $\{z_E\}$



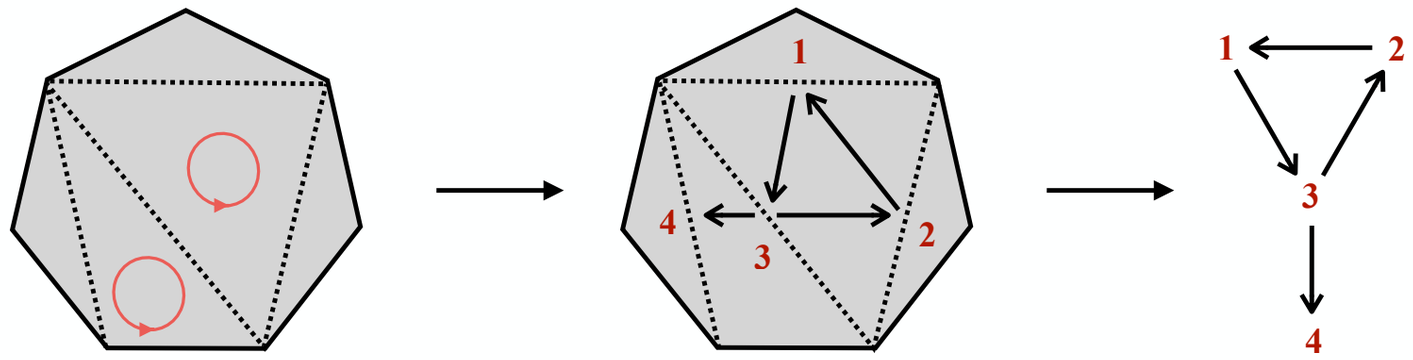
- **cross ratios** z_E associated to quadrilaterals that surround the edge E :

$$z_{ik} = -\frac{x_{ij}x_{kl}}{x_{il}x_{jk}}$$



- similar cross ratios α_E associated to R-charges; each edge has bridge length ℓ_E

From triangulations to quivers to cluster algebra



triangulation with **orientation** → **quiver** with incidence matrix

$$B_{ij} = \langle \gamma_i, \gamma_j \rangle$$

- **mutations** (flips of diagonals)

- charges γ_i :

$$\gamma_i \xrightarrow{\mu_i} \gamma_i' = -\gamma_i\,,$$

$$\gamma_j \xrightarrow{\mu_i} \gamma_j' = \gamma_j + \max\bigl(0, \langle \gamma_j, \gamma_i \rangle\bigr)\,\gamma_i\,, \qquad j \neq i\,.$$

- Fock-Goncharov coordinates $\{\chi_i\}$:

$$\{\chi_i\} = \{z_i\},\ \{\alpha_i\}$$

$$\chi_j' = \begin{cases} \chi_j^{-1}, & j = i\,, \\ \chi_j\,(1 + \chi_i^{\,\mathrm{sgn}\,B_{ij}})^{B_{ij}}, & j \neq i\,. \end{cases}$$

Strong coupling limit: TBA equations and Y-system

- At leading order in $\frac{1}{2g}$ correlation functions are given by a partition function :

$$\mathbf{Z} \approx \exp\{-2g\mathcal{A}\}\,, \qquad \mathcal{A} = \sum_{\gamma} \mathcal{A}_\gamma$$

with $$\mathcal{A}_\gamma = \int \frac{d\theta}{2\pi \cosh^2\theta} \left[R(Y_\gamma^{\rm Sph}(\theta)) + R(\bar{Y}_\gamma^{\rm Sph}(\theta)) - R(Y_\gamma^{\rm AdS}(\theta)) - R(\bar{Y}_\gamma^{\rm AdS}(\theta))\right]$$

in terms of the Rogers dilogarithm $$R(Y) \equiv {\rm Li}_2(-Y) + \frac{1}{2}\log(Y/\mathcal{Y})\log(1+Y)$$

