

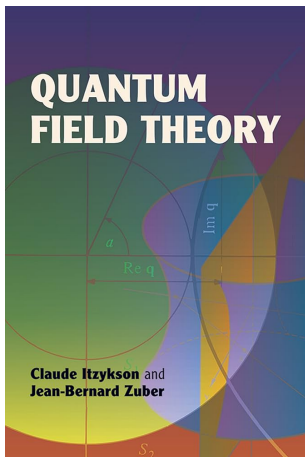
Taking a trip to cumuland with Jean-Bernard

Sylvain Lacroix – LPTHE – Sorbonne Université



Celebrating the 80th birthday of Jean-Bernard Zuber
25th September, 2026

Based on our collaboration 2508.21483



REMARKS ABOUT THE EXISTENCE OF NON-LOCAL CHARGES IN TWO-DIMENSIONAL MODELS

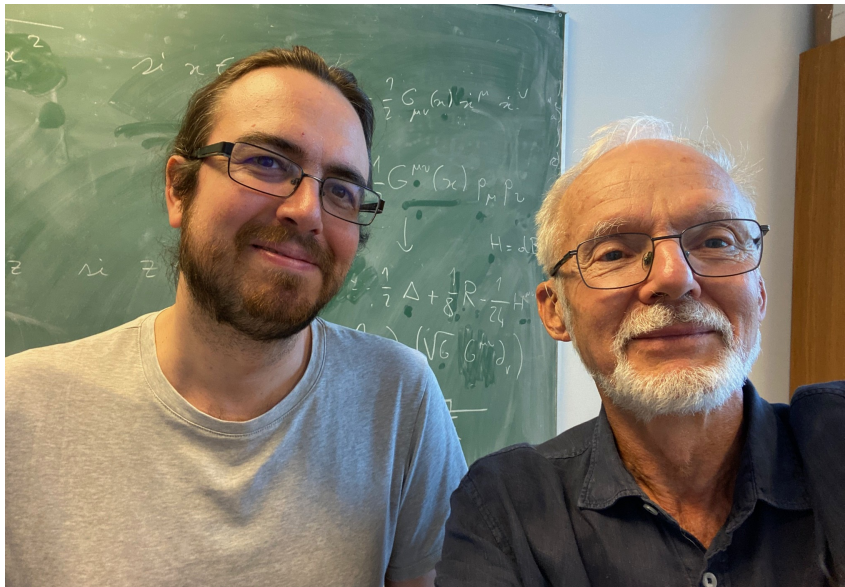
E. BRÉZIN, C. ITZYKSON, J. ZINN-JUSTIN and J.-B. ZUBER

