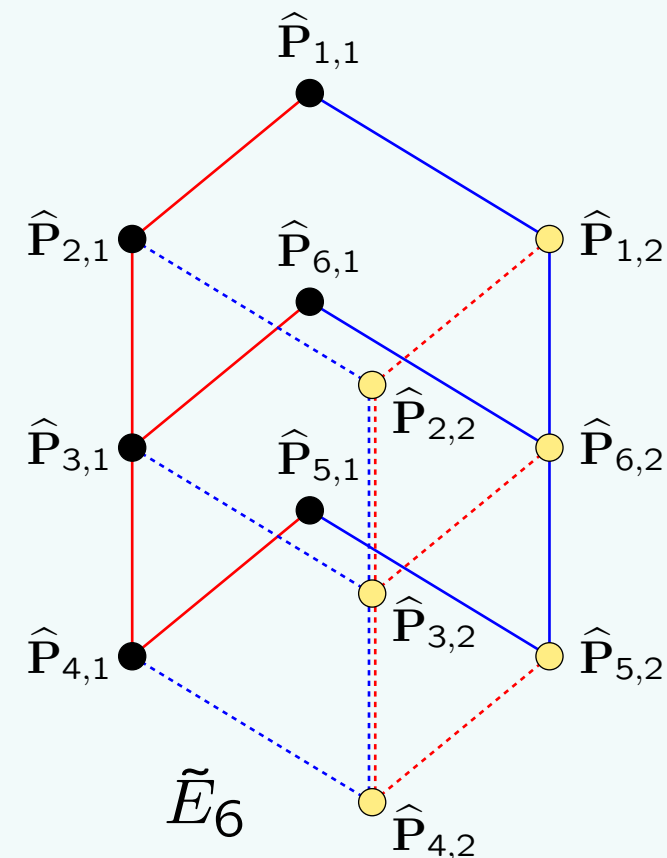
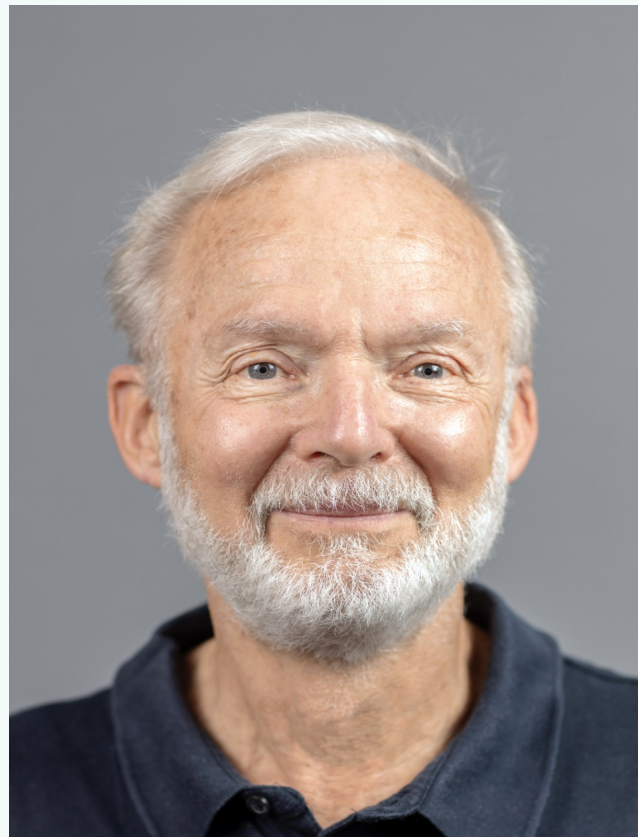


