

Cryptozoology of Special Functions

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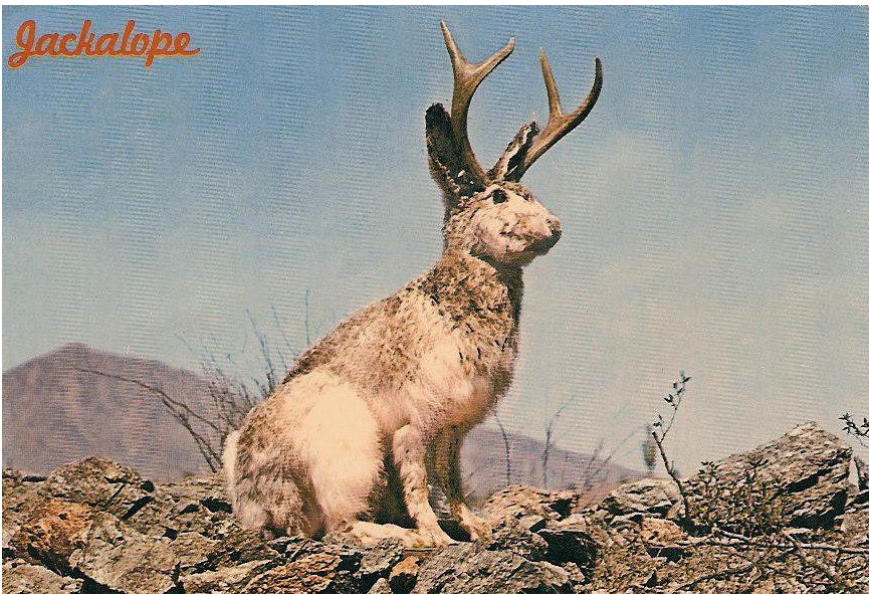
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Joint work with Siddhartha Sahi (Rutgers)

[arXiv:2606.15185](https://arxiv.org/abs/2606.15185)







Chupacabra

Large flaps of skin are sometimes reported at the back of the head. (They can alternately be reported as above and below the head)

EYES
The eyes are large and mean. They are usually reported as being black or bright red.

NOSE
The nose does not protrude. Round nostrils are set on the sides of the snout

MOUTH
Mouth is a thin slit which does not cover the sharp teeth (No Lips)

TONGUE
(Usually reported as one-pronged)

SKIN
Usually described as gray in color, the chupacabra may have scaly, reptilian skin or mangy, patchy fur. It has spines down its back.

(Mangy fur reports refer to strayed or diseased dogs)

COLOR
It's said that the chupacabra can change color like a chameleon. It may look green when seen in sunny clearings or dark brown in the middle of a forest.

(Alternatively, reports have said it was marked in both brown and green splotches at the same time)

TRACKS
Some tracks are reported as three-toed, some as resembling small human hands. Some of the three-toed tracks prove on examination to be Vulture tracks

HANDS
Hands can be reported as either four or five fingered, in the latter case with a thumb and three strongly clawed fingers

MOVEMENT
Reports vary about how the chupacabra gets around. Most agree that it moves on two feet. Some say it jumps like a kangaroo—up to 20 feet (6 m) in a single leap; others even claim it can fly. (Flying reports usually refer to bats)



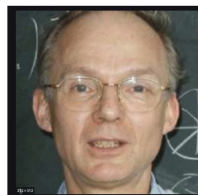
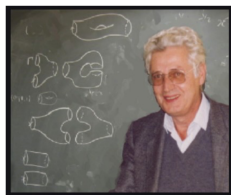
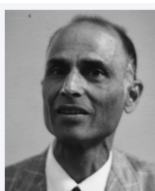