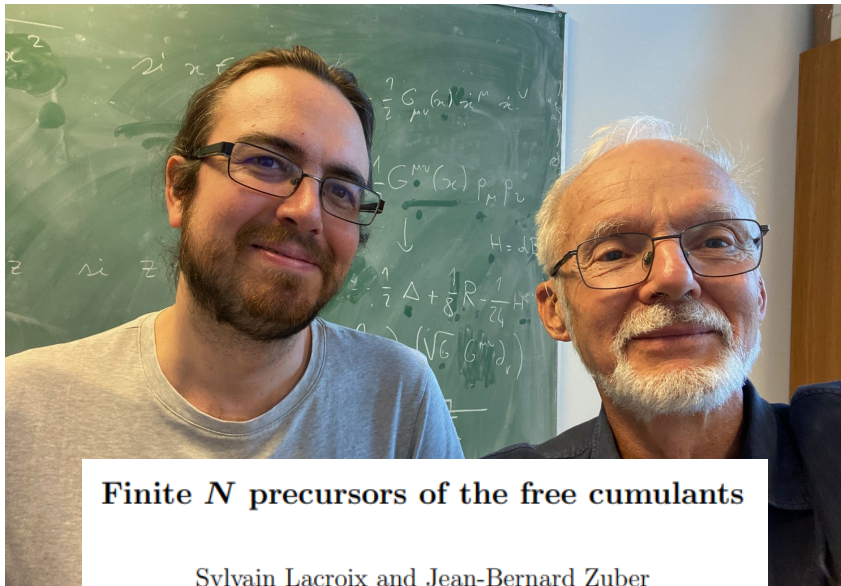
Planar Diagrams

E. Brézin, C. Itzykson, G. Parisi*, and J. B. Zuber

The A-D-E Classification of Minimal and $A_1^{(1)}$ Conformal Invariant Theories

A. Cappelli*, C. Itzykson, and J. B. Zuber





Introduction

Free probability and cumulants

- Standard probability: additivity of cumulants c_d ($d \in \mathbb{Z}_{\geq 1}$)
 a and b independent $\implies c_d(a + b) = c_d(a) + c_d(b)$
- Free probability: non-commutative random variables [Voiculescu '83]
- Freeness and free cumulants κ_d [Speicher '94]:
 a and b free $\implies \kappa_d(a + b) = \kappa_d(a) + \kappa_d(b)$
- Freeness arises in large size limit of random matrices with unitary-invariant distributions [Voiculescu '93]

Free cumulants for matrices

- In this talk: algebraic point of view
- **Free cumulant** κ_d : invariant polynomial of degree d on $N \times N$ matrices
- Naturally expressed in terms of moments $m_k(A) = \frac{1}{N} \text{Tr}(A^k)$

$$\kappa_1(A) = m_1(A), \quad \kappa_2(A) = m_2(A) - m_1^2(A)$$

$$\kappa_3(A) = m_3(A) - 3m_2(A)m_1(A) + m_1^3(A)$$

- Additivity property (averaged over $U(N)$ -orbits ; A, B deterministic):

$$\int_{U(N)} DU \, \kappa_d(A + UBU^{-1}) \approx \kappa_d(A) + \kappa_d(B) \quad \text{as } N \rightarrow \infty$$

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- **Question:** polynomials satisfying this property at finite N ?

Finite N precursors of free cumulants

- Finite precursors of free cumulants [Capitaine Casalis '06]: N -dependent invariant polynomials K_d ($d \leq N$), coinciding with κ_d when $N \rightarrow \infty$
- Additivity property at finite N :

$$\int_{U(N)} DU K_d(A + UBU^{-1}) = K_d(A) + K_d(B)$$

- Main idea of [SL Zuber '25]: alternative route to define K_d from the Harish-Chandra-Itzykson-Zuber integral
→ recover known results but also discover new ones

- 1 Definition and additivity of the precursors from HCIZ
- 2 Large N limit and Hurwitz numbers
- 3 Generating function of the precursors
- 4 Invariant polynomials and sums of conjugacy orbits
- 5 Conclusion

Definition and additivity of the precursors from HCIZ integral

HCIZ integral and precursors

- HCIZ integral [Harish-Chandra '57, Itzykson Zuber '80]:

$$Z(A, B; x) := \int_{U(N)} DU e^{Nx \operatorname{Tr}(AUBU^{-1})}$$

- Converging power series expansion in x
Coefficient of x^d : invariant polynomial of degree d in both A and B
- Free cumulant precursor: “single trace coefficient” of degree $d \leq N$

$$K_d(A) := \frac{d}{N} [x^d \operatorname{Tr}(B^d)] Z(A, B; x)$$

Invariant polynomial in A of degree d

$$K_d(A) = \frac{d}{N} [z^d \text{Tr}(B^d)] Z(A, B; z)$$

- First few precursors:

$$K_1(A) = \frac{1}{N} \text{Tr}(A)$$

$$K_2(A) = \frac{1}{N^2 - 1} (N \text{Tr}(A^2) - \text{Tr}^2(A))$$

$$K_3(A) = \frac{N}{(N^2 - 1)(N^2 - 4)} (N^2 \text{Tr}(A^3) - 3N \text{Tr}(A) \text{Tr}(A^2) + 2\text{Tr}^3(A))$$