JBZ, ADE, Defects & All That: A Lattice Perspective

JBZ80 Paris, 24–25 Sept 2026

Paul A. Pearce, University of Melbourne

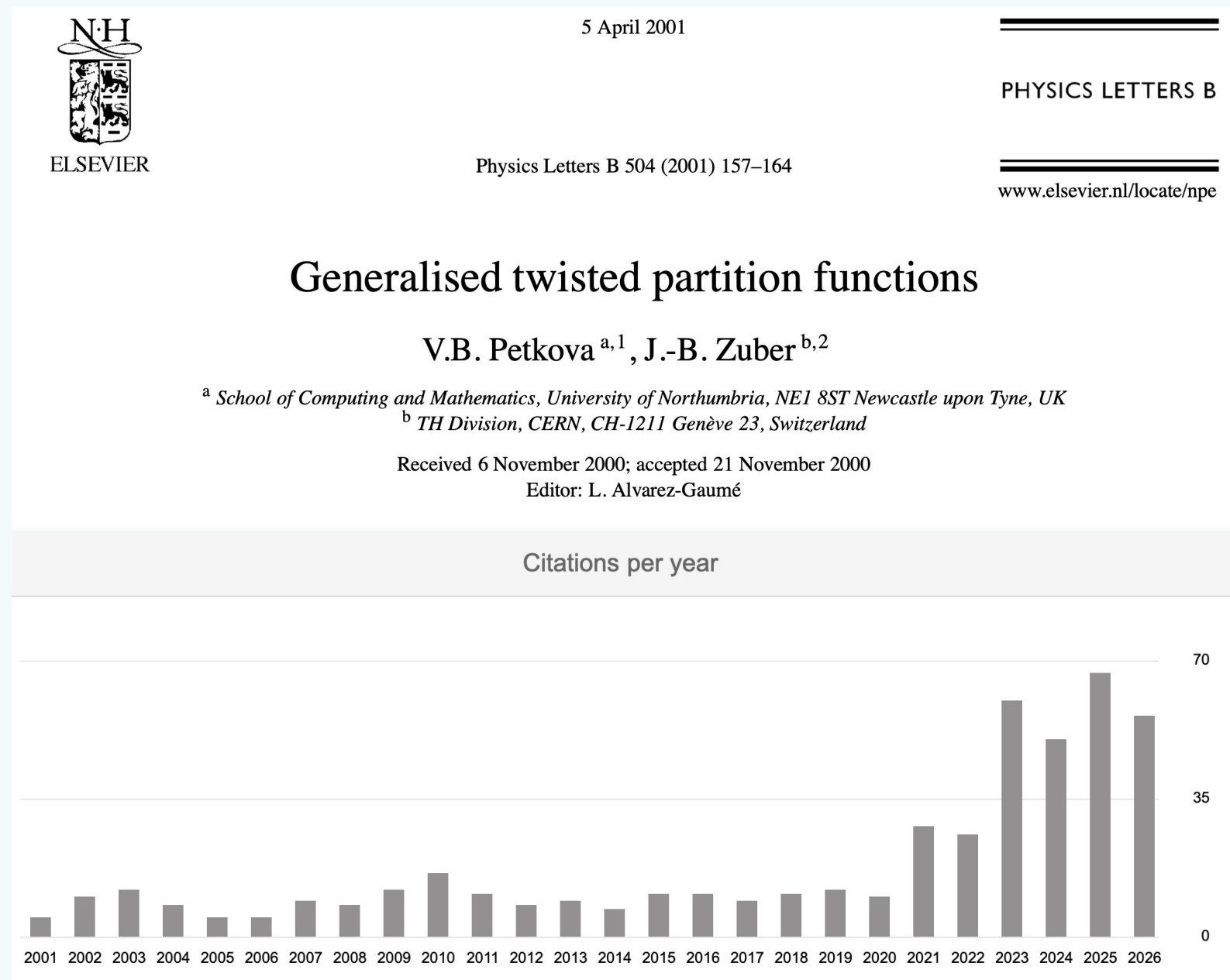


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Collaborators: *Jorgen Rasmussen, Jared Heymann, Thomas Quella*

Acknowledgments: *JBZ, Robert Coquereaux*

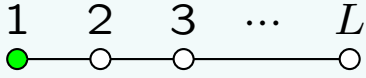
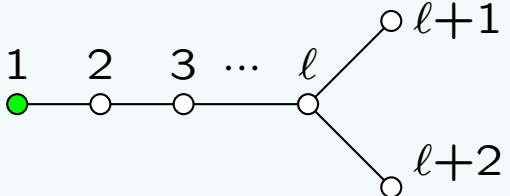
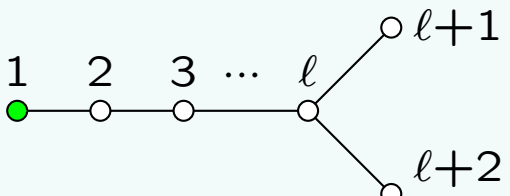
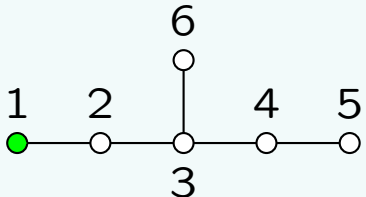
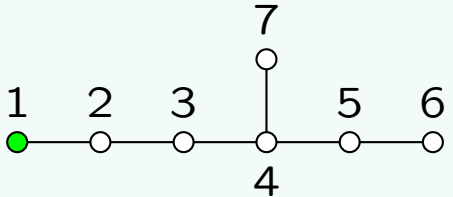
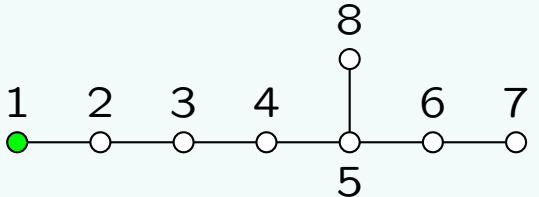
Petkova and Zuber 2001:



Topological Defect Lines $\Rightarrow$ Noninvertible Quantum Symmetries!

- (i) Can we find Ocneanu fusion rules for topological defects from the lattice?
- (ii) Are there natural finite-dimensional matrix representations of Ocneanu fusion algebras?

Simply Laced A - D - E Dynkin Diagrams

G	Dynkin Diagram	Cox g	$\text{Exp}(G)$	Type/ H	Γ
A_L		$L+1$	$1, 2, \dots, L$	I	$\mathbb{Z}_2$
$D_{\ell+2}$ (ℓ odd)		$2\ell+2$	$1, 3, \dots, 2\ell+1, \ell+1$	II/ $A_{2\ell+1}$	$\mathbb{Z}_2$
$D_{\ell+2}$ (ℓ even)		$2\ell+2$	$1, 3, \dots, 2\ell+1, \ell+1$	I	$\mathbb{Z}_2/\mathbb{S}_3$
E_6		12	$1, 4, 5, 7, 8, 11$	I	$\mathbb{Z}_2$
E_7		18	$1, 5, 7, 9, 11, 13, 17$	II/ D_{10}	1
E_8		30	$1, 7, 11, 13, 17, 19, 23, 29$	I	1

- A - D - E Dynkin diagrams G , Coxeter numbers g , Coxeter exponents $\text{Exp}(G)$, the type I or II, parent graphs $H \neq G$ and diagram automorphism group Γ .
- The eigenvalues of the adjacency matrix G are $2 \cos \frac{m\pi}{g}$ with $m \in \text{Exp}(G)$.

Fusion Matrices (nimreps)

Verlinde Matrices: $L = g-1$, $G = A_L$ ($L \times L$ Matrices): $N_i = (N_i)_j^k$, $i, j, k \in A_L$
 $N_1 = I$, $N_2 = A_L$, $N_j = N_{j-1}N_2 - N_{j-2}$, $N_L = \sigma \in \mathbb{Z}_2$

$$N_i N_j = \sum_{k \in A_{g-1}} N_{ij}^k N_k, \quad N_j = U_{j-1}(\tfrac{1}{2}N_2) = \text{Chebyshev of 2nd kind}$$

Fused Adjacency/Intertwiner Matrices:

$|G| \times |G|$ Matrices: $n_i = (n_i)_a^b$, $i = 1, 2, \dots, g-1$; $a, b \in G$
 $n_1 = I$, $n_2 = G$, $n_j = n_2 n_{j-1} - n_{j-2} = U_{j-1}(\tfrac{1}{2}n_2)$, $3 \leq j \leq g-1$

$$n_i n_j = \sum_{k \in A_{g-1}} N_{ij}^k n_k \quad n_{g-1} = \begin{cases} I, & D_{2\ell}, E_7, E_8 \\ \sigma \in \mathbb{Z}_2, & A_L, D_{2\ell-1}, E_6 \end{cases}$$

Pasquier Graph Fusion Matrices: (only for Type I models):

$|G| \times |G|$ Matrices: $\hat{N}_a = (\hat{N}_a)_b^c$, $\hat{N}_1 = I$, $\hat{N}_2 = G$, $\hat{N}_2 \hat{N}_a = \sum_{b \in G} G_{ab} \hat{N}_b$, $a, b, c \in G$

$$\hat{N}_a \hat{N}_b = \sum_{c \in G} \hat{N}_{ab}^c \hat{N}_c, \quad n_i \hat{N}_a = \sum_{b \in G} n_{ia}^b \hat{N}_b, \quad n_i = \sum_{a \in G} n_{i1}^a \hat{N}_a, \quad C = (n_i)_1^a = \text{Intertwiner}$$