The HCIZ integral

Given $N \times N$ Hermitian matrices A, B , the **HCIZ integral** is

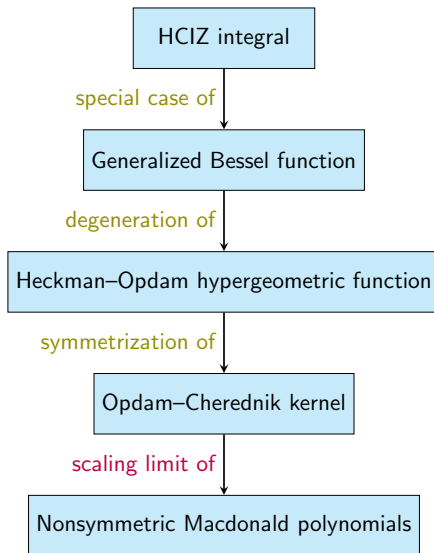
$$I_N(A, B) := \int_{U(N)} e^{\text{Tr}(AUBU^*)} dU = \left(\prod_{j=1}^{N-1} j! \right) \frac{\det [e^{a_i b_j}]_{i,j=1}^N}{\Delta(a)\Delta(b)}$$

where w.l.o.g. we can take $A = \text{diag}(a_1, \dots, a_N)$, $B = \text{diag}(b_1, \dots, b_N)$.

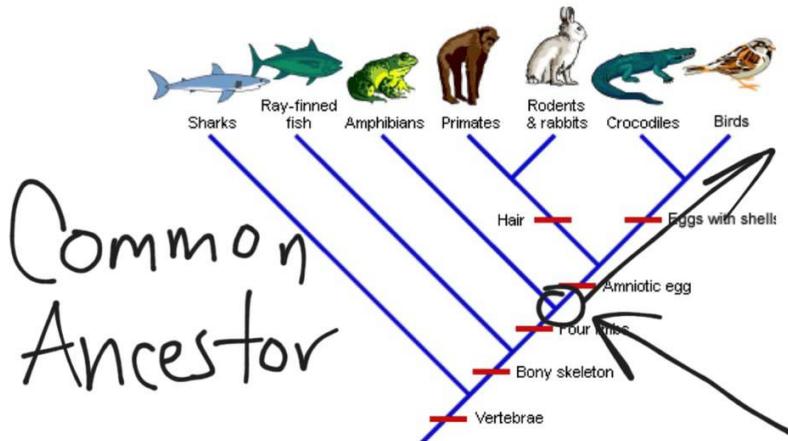


This integral is **ubiquitous** in math and physics (random matrix theory, gauge theory, statistics, representation theory, etc.).

A zoo of special functions

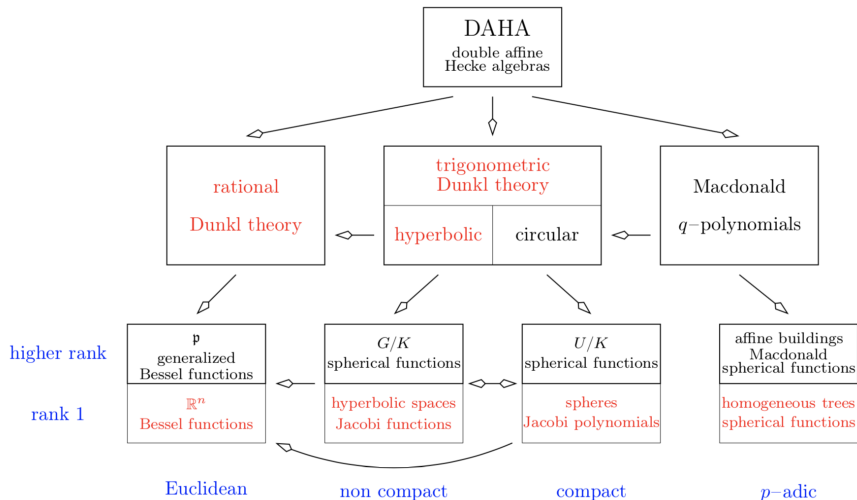


The “family tree” of vertebrates



(Figure: Cobb Learning)

The “family tree” of special functions theories



(Figure: J.-P. Anker, “An Introduction to Dunkl Theory and Its Analytic Aspects”)

Two miracles of HCIZ

From the perspective of special functions, two facts about the HCIZ integral are very strange:

- 1 It admits an **exact formula** (i.e., the integral is a determinant).
- 2 It **exists at all** (i.e., the Bessel function can be expressed as an integral over a compact group).