$$K_4(A) = \frac{N^2}{(N^2 - 1)(N^2 - 4)(N^2 - 9)} (N(N^2 + 1) \text{Tr}(A^4) - 4(N^2 + 1) \text{Tr}(A) \text{Tr}(A^3) \\ - (2N^2 - 3) \text{Tr}^2(A^2) + 10N \text{Tr}^2(A) \text{Tr}(A^2) - 5\text{Tr}^4(A))$$

$$Z(A, C; x) := \int_{U(N)} DU e^{N x \operatorname{Tr}(AUCU^{-1})}$$

- Multiplicativity of the HCIZ integral:

$$\int_{U(N)} DU Z(A + UBU^{-1}, C; x) = Z(A, C; x) Z(B, C; x)$$

Additivity of the precursors

$$Z(A, C; x) := \int_{U(N)} DU e^{N x \operatorname{Tr}(AUCU^{-1})} = 1 + \frac{N}{d} K_d(A) \operatorname{Tr}(C^d) x^d + \dots$$

- Multiplicativity of the HCIZ integral:

$$\int_{U(N)} DU Z(A + UBU^{-1}, C; x) = Z(A, C; x) Z(B, C; x)$$

- Extract single trace coefficients: **additivity of K_d**

$$\int_{U(N)} DU K_d(A + UBU^{-1}) = K_d(A) + K_d(B)$$

- Generalised precursor K_α (of degree $d \leq N$):

$$K_\alpha(A) := \frac{\prod_k k^{\hat{\alpha}_k} \hat{\alpha}_k!}{N^\ell} [x^d \operatorname{Tr}(B^{\alpha_1}) \cdots \operatorname{Tr}(B^{\alpha_\ell})] Z(A, B; x)$$

- Labelled by partition $\alpha = (\alpha_1, \dots, \alpha_\ell) \vdash d$ ($\alpha_1 \geq \dots \geq \alpha_\ell \geq 1$)

$$\alpha_1 + \dots + \alpha_\ell = d$$

Degree $d(\alpha) = d$, length $\ell(\alpha) = \ell$

- Multiplicity $\hat{\alpha}_k$: number of times k appears in α
- For length 1 partition $\alpha = (d)$: precursor $K_{(d)}(A) = K_d(A)$

Generalised precursors

$$K_{\alpha}(A) := \frac{\prod_k k^{\hat{\alpha}_k} \hat{\alpha}_k!}{N^{\ell}} [x^d \operatorname{Tr}(B^{\alpha_1}) \cdots \operatorname{Tr}(B^{\alpha_{\ell}})] Z(A, B; x)$$

- $\{K_{\alpha}\}_{\alpha \vdash d}$ basis of invariant polynomials of degree $d \leq N$
- Examples for degree 2:

$$K_{(2)}(A) = \frac{1}{N^2 - 1} (N \operatorname{Tr}(A^2) - \operatorname{Tr}^2(A))$$

$$K_{(1,1)}(A) = -\frac{1}{N(N^2 - 1)} (\operatorname{Tr}(A^2) - N \operatorname{Tr}^2(A))$$

Generalised precursors

$$K_{\alpha}(A) := \frac{\prod_k k^{\hat{\alpha}_k} \hat{\alpha}_k!}{N^{\ell}} [x^d \operatorname{Tr}(B^{\alpha_1}) \cdots \operatorname{Tr}(B^{\alpha_{\ell}})] Z(A, B; x)$$

- $\{K_{\alpha}\}_{\alpha \vdash d}$ basis of invariant polynomials of degree $d \leq N$
- Examples for degree 2:

$$K_{(2)}(A) = \frac{1}{N^2 - 1} (N \operatorname{Tr}(A^2) - \operatorname{Tr}^2(A))$$

$$K_{(1,1)}(A) = -\frac{1}{N(N^2 - 1)} (\operatorname{Tr}(A^2) - N \operatorname{Tr}^2(A))$$