Inversely, $\hat{N}_a$ are either automorphisms or simple linear combinations of n_i , e.g. for E_6 :

$$\hat{N}_a = n_a, \quad a = 1, 2, 3; \quad \hat{N}_4 = n_6 - n_2, \quad \hat{N}_5 = n_5 - n_3, \quad \hat{N}_6 = n_2 + n_4 - n_6$$

● Minimal Model Extended Characters: $\hat{\chi}_{r,a}(q) = \sum_{s=1}^{g-1} n_{s1}^a \chi_{r,s}(q)$, Fix: $r = 1$

Critical $A-D-E$ Lattice Models

- The face weights of the critical $A-D-E$ lattice models [ABF84,Pasquier87] are

$$W\left(\begin{array}{cc|c} d & c & \\ a & b & u \end{array}\right) = \begin{array}{c} d \quad c \\ \square \\ a \quad b \end{array} = \frac{\sin(\lambda - u)}{\sin \lambda} \delta_{ac} + \frac{\sin u}{\sin \lambda} \sqrt{\frac{\psi_a \psi_c}{\psi_b \psi_d}} \delta_{bd}, \quad \lambda = \frac{\pi}{g}, \quad 0 \leq u \leq \lambda$$

where u is the spectral parameter. The face weights vanish if $G_{ab}G_{bc}G_{cd}G_{da} = 0$ and satisfy the [Yang-Baxter equation](#) implying commuting transfer matrices and [integrability](#).

- The largest eigenvalue of the adjacency matrix G is $[2]_x = 2 \cos \lambda$ with eigenvectors ψ

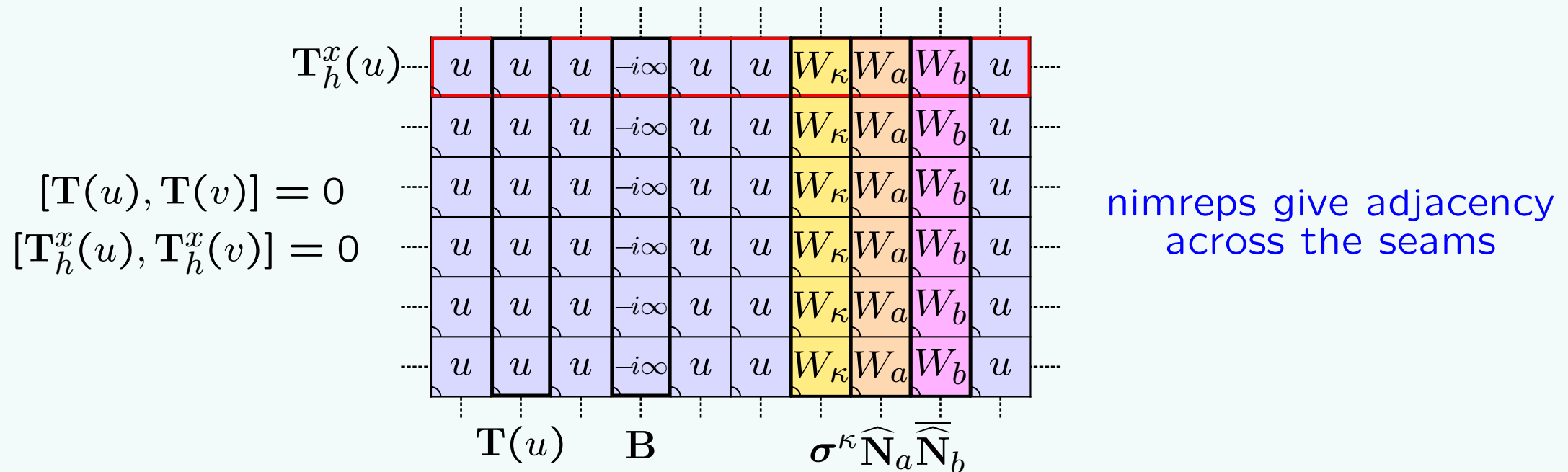
$$G\psi = [2]_x \psi, \quad \psi = (\psi_a)_{1 \leq a \leq |G|} = \begin{cases} ([1]_x, [2]_x, \dots, [L]_x), & G = A_L \\ ([1]_x, [2]_x, \dots, [\ell]_x, \frac{[\ell]_x}{[2]_x}, \frac{[\ell]_x}{[2]_x}), & G = D_{\ell+2} \\ ([1]_x, [2]_x, [3]_x, [2]_x, [1]_x, \frac{[3]_x}{[2]_x}), & G = E_6 \\ ([1]_x, [2]_x, [3]_x, [4]_x, \frac{[6]_x}{[2]_x}, \frac{[4]_x}{[3]_x}, \frac{[4]_x}{[2]_x}), & G = E_7 \\ ([1]_x, [2]_x, [3]_x, [4]_x, [5]_x, \frac{[7]_x}{[2]_x}, \frac{[5]_x}{[3]_x}, \frac{[5]_x}{[2]_x}), & G = E_8 \end{cases}$$

Here $S_a = [a]_x = \frac{x^a - x^{-a}}{x - x^{-1}}$ with $x = e^{i\lambda}$ and $\psi_a = S_a$ for the A series.

- Setting $\rho(u) = \sin(\lambda - u)/\sin(\lambda)$, the braid limits of the $A-D-E$ face weights are given by the complex conjugate weights

$$\begin{aligned} B\left(\begin{array}{cc|c} d & c & \\ a & b & \end{array}\right) &= \begin{array}{c} d \quad c \\ \square \\ a \quad b \end{array} = - \lim_{u \rightarrow -i\infty} \frac{x^{\frac{1}{2}}}{\rho(u)} W\left(\begin{array}{cc|c} d & c & \\ a & b & u \end{array}\right) = i \left(-x^{-\frac{1}{2}} \delta_{ac} + x^{\frac{1}{2}} \sqrt{\frac{\psi_a \psi_c}{\psi_b \psi_d}} \delta_{bd} \right) \\ \overline{B}\left(\begin{array}{cc|c} d & c & \\ a & b & \end{array}\right) &= \begin{array}{c} d \quad c \\ \square \\ a \quad b \end{array} = - \lim_{u \rightarrow i\infty} \frac{x^{-\frac{1}{2}}}{\rho(u)} W\left(\begin{array}{cc|c} d & c & \\ a & b & u \end{array}\right) = i \left(x^{\frac{1}{2}} \delta_{ac} - x^{-\frac{1}{2}} \sqrt{\frac{\psi_a \psi_c}{\psi_b \psi_d}} \delta_{bd} \right) \end{aligned}$$