- The **Y-functions** are determined by a **TBA system** :

$$\log Y_\gamma^I(\theta) = \log \mathcal{Y}_\gamma^I(\theta) + \sum_{\gamma'}\sum_J (-1)^{F_J+1}\langle\gamma,\gamma'\rangle K^{IJ} \star \log(1+Y_{\gamma'}^J(\theta))$$

$$\log \mathcal{Y}^{\rm AdS}(\theta) = i\phi^{\rm AdS} + i\sigma^{\rm AdS}\sinh\theta - l\cosh\theta\,,$$
$$\log \bar{\mathcal{Y}}^{\rm AdS}(\theta) = -i\phi^{\rm AdS} + i\sigma^{\rm AdS}\sinh\theta - l\cosh\theta \qquad \cdots \qquad I \in \{\rm AdS,\ \overline{AdS},\ Sph,\ \overline{Sph}\}$$

similarity with the Thermodynamic Bubble Ansatz for amplitudes

[Alday, Gaiotto, Maldacena, 09; Alday, Maldacena, Sever, Vieira, 10]

Strong coupling limit: TBA equations and Y-system

- with a bit of massaging the kernel can be put in a **block diagonal form**

$$K^{IJ}_{\gamma\gamma'} = (-1)^{1+F_J}(\tilde{K}^{IJ}_{\gamma\gamma'} + \delta\tilde{K}^{IJ}_{\gamma\gamma'}) \qquad \text{with} \qquad \tilde{K}^{IJ} = \begin{pmatrix} K & \bar{K} & 0 & 0 \\ \bar{K} & K & 0 & 0 \\ 0 & 0 & K & \bar{K} \\ 0 & 0 & \bar{K} & K \end{pmatrix}$$

$$K(\theta,\theta') = \frac{i}{2}\cosh\theta\cosh\theta'\coth\left(\frac{\theta-\theta'}{2}\right), \quad \bar{K}(\theta,\theta') = -K(\theta+i\pi,\theta')$$

- the kernel is of **Cauchy type** with singularity $K \sim \dfrac{i\cosh^2\theta}{\theta-\theta'}$

$\longrightarrow$ regularisation by **splitting the contours of integration** $\mathcal{C}_\gamma$

ordering of the contours $\longleftrightarrow$ ordering of the roots $\quad \gamma \succ \gamma' \text{ for } \mathrm{Im}\,\mathcal{C}_\gamma > \mathrm{Im}\,\mathcal{C}_{\gamma'}$

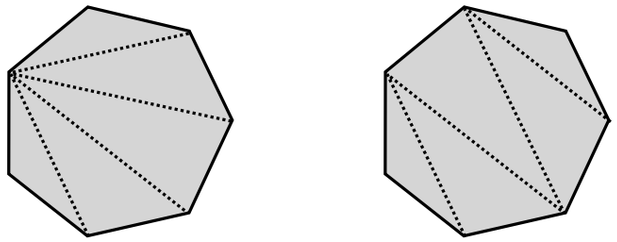
- **Y-system**, with $Y^{++}_\gamma \equiv Y_\gamma(\theta+i\pi)$

$$Y^{++}_\gamma \bar{Y}_\gamma = \prod_{\gamma'\succ\gamma}(1+Y^{++}_{\gamma'})^{\langle\gamma',\gamma\rangle} \prod_{\gamma'\prec\gamma}\left(1+\bar{Y}_{\gamma'}\right)^{\langle\gamma,\gamma'\rangle}$$

both for AdS and Sph sector

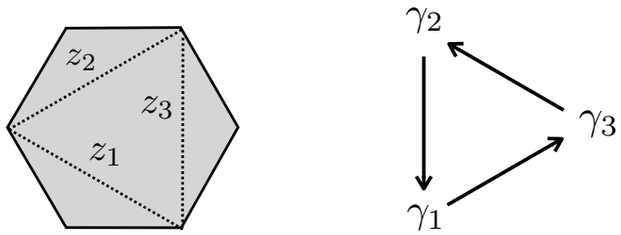
Quivers with cycles: wall-crossing

- the construction above **works for the linear quivers** but **not for those with cycles** (unavoidable for the sphere geometry)



linear quivers

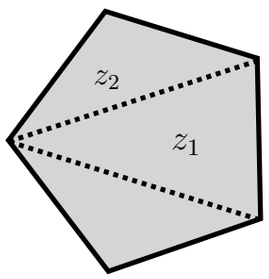
$\gamma_i \succ \gamma_j \quad \text{if} \quad \langle \gamma_i, \gamma_j \rangle > 0$