- General expressions of K_{α} in terms of $\operatorname{Tr}(A^k)$ or Schur polynomials
 - representation theory of $U(N)$ and S_d , Schur-Weyl duality
 - matches with “matrix cumulants” of [Casalis Capitaine '06 '08]

Large N limit and Hurwitz numbers

- **Moment** (normalised trace):

$$m_k(A) := \frac{1}{N} \text{Tr}(A^k)$$

- Sequence $(A_N)_{N \in \mathbb{Z}_{\geq 1}}$ with limiting eigenvalue distribution as $N \rightarrow \infty$
 $\rightarrow$ moments $m_k(A_N)$ stay finite
- In this talk, **large N limit** of invariant polynomial $f(A)$:
 - rewrite $f(A)$ in terms of $m_k(A)$, with N -dependent coefficients
 - **take the limit $N \rightarrow \infty$ while keeping $m_k(A)$ finite**

Precursors in terms of moments

- Precursors in terms of moments:

$$K_1 = m_1$$

$$K_2 = \frac{N^2}{N^2 - 1} (m_2 - m_1^2)$$

$$K_3 = \frac{N^4}{(N^2 - 1)(N^2 - 4)} (m_3 - 3m_2m_1 + 2m_1^3)$$

$$K_4 = \frac{N^4}{(N^2 - 1)(N^2 - 4)(N^2 - 9)} \\ ((N^2 + 1)(m_4 - 4m_3m_1 - 2m_2^2 + 10m_2m_1^2 - 5m_1^4) + 5(m_2 - m_1^2)^2)$$

$$K_5 = \dots$$

Large N limits of precursors and free cumulants

- Large N limits of first precursors:

$$\lim_{N \rightarrow \infty} K_1 = \kappa_1 := m_1$$

$$\lim_{N \rightarrow \infty} K_2 = \kappa_2 := m_2 - m_1^2$$

$$\lim_{N \rightarrow \infty} K_3 = \kappa_3 := m_3 - 3m_2m_1 + 2m_1^3,$$

$$\lim_{N \rightarrow \infty} K_4 = \kappa_4 := m_4 - 4m_3m_1 - 2m_2^2 + 10m_2m_1^2 - 5m_1^4$$

Large N limits of precursors and free cumulants

- Large N limits of first precursors:

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$$\lim_{N \rightarrow \infty} K_4 = \kappa_4 := m_4 - 4m_3m_1 - 2m_2^2 + 10m_2m_1^2 - 5m_1^4$$

- More generally, from HCIZ [Itzykson Zuber '80, Zinn-Justin '02, Guionnet Maïda '05, Collins Sniady '05, ...]:

$$\lim_{N \rightarrow \infty} K_d = \kappa_d := \sum_{\beta \vdash d} (-1)^{1+\ell(\beta)} \frac{(d + \ell(\beta) - 2)!}{(d-1)! \prod_{k=1}^n \hat{\beta}_k!} m_{\beta_1} \cdots m_{\beta_{\ell(\beta)}}$$

κ_d free cumulant of degree d

Topological expansion of generalised precursors

- $\frac{1}{N}$ corrections as $N \rightarrow \infty$?
- “Topological” expansion (see Alice’s talk) of the generalised precursors:

$$K_\alpha = \frac{\prod_k k^{\hat{\alpha}_k} \hat{\alpha}_k!}{d(\alpha)!} \sum_{g \geq 1 - \ell(\alpha)} N^{2(1 - \ell(\alpha) - g)} \left(\sum_{\beta \vdash d} (-1)^{\ell(\alpha) + \ell(\beta)} H_g^{\bullet, \leq}(\alpha, \beta) m_\beta \right)$$