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Long-standing question: When can special functions in the “vertebrate” taxonomy be written as an integral of *something*?

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Answer (this talk): Always! In fact, they can always be written as the Laplace transform of a probability measure.

What's “special” about special functions?

When we say that a function is “special,” usually we mean that it is

- 1 **Ubiquitous**, and
- 2 **Difficult to study**.

They appear in many applications, so we need to understand them, but they are typically transcendental, with no exact formulas.

Only **indirect** descriptions are available. So we need to be clever.

This is both interesting and annoying.

A classical tool: integral representations

To work with objects like these, we need to relate them to something more tangible.

One of the oldest tricks: Figure out a way to write it as an integral of something that you can actually analyze.

Classical example: Poisson integral for Bessel functions of the first kind:

$$J_{\alpha}(x) = \frac{1}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \left(\frac{x}{2}\right)^{\alpha} \int_{-1}^1 (1 - t^2)^{\alpha - \frac{1}{2}} \cos(xt) dt, \quad \operatorname{Re}(\alpha) > -\frac{1}{2}$$

Interpretation: Up to normalization, $(x/2)^{-\alpha} J_{\alpha}(x)$ is the cosine transform of a measure with density $(1 - t^2)^{\alpha - 1/2}$ on $[-1, 1]$. Easily gives global bounds, asymptotics at infinity, and other properties.

Root systems

For $\alpha \in \mathbb{R}^N$, let s_α be the reflection through $\{x \in \mathbb{R}^N \mid \langle \alpha, x \rangle = 0\}$:

$$s_\alpha x = x - 2 \frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} \alpha, \quad x \in \mathbb{R}^N.$$

A finite set of non-zero vectors $\Phi \subset \mathbb{R}^N$ is called a (crystallographic) *root system* if it satisfies the following properties:

- 1 $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$, and
- 2 $2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

The root system Φ is *reduced* if $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in \Phi$, and is *irreducible* if it cannot be decomposed as $\Phi_1 \oplus \Phi_2$.

Root systems: Examples

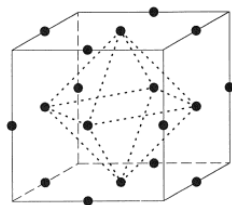
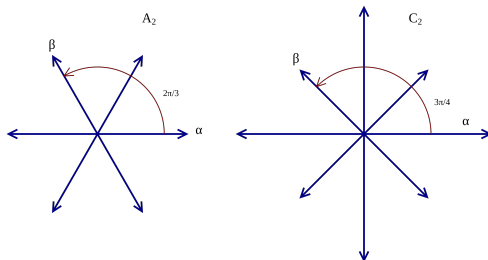
The four classical families of root systems in $\mathbb{R}^N$:

- $A_{N-1} = \{ \pm (e_i - e_j) \mid 1 \leq i < j \leq N \}$
- $D_N = A_{N-1} \cup \{ \pm (e_i + e_j) \mid 1 \leq i < j \leq N \}$
- $B_N = D_N \cup \{ \pm e_i \mid 1 \leq i \leq N \}$
- $C_N = D_N \cup \{ \pm 2e_i \mid 1 \leq i \leq N \}$

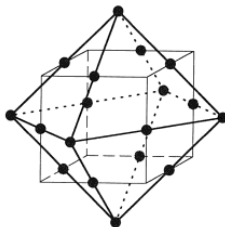
The **rank** of a root system is the dimension of its span.

Every reduced, irreducible root system with rank ≥ 9 is one of the above.