Commuting Row and Column Transfer Matrices



- An $(N, M) = (10, 6)$ periodic lattice showing (i) the row transfer matrix $T_h^x(u)$ with seam segment $x = (a, b, \kappa)$, (ii) the column/seam transfer matrices $T(u)$, $\widehat{N}_a$, $\overline{\widehat{N}}_b$ and (iii) the automorphism seam σ^κ with $\kappa = 0, 1$. The composite seam is the matrix product $T_x = \sigma^\kappa \widehat{N}_a \overline{\widehat{N}}_b$.
- The braid seams $B, \overline{B}$ are the limits $\lim_{u \rightarrow \mp i\infty} T(u)$, that is, the face weights are replaced by the complex braid face weights $B, \overline{B}$. The seams $B, \overline{B}$ commute with $T(u)$ and with each other. The fused s -type braid seams n_s (and their complex conjugates $\bar{n}_s$) are defined recursively by

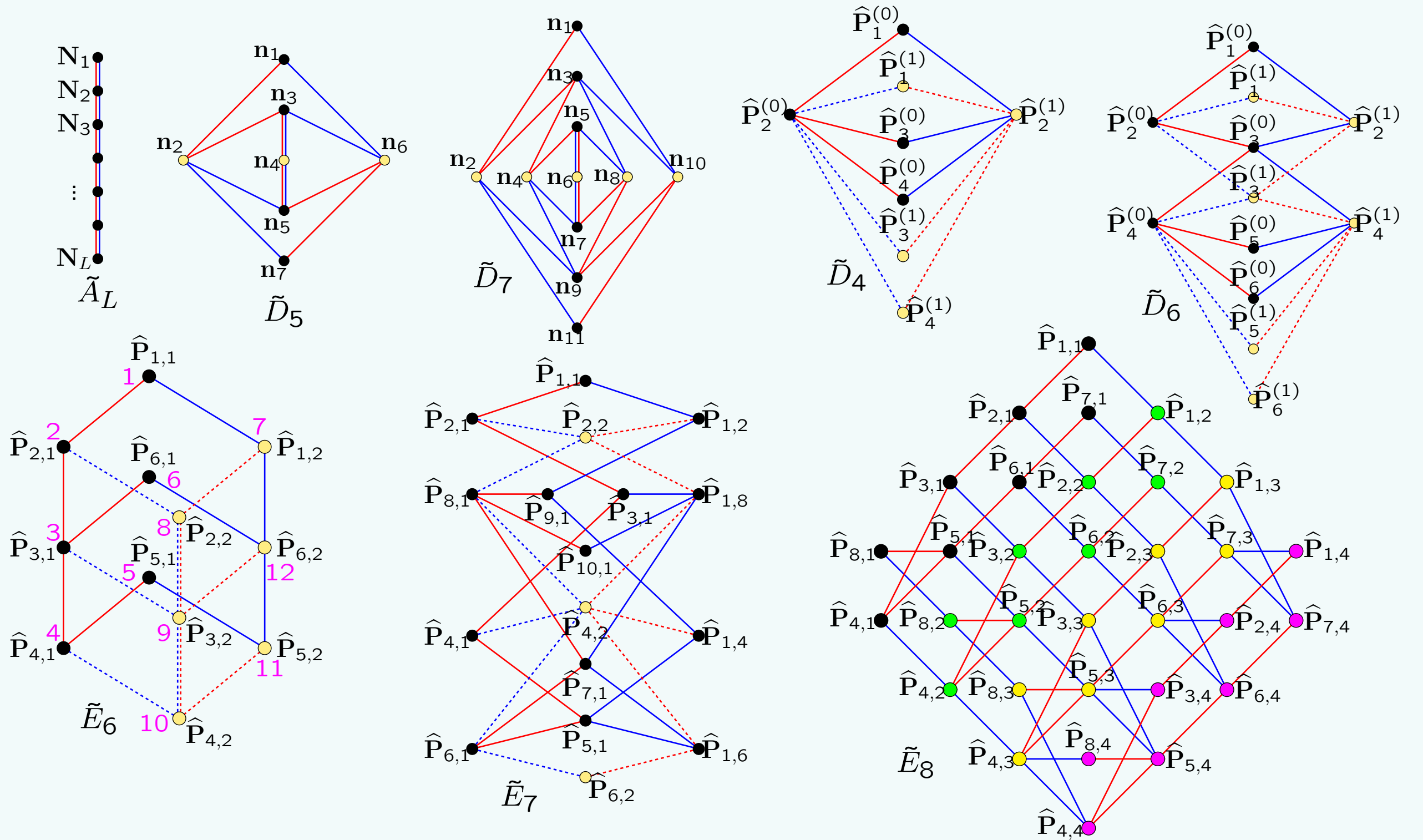
$$n_1 = I, \quad n_2 = B, \quad n_s = n_2 n_{s-1} - n_{s-2} = U_{s-1}(\tfrac{1}{2} n_2), \quad 3 \leq s \leq g-1$$

The $\widehat{N}_a$ seams are given by the same linear combinations of n_s as the graph fusion matrices. The eigenvalues of B are $2 \cos \frac{m\pi}{g}$ with $m \in \text{Exp}(G)$.

- For each A - D - E lattice model, the seams $n_s, \bar{n}_s$ and $\widehat{N}_a, \overline{\widehat{N}}_b$ satisfy the Verlinde and graph fusion algebras respectively for arbitrary system sizes:

$$n_i n_j = \sum_{k \in A_{g-1}} N_{ij}^k n_k, \quad 1 \leq i, j \leq g-1, \quad \widehat{N}_a \widehat{N}_b = \sum_{c \in G} \widehat{N}_{ab}^c \widehat{N}_c, \quad a, b \in G$$

Ocneanu Fusion Graphs $\tilde{G}$



● The red/blue lines show the action of the left/right chiral fundamentals labelled by $a, \bar{a}=2$.