- $H_g^{\bullet, \leq}(\alpha, \beta)$: disconnected weakly monotone double Hurwitz number of type (α, β) and genus g (counts certain covers $\Sigma_g \rightarrow \mathbb{CP}^1$ of degree d)
- HCIZ: generating func of $H_g^{\bullet, \leq}(\alpha, \beta)$ [Goulden Guay-Paquet Novak '11]

Moments-cumulant relation in free probability

- Inverted relation between moments and generalised precursors:

$$m_d = \sum_{g \geq 0} N^{-2g} \sum_{\alpha \vdash d} P_g(\alpha) K_\alpha$$

$P_g(\alpha) := | \text{permutations } \sigma \in S_d \text{ of cycle type } \alpha \text{ and genus } g |$

- Large N limit recovers moment-cumulant relation of free probability:

$$m_d = \sum_{\alpha \vdash d} P_0(\alpha) \kappa_\alpha$$

$P_0(\alpha) = | \text{non-crossing partitions of } \{1, \dots, d\} \text{ by subsets of size } \alpha_i |$

- Counting permutations/partitions by genus and type [Kreweras '72, Cori '75, Harer Zagier '86, Cori Hetyei '13, Coquereaux Zuber '23, Hock '23]

Generating function of the precursors

Generating function of the precursors

- Generating function of K_d :

$$\begin{aligned}\mathcal{K}(A; x) &:= \frac{1}{N} \int_{U(N)} DU e^{N \text{Tr}(AU^{-1})} \text{Tr} \left(\frac{1}{1 - xU} \right) \\ &= 1 + \sum_{d=1}^N x^d K_d(A) + O(x^{N+1})\end{aligned}$$

- Proof: extract K_d from HCIZ by orthogonality of Newton polynomials
- Consequence: $\frac{\mathcal{K}(A; x) - 1}{x} \xrightarrow{N \rightarrow \infty} \text{Voiculescu } \mathcal{R}\text{-transform}$

Invariant polynomials and sums of conjugacy orbits

Algebra of invariant polynomials

- $\mathcal{A}$: graded algebra of **invariant polynomials on $N \times N$ matrices**:

$$\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}_d, \quad \mathcal{A}_d \cdot \mathcal{A}_{d'} \subset \mathcal{A}_{d+d'}$$

$\mathcal{A}_d$: degree d polynomials

- **Netwon basis $\{p_\alpha\}_{\alpha \in \mathcal{P}}$** : $(p_\emptyset = 1)$

$$p_\alpha(A) = \text{Tr}(A^{\alpha_1}) \cdots \text{Tr}(A^{\alpha_\ell}), \quad \mathcal{P} = \left\{ \text{partitions } \alpha \mid \alpha_i \leq N \right\}$$

- Structure constants:

$$p_\alpha p_\beta = \sum_{\gamma \in \mathcal{P}} \Pi_{\alpha\beta}^\gamma p_\gamma$$

- Average of $f \in \mathcal{A}_d$ over sum of $U(N)$ -conjugacy orbits:

$$\Delta f(A, B) := \iint_{U(N)^2} DU DV f(UAU^{-1} + VB V^{-1}) = \int_{U(N)} DU f(A + UBU^{-1})$$

Sum of products of invariant polynomials in A and B

- Defines **orbit coproduct** $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$

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Sum of products of invariant polynomials in A and B

- Defines **orbit coproduct** $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ ($(\mathcal{A}, \cdot, \Delta)$ not a bialgebra)
 - Graded: $\Delta(\mathcal{A}_d) = \bigoplus_{k=0}^d \mathcal{A}_k \otimes \mathcal{A}_{d-k}$
 - Cocommutative: $\Delta f(A, B) = \Delta f(B, A)$
 - Coassociative: $(\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta = \Delta$

$$\Delta f(A, B, C) = \iiint_{U(N)^3} DU DV DW f(UAU^{-1} + VB V^{-1} + WC W^{-1})$$