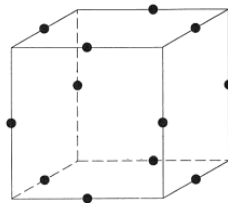
Classical root systems of rank 2 and 3



B_3



C_3



$A_3 = D_3$

(Images: Wikipedia)

A few last definitions

- The **positive roots** are a fixed choice $\Phi^+ \subset \Phi$ such that $\alpha \in \Phi^+ \iff -\alpha \notin \Phi^+$.
- The **Weyl group** W is the group of isometries of $\mathbb{R}^N$ generated by the reflections s_α , $\alpha \in \Phi^+$.
- A **multiplicity parameter** is a function $k : \Phi \rightarrow \mathbb{C}$ such that $k(w \cdot \alpha) = k(\alpha)$ for $w \in W$.
- The **Weyl vector** is a weighted sum of the positive roots:
$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} k(\alpha) \alpha.$$

The Opdam–Cherednik kernel

Given a root system $\Phi \subset \mathbb{R}^N$, a multiplicity parameter k , and a point $\lambda \in \mathbb{C}^N$, we can write down a system of differential-difference equations that define the **Opdam–Cherednik kernel** $G_{k,\lambda} : \mathbb{R}^N \rightarrow \mathbb{C}$.

This is a kind of master special function that subsumes many others.

Via specializations, degenerations, and symmetrizations, it recovers:

- Many classical special functions and orthogonal polynomials in 1D
- Multivariable Bessel and hypergeometric functions
- Multivariable Jacobi polynomials, including Jack polynomials
- Spherical functions on all Riemannian symmetric spaces
- Irreducible characters of compact Lie groups

... and other stuff.

Back to the zoo, and back to HCIZ

- The **Heckman–Opdam hypergeometric function** is the W -symmetric part of the OC kernel:

$$F_{k,\lambda}(x) = \frac{1}{|W|} \sum_{w \in W} G_{k,\lambda}(w(x))$$

- The **generalized Bessel function** is the *rational limit* of the HOHGF:

$$J_{k,\lambda}(x) = \lim_{\varepsilon \rightarrow 0} F_{k,\varepsilon^{-1}\lambda}(\varepsilon x)$$

- The **HCIZ integral** is the GBF with $\Phi = A_{N-1}$ and $k \equiv 1$:

$$J_{1,\lambda}^{A_{N-1}}(x) = \int_{U(N)} e^{\text{Tr}(\Lambda U X U^*)} dU$$

where $\Lambda = \text{diag}(\lambda)$, $X = \text{diag}(x)$.

Spherical integrals as Bessel functions and MGFs

When $\Phi = A_{N-1}$, the Weyl group $W = S_N$ acts transitively on the roots, so k only takes one value. In random matrix theory, we write $k \equiv \beta/2$.

For $\beta = 1, 2, 4$, let $G_\beta = O(N), U(N), \text{USp}(N)$ respectively. Then

$$J_{\beta/2, \lambda}^{A_{N-1}}(x) = \int_{G_\beta} e^{\text{Tr}(\Lambda U X U^*)} dU = \int_{G_\beta} e^{\sum_i \lambda_i (U X U^*)_{ii}} dU$$

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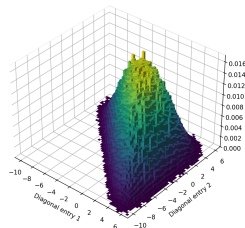
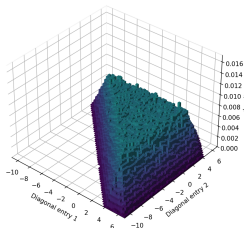
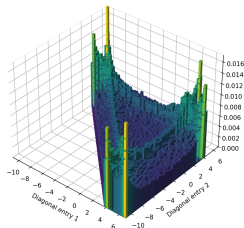
Pushing dU forward to $\mathbb{R}^N$ via $U \mapsto ((UXU^*)_{11}, \dots, (UXU^*)_{NN})$:

$$J_{\beta/2, \lambda}^{A_{N-1}}(x) = \int_{\mathbb{R}^N} e^{\langle \lambda, y \rangle} \sigma_{\beta, x}(dy)$$

where $\sigma_{\beta, x}$ is the distribution of the diagonal entries of UXU^* .