Twisted partition functions:
$$Z_{a,b}(q) = \sum_{c,d \in G} [\hat{P}_{a,b}]_{c,d} \hat{\chi}_{1,c}(q) \hat{\chi}_{1,d}(\bar{q}), \quad \hat{P}_{a,b} = \hat{N}_a \hat{N}_{\bar{b}}, \quad G = E_6, E_8$$

E_6 Ocneanu Graph Fusion Algebra (nimreps)

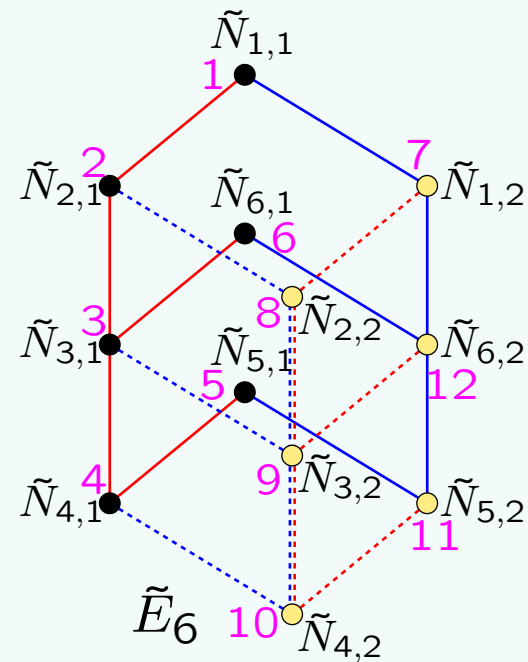
- The 12 Ocneanu graph fusion matrices (nimreps) $\tilde{N}_a$ for $\tilde{E}_6$ are [PetkovaZ2001]:

$$\tilde{N}_{a,1} = \begin{pmatrix} \hat{N}_a & 0 \\ 0 & \hat{N}_a \end{pmatrix}, \quad \tilde{N}_{a,2} = \begin{pmatrix} 0 & \hat{N}_a \\ \hat{N}_a & \hat{N}_a \hat{N}_6 \end{pmatrix}, \quad a = 1, 2, \dots, 6$$

$$\hat{N}_a = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad a = 1, 2, \dots, 6$$

- Impose Order: $\{\tilde{N}_\eta\} = \{\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_{12}\} = \{\tilde{N}_{1,1}, \tilde{N}_{2,1}, \dots, \tilde{N}_{6,1}, \tilde{N}_{1,2}, \tilde{N}_{2,2}, \dots, \tilde{N}_{6,2}\}$

$$\Sigma = (\delta_{\eta, \bar{\mu}}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



$$\tilde{N}_\eta \tilde{N}_\mu = \sum_{\nu=1}^{12} \tilde{N}_{\eta\mu}{}^\nu \tilde{N}_\nu$$

$$\tilde{N}_a = \begin{cases} \hat{N}_a \oplus \hat{N}_a, & a = 1, 2, \dots, 6 \\ \tilde{N}_7 = \Sigma \tilde{N}_2 \Sigma, \\ \tilde{N}_7 \tilde{N}_{a-6}, & a = 8, 9, \dots, 12 \end{cases}$$

- $\Sigma = \mathbb{Z}_2$ chiral conjugation; $\tilde{N}_{\bar{a}} = \Sigma \tilde{N}_a \Sigma$; $\bar{2}=7, \bar{3}=12, \bar{4}=11$; $\bar{a}=a$ for $a \in \{1, 5, 6, 8, 9, 10\}$.
- In fact everything factorizes [PRasm24] into left/right chiral halves with $\tilde{N}_1 = \tilde{N}_{\bar{1}} = I$:

$$\tilde{N}_{a,b} = \tilde{N}_a \tilde{N}_{\bar{b}}, \quad b = 1, 2; \quad \tilde{N}_2^2 = \tilde{N}_7^2 = I + \tilde{N}_6 \tilde{N}_{\bar{2}}; \quad \tilde{N}_a \tilde{N}_b = \sum_{c=1}^6 \hat{N}_{ab}{}^c \tilde{N}_c, \quad a, b = 1, 2, \dots, 6$$

All products of the $\tilde{N}_{a,b}$ then follow from the commutative, associative and distributive laws!

E_6 Ocneanu Algebra of Seams

- Following the relations for $\hat{N}_a$ (from intertwiners):

$$\hat{N}_a = \mathbf{n}_a, \quad a = 1, 2, 3; \quad \hat{N}_4 = \mathbf{n}_6 - \mathbf{n}_2, \quad \hat{N}_5 = \mathbf{n}_5 - \mathbf{n}_3 = \sigma, \quad \hat{N}_6 = \mathbf{n}_2 + \mathbf{n}_4 - \mathbf{n}_6$$

with $\overline{\hat{N}}_b = \hat{N}_{\bar{b}}$, $b = 1, 2, \dots, 6$. The 12 E_6 seams $\hat{P}_{a,b} = \hat{N}_a \overline{\hat{N}}_b$ ($a = 1, 2, \dots, 6$; $b = 1, 2$) are

$$\{\hat{P}_\eta\} = \{\hat{P}_1, \hat{P}_2, \dots, \hat{P}_{12}\} = \{\hat{P}_{1,1}, \hat{P}_{2,1}, \dots, \hat{P}_{6,1}, \hat{P}_{1,2}, \hat{P}_{2,2}, \dots, \hat{P}_{6,2}\} \quad (\text{Basis Seams})$$

- The seams satisfy the Ocneanu algebra (proof by simultaneous diagonalization)

$$\hat{P}_\eta \hat{P}_\mu = \sum_{\nu=1}^{12} \tilde{N}_{\eta\mu}{}^\nu \hat{P}_\nu, \quad \tilde{N}_\eta \tilde{N}_\mu = \sum_{\nu=1}^{12} \tilde{N}_{\eta\mu}{}^\nu \tilde{N}_\nu, \quad \eta, \mu = 1, 2, \dots, 12$$

- Cayley table of the commutative $\tilde{E}_6$ Ocneanu algebra using the notation $\eta = \hat{P}_\eta$:

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	1+3	2+4+6	3+5	4	3	8	7+9	8+10+12	9+11	10	9
3	3	2+4+6	1+2(3)+5	2+4+6	3	2+4	9	8+10+12	7+2(9)+11	8+10+12	9	8+10
4	4	3+5	2+4+6	1+3	2	3	10	9+11	8+10+12	7+9	8	9
5	5	4	3	2	1	6	11	10	9	8	7	12
6	6	3	2+4	3	6	1+5	12	9	8+10	9	12	7+11
7	7	8	9	10	11	12	1+12	2+9	3+8+10	4+9	5+12	6+7+11
8	8	7+9	8+10+12	9+11	10	9	2+9	1+3+8 +10+12	2+4+6 +7+2(9)+11	3+5+8 +10+12	4+9	3+8+10
9	9	8+10+12	7+2(9)+11	8+10+12	9	8+10	3+8+10	2+4+6 +7+2(9)+11	1+2(3)+5 +2(8)+2(10)+2(12)	2+4+6 +7+2(9)+11	3+8+10	2+4+2(9)
10	10	9+11	8+10+12	7+9	8	9	4+9	3+5+8 +10+12	2+4+6 +7+2(9)+11	1+3+8 +10+12	2+9	3+8+10
11	11	10	9	8	7	12	5+12	4+9	3+8+10	2+9	1+12	6+7+11
12	12	9	8+10	9	12	7+11	6+7+11	3+8+10	2+4+2(9)	3+8+10	6+7+11	1+5+2(12)

- For $1 \leq \eta \leq 6$ and $3 \leq \mu \leq 6$, $\hat{P}_{\eta,\mu}$ is given in terms of the Basis Seams by the Quantum Symmetry

$$\hat{P}_{\eta,\mu} = \hat{P}_\eta \hat{P}_{\bar{\mu}} = \sum_{\nu=1}^{12} \tilde{N}_{\eta\bar{\mu}}{}^\nu \hat{P}_\nu, \quad \eta = 1, 2, \dots, 6; \quad \mu = 3, 4, 5, 6$$

Untwisted Affine A - D - E Dynkin Diagrams

G	Cox g	Dynkin Diagram ψ_a	$m \in \text{Exp}(G)$
$A_{p-1}^{(1)}, p \geq 1$	p		$0, 2, 4, \dots, 2p-2$
$D_{p+2}^{(1)}, p \geq 2$	$2p$		$0, 2, 4, \dots, 2p, p, p$
$E_6^{(1)}$	6		$0, 2, 2, 3, 4, 4, 6$
$E_7^{(1)}$	12		$0, 3, 4, 6, 6, 8, 9, 12$
$E_8^{(1)}$	30		$0, 6, 10, 12, 15, 18, 20, 24, 30$

- All these (Type I) Dynkin diagrams admit Pasquier graph fusion algebras.
 - The eigenvalues of the adjacency matrices are $2 \cos \frac{m\pi}{g}$ with $m \in \text{Exp}(G)$.
 - The ψ_a are the entries of the Perron Frobenius eigenvector of the adjacency matrices.
- These quantum dimensions = Coxeter labels give 1-d representations of the Pasquier algebras.

Twisted Affine A - D - E Dynkin Diagrams

G	Cox h	Dynkin Diagram $\psi_a, \psi_a^\vee$	$m \in \text{Exp}(G)$
$D_{p+1}^{(2)}, p \geq 2$	$2p$		$0, 2, 4, \dots, 2p$
$A_{2p}^{(2)}, p \geq 2$	$2p$		$0, 2, 4, \dots, 2p$
$A_{2p-1}^{(2)}, p \geq 3$	$2p-2$		$0, 2, 4, \dots, 2p-2, p-1$
$A_2^{(2)}$	2		$0, 2$
$D_4^{(3)}$	4		$0, 2, 4$
$E_6^{(2)}$	6		$0, 2, 3, 4, 6$

- The twisted affine A - D - E Dynkin diagrams $G^{(k)}$ [Kac,FuchsSchw] are obtained by foldings of the untwisted affine Dynkin diagrams.
- The eigenvalues of the adjacency matrices are $2 \cos \frac{m\pi}{g}$ with $m \in \text{Exp}(G)$.
- The $\psi_a(\text{top}), \psi_a^\vee(\text{bottom})$ are the Coxeter/dual Coxeter exponents (entries of the left and right eigenvectors of the adjacency matrices). Both entries are only shown when they differ.

Yang-Baxter Integrable Affine $A-D-E$ Lattice Models

- The face weights of the critical $c=1$ affine $A-D-E$ lattice models [Pasquier87] are

$$W\left(\begin{array}{cc|c} d & c & \\ a & b & u \end{array}\right) = \boxed{u}_{\begin{array}{cc} d & c \\ a & b \end{array}} = (1-u)\delta_{ac} + u\sqrt{\frac{\psi_a\psi_c}{\psi_b\psi_d}}\delta_{bd}, \quad \lambda = \frac{\pi}{g}, \quad 0 \leq u \leq \lambda$$

The ψ_a are the entries of the Perron-Frobenius eigenvector of the affine $A-D-E$ adjacency matrix.

- The real braid limits $\mathbf{B} = \overline{\mathbf{B}}$ of the commuting periodic transfer matrices are

$$\mathbf{B} = \mathbf{B}_1 = (-1)^{N/2} \lim_{u \rightarrow \pm i\infty} u^{-N} \mathbf{T}(u), \quad \mathbf{B}_s = U_s(\tfrac{1}{2}\mathbf{B}) = \text{Chebyshev}, \quad s = 0, 1, \dots$$

corresponding to the fundamental and fused (ambichiral) integrable braid seams.

The eigenvalues of the braid matrix $\mathbf{B}$ are $2\cos\frac{m\pi}{g}$ with $m \in \text{Exp}(G)$.

- An Example:** The critical 4-state Potts model is $D_4^{(1)}$ with 5×5 nimreps:

$$D_4^{(1)} : \begin{array}{c} \begin{array}{ccccc} & 2 \circ 1 & & 1 \circ 3 & \\ & \diagdown & & \diagup & \\ & 2 & & & \\ & \diagup & & \diagdown & \\ 0 \bullet 1 & & 1 & & 1 \circ 4 \end{array} \end{array} \quad \hat{N}_{0,1,2,3,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{N}_a \hat{N}_b = \sum_{c=0,1,\dots,4} \hat{N}_{ab}^c \hat{N}_c, \quad \hat{N}_2^2 = \hat{N}_3^2 = \hat{N}_4^2 = I, \quad \hat{N}_2 = \hat{N}_3 \hat{N}_4 \quad (\mathbb{Z}_2 \times \mathbb{Z}_2 = \text{Klein-4 group})$$

- The braid $\mathbf{B} = \hat{\mathbf{N}}_1$ and automorphism seams $\hat{\mathbf{N}}_a$, $a = 2, 3, 4$, satisfy the same fusion algebra.
- The symmetry can be enlarged to $\mathbb{S}_4$ by adding the $\mathbb{Z}_2$ transposition $3 \leftrightarrow 4$ which leads to a noncommutative graph fusion algebra but we do not do this here.