- Average of $f \in \mathcal{A}_d$ over sum of $U(N)$ -conjugacy orbits:

$$\Delta f(A, B) := \iint_{U(N)^2} DU DV f(UAU^{-1} + VBV^{-1}) = \int_{U(N)} DU f(A + UBU^{-1})$$

Sum of products of invariant polynomials in A and B

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 - Coassociative: $(\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta = \Delta$
- Precursors are primitive elements: $\Delta K_d = K_d \otimes 1 + 1 \otimes K_d$

Orbit coproduct in dual Newton basis

- Dual Newton polynomials $\{p_\star^\alpha\}_{\alpha \in \mathcal{P}}$ from HCIZ integral:

$$Z(A, B; x) = \sum_{\alpha \in \mathcal{P}} x^{d(\alpha)} p_\star^\alpha(A) p_\alpha(B)$$

- For degree $d(\alpha) \leq N$, proportional to generalised precursor:

$$p_\star^\alpha = \frac{N^{\ell(\alpha)}}{\prod_k k^{\hat{\alpha}_k} \hat{\alpha}_k!} K_\alpha$$

- Theorem:** orbit coproduct in dual Newton basis

$$\Delta p_\star^\gamma = \sum_{\alpha, \beta \in \mathcal{P}} \Pi_{\alpha\beta}^\gamma p_\star^\alpha \otimes p_\star^\beta$$

Proof: $\int_{U(N)} DU Z(A + UBU^{-1}, C; x) = Z(A, C; x) Z(B, C; x)$

$$Z(A, B; x) = \sum_{\alpha, \beta \in \mathcal{P}} x^{d(\alpha)} \eta^{\alpha\beta} p_{\alpha}(A) p_{\beta}(B) = \sum_{\alpha, \beta \in \mathcal{P}} x^{d(\alpha)} \eta_{\alpha\beta} p_{\star}^{\alpha}(A) p_{\star}^{\beta}(B)$$

- Symmetric tensors $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$ (N -dependent):
 - Block-diagonal: $\eta^{\alpha\beta} = \eta_{\alpha\beta} = 0$ if $d(\alpha) \neq d(\beta)$
 - Inverse: $\sum_{\gamma \in \mathcal{P}} \eta^{\alpha\gamma} \eta_{\gamma\beta} = \delta^{\alpha}_{\beta}$
 - Expressions in terms of S_d -characters or Weingarten coefficients
 - Gen func for weakly/strictly monotone disconnected Hurwitz numbers
- Scalar product: $\eta(p_{\alpha}, p_{\beta}) := \eta_{\alpha\beta}$ ($\{p_{\alpha}\}_{\alpha \in \mathcal{P}}, \{p_{\star}^{\alpha}\}_{\alpha \in \mathcal{P}}$ dual bases)
- **Corollary:** $\eta(\cdot, \cdot)$ induces algebra isomorphism $(\mathcal{A}^{\star}, \Delta^{\star}) \longrightarrow (\mathcal{A}, \cdot)$

Conclusion

Conclusion: finite N precursors of the free cumulants

- **Finite N precursors of the free cumulants** from HCIZ integral:
 - additive with respect to average over sums of $U(N)$ -orbits
 - tend to free cumulants at large N
 - $\frac{1}{N^2}$ -corrections in terms of monotone Hurwitz numbers
- Other points discussed in [SL Zuber '25]:
 - expansions in terms of Newton and Schur polynomials
 - average over GUE, relation to matrix integrals, Horn problem ...
 - probabilistic aspects and relation with [Collins Gurau Lionni '24]
 - no finite N convolution à la [Marcus Spielman Srivastava '15] associated with K_d
- **Future perspectives:**
 - other classical groups than $U(N)$?
 - deformations, quantum group?