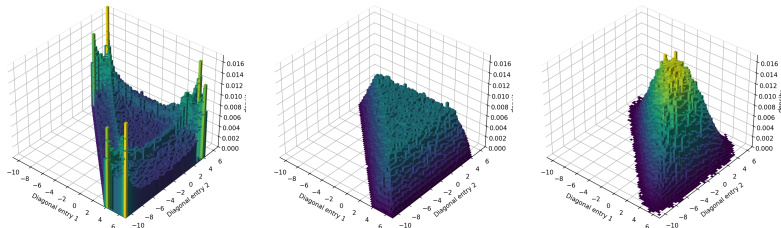
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Diagonal entries of UXU^* for $x = (7, 4, -11)$, $\beta = 1, 2, 4$:



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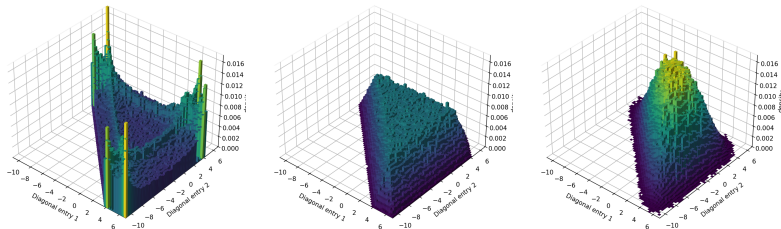
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Interpretation: The OC kernel $G_{k,\lambda}(x)$ is some kind of distant generalization of the MGF of the diagonal entries of a random matrix with fixed spectrum.

But is $G_{k,\lambda}(x)$ actually the MGF of something?

Does a random variable with that MGF actually exist?

Integral representation for the Opdam–Cherednik kernel

Theorem (M.–Sahi, '26)

Fix a root system $\Phi \subset \mathbb{R}^N$ and a nonnegative multiplicity parameter $k : \Phi \rightarrow \mathbb{R}_{\geq 0}$. For each $x \in \mathbb{R}^N$, there is a unique probability measure $\pi_{k,x}$ supported on the convex hull of the orbit $W \cdot x$ such that

$$G_{k,\lambda-\rho}(x) = \int_{\mathbb{R}^N} e^{\langle \lambda, y \rangle} \pi_{k,x}(dy), \quad \lambda \in \mathbb{C}^N.$$

Conjectured for ~ 30 years. Can interpret as a simultaneous extension of:

- Poisson integral formula for the 1D Bessel function
- Harish-Chandra integral formula for spherical functions
- Kostant convexity theorem for the Iwasawa decomposition
- MGF of diagonal entries of UXU^*

Proof sketch

Steps:

- 1 Prove a different theorem expressing the OC kernel as a scaling limit of normalized **nonsymmetric Macdonald polynomials**:

$$G_{k,\lambda}(x) = \lim_{q \rightarrow 1} E_{\lfloor (\log q)^{-1} x \rfloor}^{(q,k)}(\lambda)$$

- 2 The **Ram–Yip formula** gives a positive expansion:

$$E_{\eta}^{(q,k)}(x) = \sum_{\nu \in \text{conv}(W \cdot \eta)} c_{\eta\nu} q^{\langle \nu, x \rangle}, \quad c_{\eta\nu} \geq 0.$$

Rewrite:

$$E_{\eta}^{(q,k)}(\lambda - \rho) = \int_{\mathbb{R}^N} e^{\langle \lambda, y \rangle} \varpi_{q,\eta}(dy)$$

for a discrete probability measure $\varpi_{q,\eta}$ on $\text{conv}(W \cdot \eta)$.

- 3 Set $\eta = \lfloor (\log q)^{-1} x \rfloor$ and take $q \rightarrow 1$. Convergence of MGFs gives existence and uniqueness of the limiting measure.