4-State Potts Lattice Model: Two CFTs

- The known operator content [DVVV89,CappD'App02] consists of 11 fields (5 untwisted, 6 twisted) labelled by nodes $a = 0, 1, 2, \dots, 10$, j -spins or conformal weights $\Delta = \Delta_j$:

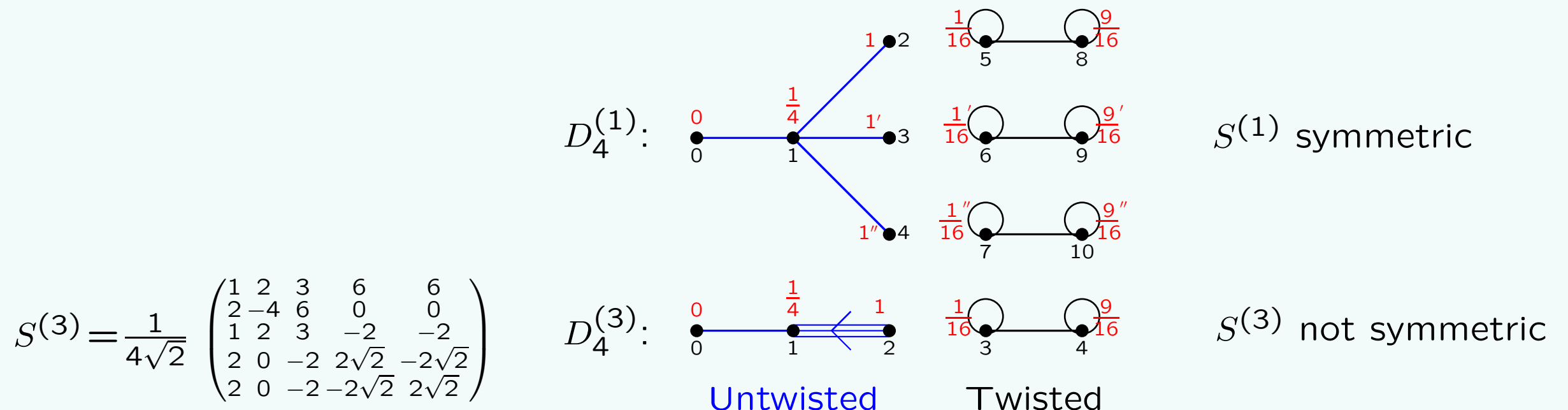
Node Label a	0	2	3	4	5	1	8	6	7	9	10
j -spin	0	4	4'	4''	1	2	3	1'	1''	3'	3''
$\Delta_j = \frac{j^2}{16}$	0	1	1'	1''	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{9}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{9}{16}$	$\frac{9}{16}$
Character $= \chi_a$	$\chi_0 - \frac{1}{2}\chi_4$	$\frac{1}{2}\chi_4$	$\frac{1}{2}\chi_4$	$\frac{1}{2}\chi_4$	χ_1	χ_2	χ_3	χ_1	χ_1	χ_3	χ_3

- The modular invariant partition function (MIPF) and affine $u(1)$ characters χ_j are

$$Z_0^{(1)}(q) = |\chi_0(q)|^2 + |\chi_1(q)|^2 + 3|\chi_2(q)|^2 + 3|\chi_5(q)|^2 + 3|\chi_8(q)|^2$$

$$\chi_j = \chi_j^4, \quad \chi_j^n = \chi_j^n(q) = \frac{1}{\eta(q)} \sum_{k \in \mathbb{Z}} q^{n(k + \frac{j}{2n})^2}, \quad \eta(q) = q^{1/24} \prod_{k=1}^{\infty} (1 - q^k)$$

- The 11×11 nimreps $\hat{N}_a^{(1)}$ follow from the known modular $S^{(1)}$ matrix using the Verlinde formula. Identifying the 3-fold degenerate fields gives the $S^{(3)}$ modular matrix and 5×5 nimreps of the $D_4^{(3)}$ twisted affine CFT. The adjacency graphs of the fundamental nimreps are



4-State Potts: Twisted Partition Functions

- The integrable seams and relations

$$\begin{aligned} \mathbf{I} &= \hat{\mathbf{N}}_0, & \mathbf{B} &= \mathbf{B}_1 = \hat{\mathbf{N}}_1, & \mathbf{B}_2 &= \hat{\mathbf{N}}_2 + \hat{\mathbf{N}}_3 + \hat{\mathbf{N}}_4 \\ \mathbf{B}^2 - \mathbf{I} &= \hat{\mathbf{N}}_2 + \hat{\mathbf{N}}_3 + \hat{\mathbf{N}}_4 = \mathbf{B}_2, & \mathbf{B}\mathbf{B}_2 &= \mathbf{B}(\hat{\mathbf{N}}_2 + \hat{\mathbf{N}}_3 + \hat{\mathbf{N}}_4) = 3\mathbf{B}_2 \end{aligned}$$

now give the $D_4^{(3)}$ graph fusion algebra with nonsymmetric 3×3 nimreps

$$\hat{N}_i^{(3)} \hat{N}_j^{(3)} = \sum_{k=0}^2 \hat{N}_{ij}^{(3)k} \hat{N}_k^{(3)}, \quad \hat{N}_{0,1,2}^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

- The $D_4^{(1)}$ twisted partition functions (TPFs) resulting from inserting a single defect $\hat{\mathbf{N}}_a$ are

$$Z_a^{(1)}(q) = \sum_{b,c=0,1,\dots,10} \hat{N}_{ab}^{(1)c} \chi_b(q) \chi_c(\bar{q}), \quad a = 0,1,2,3,4$$

The TPFs associated with composite defects $a \times b$ are

$$Z_{ab}^{(1)}(q) = \sum_{c=0,1,2,3,4} \hat{N}_{ab}^c Z_c^{(1)}(q), \quad a, b = 0,1,2,3,4$$