Conclusion: orbit coproduct

- **Orbit coproduct:** behaviour of general invariant polynomials with respect to averaging over sums of $U(N)$ -orbits
- Perspective: **more general orbit coproduct for V rep of G** (compact)
- $\Delta_V : P(V)^G \rightarrow P(V)^G \otimes P(V)^G$ defined by

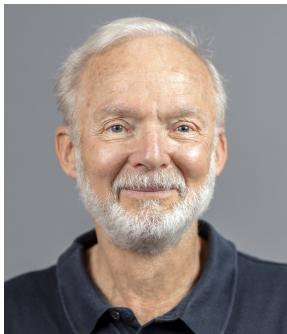
$$\Delta_V f(a, b) := \int_G DU f(a + U.b), \quad a, b \in V$$

- Study Δ_V by replacing HCIZ integral by

$$Z_V(a, b; x) := \int_G DU e^{x \langle a, U.b \rangle}$$

$$\int_G DU Z_V(a + U.b, c; x) = Z_V(a, c; x) Z_V(b, c; x)$$

Thank you for your attention
and happy birthday Jean-Bernard!



Representation theory of symmetric and unitary groups

- Permutation σ in **symmetric group** S_d with cycle type $[\sigma] \vdash d$:

$$[\sigma] = (\alpha_1, \dots, \alpha_\ell) \iff \sigma \text{ has } \ell \text{ cycles of sizes } \alpha_i$$

- Characterises conjugacy classes:

$$\text{Cl}_\alpha = \{\sigma \in S_d \text{ with cycle type } [\sigma] = \alpha\}, \quad |\text{Cl}_\alpha| = \frac{d!}{\prod_k k^{\hat{\alpha}_k} \hat{\alpha}_k!}$$

- **Irreducible reps** $V_\lambda^{S_d}$ ($\lambda \vdash d$), with **characters** $\chi_\lambda(\alpha)$
- **Irreducible polynomial reps** $V_\lambda^{U(N)}$ ($\ell(\lambda) \leq N$) of **unitary group** $U(N)$
- Character: **Schur polynomial** $s_\lambda(A)$, of degree $d(\lambda)$
→ form a basis of invariant polynomials on $N \times N$ matrices

Generalised precursors in terms of Newton polynomials

- Newton polynomials $p_\alpha(A) := \text{Tr}(A^{\alpha_1}) \cdots \text{Tr}(A^{\alpha_\ell})$
- Generalised precursor $K_{[\sigma]}$ ($\sigma \in S_d$) in Newton basis:

$$K_{[\sigma]}(A) = N^{d-\ell([\sigma])} \sum_{\tau \in S_d} \text{Wg}([\sigma\tau^{-1}]) p_{[\tau]}(A)$$

- Weingarten coefficient of $\alpha \vdash d$ (N -dependent):

$$\text{Wg}(\alpha) := \frac{1}{d!} \sum_{\substack{\lambda \vdash d \\ \ell(\lambda) \leq N}} \frac{(\dim V_\lambda^{S_d})^2}{\dim V_\lambda^{U(N)}} \chi_\lambda(\alpha)$$

[Weingarten '78, Samuel '80, Collins '02]

- $K_{[\sigma]}(A)$: convolution of $\sigma \mapsto \text{Wg}([\sigma])$ and $\sigma \mapsto p_{[\sigma]}(A)$ in $\mathbb{C}[S_d]$
→ recovers matrix cumulants of [Casalis Capitaine '06 '08]

Generalised precursors in terms of Schur polynomials

- Frobenius-Schur relation: Newton $\leftrightarrow$ Schur polynomials

$$p_\alpha(A) = \sum_{\substack{\lambda \vdash d \\ \ell(\lambda) \leq N}} \chi_\lambda(\alpha) s_\lambda(A)$$