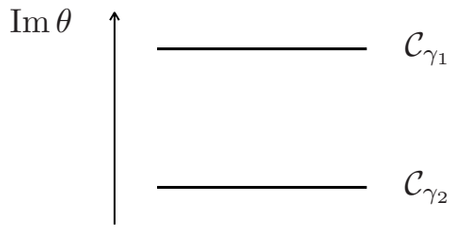


quiver with a cycle

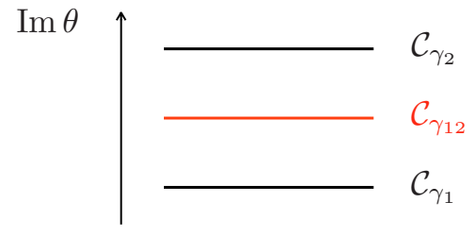
no natural ordering

- the solution is **wall-crossing** : exchanging the order of contour of integration creates a composite state (exemple below for A_2)





$\gamma_1 \succ \gamma_2$



$\gamma_2 \succ \gamma_{12} \succ \gamma_1$

$\mathcal{Y}_{\gamma_{12}} = \mathcal{Y}_{\gamma_1}\,\mathcal{Y}_{\gamma_2}$

$\gamma_{12} = \gamma_1 + \gamma_2$

can be done systematically on any quiver to remove cycles **[Alim et al., 11]**

$\longrightarrow$ chamber $\mathcal{B} = \{\gamma\}$

From Y-functions to χ-functions

- we write now the Y-system for a collection of **piecewise continuous functions** X_γ

$$\log X_\gamma(\theta) = \log \boldsymbol{\chi}_\gamma(\theta) + \sum_{\gamma' \in \mathcal{B}} \langle \gamma, \gamma' \rangle \left[G * \log\big(1 + X_{\gamma'}(\theta)\big) + \bar{G} * \log\big(1 + \bar{X}_{\gamma'}(\theta)\big) \right] \qquad G, \bar{G} \sim K, \bar{K}$$

$\sim$ **[Gaiotto, Moore, Neitzke, 08]** for the Hitchin description of BPS spectra of 4D $\mathcal{N} = 2$ theories

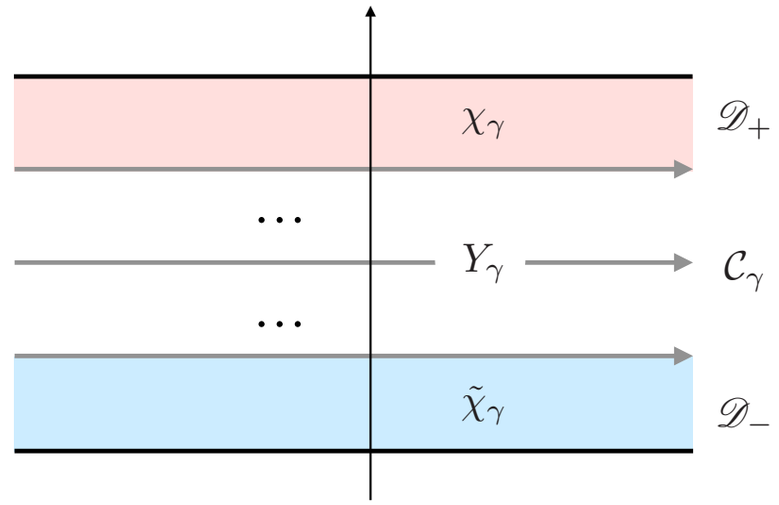
- jumps are controlled by the **Kontsevich-Soibelman** transformations

$$\mathcal{K}_{\gamma'} : \qquad X_\gamma \;\mapsto\; X_\gamma (1 + X_{\gamma'})^{\langle \gamma', \gamma \rangle}$$

- Y_γ and χ_γ are **two determinations of the same function** X_γ

$$\chi_\gamma \equiv X_\gamma\big|_{\mathscr{D}_+} , \qquad Y_\gamma \equiv X_\gamma\big|_{\mathcal{C}_\gamma}$$

$$Y_\gamma = \prod_{\gamma' \succ \gamma}^{\curvearrowleft} \mathcal{K}_{\gamma'}\ \chi_\gamma$$



- χ_γ satisfy the multiplicative relation $\chi_{\gamma+\gamma'} = \chi_\gamma \chi_{\gamma'}$