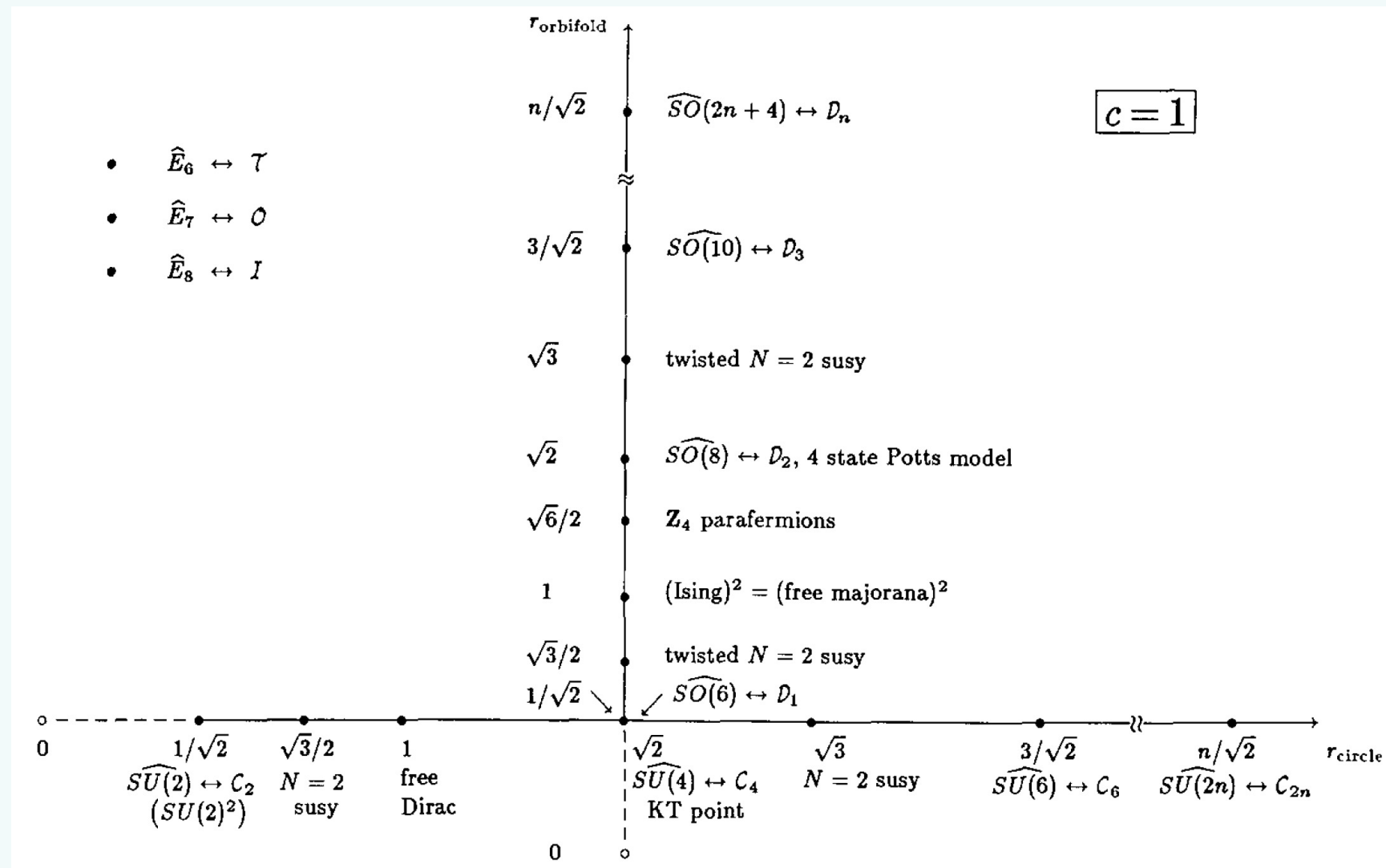
where the $\hat{N}_a$ refer to the 5×5 nimreps of the $D_4^{(1)}$ untwisted nodes. By linearity, the TPFs resulting from inserting the $D_4^{(3)}$ defects $\mathbf{B} = \hat{\mathbf{N}}_1$ and $\mathbf{B}_2 = \hat{\mathbf{N}}_2 + \hat{\mathbf{N}}_3 + \hat{\mathbf{N}}_4$ are

$$Z_1^{(3)} = Z_1^{(1)}(q), \quad Z_{2+3+4}^{(3)}(q) = Z_2^{(1)}(q) + Z_3^{(1)}(q) + Z_4^{(1)}(q)$$

- Note that the multiplicity of the untwisted field j in the MIPF $Z_0^{(1)}(q)$ is given by $\frac{\psi_1 \psi_j^\vee}{\psi_j \psi_1^\vee}$.
- These formulas generalize to all (untwisted and twisted) affine A - D - E CFTs!

Ginsparg Zoo

- But where are the Ocneanu-type fusion graphs/algebras for the affine A - D - E models?



- The affine A - D - E Pasquier RSOS models are all **isolated points** in the $c=1$ Ginsparg zoo!
- To get Ocneanu integrable seams $\hat{\mathbf{P}}_{a,b}$ we need **trigonometric** not linear face weights. Such affine A and D Yang-Baxter integrable RSOS models exist at a dense set of “rational points” ($\lambda/\pi \in \mathbb{Q}$). These models are built on **affine Hecke algebras** not Temperley-Lieb algebras.
- The lattice methods developed here can be used to find the **Ocneanu graphs/algebras** describing their fusion rules. But this is a story for another day!

Conclusion

- On the torus, the algebra of integrable seams gives natural [matrix representations](#) of the Ocneanu algebra which encode how the left and right chiral halves of a CFT are glued together.
- For all critical classical A - D - E lattice models and arbitrary system sizes M :
 - (i) the s -type braid transfer matrices $\mathbf{n}_s$ satisfy the [Verlinde algebra](#),
 - (ii) the a -type integrable seams $\widehat{\mathbf{N}}_a$ satisfy the [Pasquier graph fusion algebra](#),
 - (iii) the integrable seams $\widehat{\mathbf{P}}_{a,b}$ satisfy the [Ocneanu algebra](#) and [quantum symmetry](#).The integrable seams are automorphisms or polynomials in the braid transfer matrices $\mathbf{n}_2, \bar{\mathbf{n}}_2$.
- Similar results are expected for all critical (untwisted/twisted) affine A - D - E lattice models.
- Since the integrable seams commute with the transfer matrix $\mathbf{T}(u)$, they are in some sense generalized symmetries. The seams that admit zero eigenvalues are [noninvertible symmetries](#).
- In the continuum scaling limit, integrable seams become [topological defect lines](#) $\mathcal{L}_a$ which inherit the same fusion rules.
- The new insights from the lattice provide tools to systematically investigate Ocneanu algebras in other A - D - E theories such as superconformal, $\mathbb{Z}_n$ parafermionic and higher rank CFTs.



Belated Happy 80th Birthday wishes Jean-Bernard!