- Generalised precursor K_α ($\alpha \vdash d$) in Schur basis:

$$K_\alpha(A) = \sum_{\substack{\lambda \vdash d \\ \ell(\lambda) \leq N}} D_\lambda \chi_\lambda(\alpha) s_\lambda(A), \quad D_\lambda := \frac{\dim V_\lambda^{S_d}}{d! \dim V_\lambda^{U(N)}}$$

- Precursor K_d from hook partitions $\lambda_{d,t} = (d-t, 1, \dots, 1)$:

$$K_d(A) = \sum_{t=0}^{d-1} (-1)^t D_{\lambda_{d,t}} s_{\lambda_{d,t}}(A), \quad D_{\lambda_{d,t}} = \frac{N^{d-1} (N-t-1)!}{(N+d-t-1)!}$$

Topological expansion of generalised precursors

- “Topological” expansion of the generalised precursors as $N \rightarrow \infty$:

$$K_\alpha = \frac{\prod_k k^{\hat{\alpha}_k} \hat{\alpha}_k!}{d(\alpha)!} \sum_{g \geq 1 - \ell(\alpha)} N^{2(1 - \ell(\alpha) - g)} \left(\sum_{\beta \vdash d} (-1)^{\ell(\alpha) + \ell(\beta)} H_g^{\bullet, \leq}(\alpha, \beta) m_\beta \right)$$

- $H_g^{\bullet, \leq}(\alpha, \beta)$: disconnected weakly monotone double Hurwitz number of type (α, β) and genus g
- Counts covers $\Sigma_g \rightarrow \mathbb{CP}^1$ of degree d with:
 - Σ_g surface of genus g (potentially disconnected)
 - 2 branch points with ramification profiles α and β
 - $r = 2g - 2 - \ell(\alpha) - \ell(\beta)$ simple branch points with monodromies $(a_i b_i)$ such that $a_i < b_i$ and $b_1 \leq \dots \leq b_r$
- HCIZ generating func of $H_g^{\bullet, \leq}(\alpha, \beta)$ [Goulden Guay-Paquet Novak '11]

Average of generalised precursors over Gaussian weight

- Gaussian Unitary Ensemble:

$$\mathbb{E}_{A \sim \text{GUE}(N, \sigma)}(f(A)) := \frac{1}{Z_{\text{GUE}}} \int DA f(A) e^{-\frac{1}{2\sigma^2} N \text{Tr}(A^2)}$$

- Wick-like relation for generalised precursors:

$$\mathbb{E}_{A \sim \text{GUE}(N, \sigma)}(K_\alpha(A)) = \delta_{\alpha, [2^d]} \sigma^{2d}$$

Similar to ordinary/free cumulants with normal/semi-circle law

- Proof: compute $\mathbb{E}_A(Z(A, B; x))$ (Fourier transform of Gaussian dist)

Orbit coproduct in Schur basis

- Schur basis of $\mathcal{A}$: $\{s_\lambda\}_{\ell(\lambda) \leq N}$
- Self-dual up to content coefficient C_λ : $\eta(s_\lambda, s_\mu) = C_\lambda \delta_{\lambda\mu}$

$$Z(A, B; x) = \sum_{\ell(\lambda) \leq N} \frac{x^{d(\lambda)}}{C_\lambda} s_\lambda(A) s_\lambda(B), \quad C_\lambda := \frac{d(\lambda)! \dim V_\lambda^{U(N)}}{N^{\ell(\lambda)} \dim V_\lambda^{S_d}}$$

- **Corollary:** orbit coproduct in Schur basis

$$\Delta s_\lambda = \sum_{\ell(\mu), \ell(\nu) \leq N} \frac{C_\lambda}{C_\mu C_\nu} c_{\mu\nu}^\lambda s_\mu \otimes s_\nu$$

Littlewood-Richardson coefficients: $s_\mu s_\nu = \sum_{\ell(\lambda) \leq N} c_{\mu\nu}^\lambda s_\lambda$