χ-system

- the integral equations can be transformed into difference equations; using the **spectrum generator**, independent of the chamber $\mathcal{B}$

$$\boldsymbol{S} = \prod_{\gamma \in \mathcal{B}}^{\curvearrowleft} \mathcal{K}_\gamma \qquad\qquad \boldsymbol{S}(\chi_\gamma) = \chi_\gamma \prod_{\gamma' \in \mathcal{B}} (1 + Y_{\gamma'})^{\langle \gamma', \gamma \rangle}$$

the Y-system becomes $\qquad \chi_\gamma^{++} \boldsymbol{S}(\bar{\chi}_\gamma) = 1\,, \qquad \bar{\chi}_\gamma^{++} \boldsymbol{S}(\chi_\gamma) = 1$

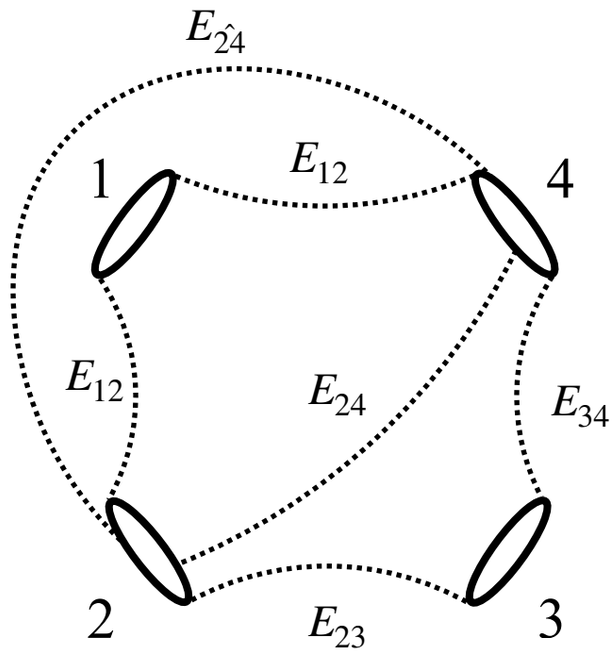
- reducing the equations to the **fundamental charges** γ_E one gets

$$\bar{\chi}_E^{++} \chi_E = \prod_{E'} (1 + A_{E'})^{\langle E, E' \rangle}$$

with $\qquad 1 + A_E = \prod_{\gamma \ni \gamma_E} (1 + Y_\gamma) = \prod_{\gamma} (1 + Y_\gamma)^{\delta_{\gamma, E}} \qquad$ (for polygons)

χ-system for closed geometry: the four point function

- for the **four-point function** in the triangulation in the figure we obtain



$$\bar{\chi}^{++}_{12}\chi_{12} = \left(\bar{\chi}^{++}_{14}\chi_{14}\right)^{-1} = \bar{\chi}^{++}_{34}\chi_{34} = \left(\bar{\chi}^{++}_{23}\chi_{23}\right)^{-1} = \frac{(1+A_{24})}{(1+A_{\hat{2}\hat{4}})}$$

$$\bar{\chi}^{++}_{24}\chi_{24} = \left(\bar{\chi}^{++}_{\hat{2}\hat{4}}\chi_{\hat{2}\hat{4}}\right)^{-1} = \frac{(1+A_{23})(1+A_{14})}{(1+A_{34})(1+A_{12})}$$

$$A_{ij} = A_{ij}(\{\chi_{kl}\})$$

- the same equations hold for the AdS and the sphere part
- to get the full result, summation over the bridge lengths is necessary
- in the AdS_2 kinematics identical to the **χ-system for correlation functions** **[Caetano, Toledo, 12]** obtained from the **semiclassical string analysis**

Conclusion

- correlation functions at strong coupling in $\mathcal{N} = 4$ SYM have the **structure expected from the duality with classical string theory**

- we obtained this description based on the **integrability representation** (hexagonalization) valid at any coupling

- the expressions depend only on the **kinematical data** through the cross ratios, entering in the TBA equations via the driving terms

- the next step would be to consider correlation functions of **excited classical operators** and understand how the spectral data combines with the kinematical one

- it would be interesting to see how the wall-crossing phenomena manifest themselves at **finite coupling**



✳ generated with help from Claude (Anthropic)

Claude: “please give my regards to Zuber, even if he'll never know that a mathematical assistant sent them via one of his mathematical descendants”

Happy birthday, Jean-Bernard, and thank you for being our highest weight!