

NIMREPS, old and new

Contribution to the conference « Jean-Bernard is eighty »

Robert Coquereaux

The landscape

An affine semi-simple Lie algebra A at some level k

A modular invariant Z

Remember famous papers...

A. Cappelli, C. Itzykson, J.-B. Zuber classification of $SU(2)$ modular invariant partition functions (ADE) – 1986

T. Gannon classification of $SU(3)$ modular invariant partition functions – 1992

P. Di Francesco, J.-B. Zuber classification of $SU(3)$ type graphs (generalized ADE) – 1998

V.B. Petkova, J.-B. Zuber definition of generalised twisted partition functions – 2000

A. Ocneanu Paths on Coxeter diagrams < 2000, Classification of subgroups of quantum $SU(N)$ – 2000

A talk with two parts

Generalities and old examples

Nimreps and quantum symmetries for an exotic example

An exotic exceptional example associated with F_4 at level 6

B. Schellekens, S. Yankielowicz 1989 Not new !

An email from Jean-Bernard (november 2025)

New developments : Nimreps and quantum symmetries for this exotic case

More precisely:

Part 1 General methods and a few examples, with no attempt to give references !

Part 2 An exotic exceptional example associated with F_4 at level 6

The modular invariant was written by B. Schellekens, S. Yankielowicz 1989 (not new !)

It cannot be obtained from a conformal embedding of affine Lie algebras

It can be constructed from a Vertex Operator Algebra (VOA). C.H. Lam & X. Lin, 2018

It is associated with a « quantum subgroup » (see discussion T. Gannon, 2023-2025)

Jean-Bernard wanted to see the Nimreps (November 2025)

Part 2 - New stuff (this talk) :

Generalized twisted partition functions (defect lines = toric matrices)

The Ocneanu graphs for this exotic modular invariant of $(F_4, 9)$

Two (explicit) Nimreps : one quantum subgroup, one quantum module

Explain the title of the talk... What are Nimreps ?

Consider the family of representations (up to equivalence) of a semi-simple Lie algebra, « truncated at level k » (a non-negative integer).

Such a family of representations can be properly defined within the theory of affine Lie algebras.

Multiplication of simple objects (irreducible representations, called irreps — there is a finite number of them) is encoded by a family of finite size matrices called fusion matrices. They generate a ring, or an algebra, called the Verlinde algebra. It will be called A . The fusion matrices have non-negative integer entries.

A « theory » is characterized by the choice of a module M for the ring A . It is encoded by a family of finite size matrices that represent this ring. They have non-negative integer entries. M can be A itself.

First definition of Nimreps (stricto sensu) : the choice of M .

Second definition of Nimreps (lato sensu) : any family of matrices with non-negative integer entries that represent the fusion ring.

Classical analogy

Think of A as the family of representations of a finite group

Think of M as the family of representations of a subgroup

A more precise definition uses category theory (it will be mentioned later)

Comments, generalities
and examples

What happened in the last 20-25 years ?

- The underlying mathematical problem was better understood
- New classifications (of modular invariants, and of graphs) were obtained
- « Classifying modular invariants » is not the same as « classifying graphs » (or modules)
- Generalizations of the modular invariant partition function Z were introduced (toric matrices)
- The role/meaning of the « algebra of quantum symmetries » (Ocneanu) was better understood
- Many other things...

The underlying mathematical description was better understood

(in particular, in the framework of category theory)

Given an affine Lie algebra A at level k

- (A,k) determines a representation of $SL(2,\mathbb{Z})$: generators S and T

What comes first ?

The quantum module (described by Nimreps) ?

The modular invariant ?

(several quantum modules may be associated with the same modular invariant)

Conceptually

It is easier to start from a « given » quantum module

In practice

Start from a modular invariant

Where does the modular invariant Z come from ?

- (A,k) determines a modular invariant which is diagonal : $(1,1,1, \dots, 1)$.

Diagonal case (A) :

- (A,k) determines the Nimreps of the diagonal case (the module is itself !)

Orbifold cases (D) : « Quotient and/or « blow-up » of the A case.

Exceptional cases (E) :

Conformal embedding of (A,k) into a larger affine algebra (B,1)

Conformal embedding followed by contraction

rank-level duality, automorphisms

Exotic cases :

Conformal embedding in a VOA (?)

Check commutation
of Z with S and T

We start with an affine Lie algebra A at level k

Once we have a modular invariant Z

How to get the « quantum module » (or the Nimreps, or the graphs) ?

Several ad hoc techniques

CAF = « Computed Aided Flair » (P. di F.)

Modular splitting technique

In a nutshell...

$Z = W(0) \rightarrow$ (solve eqs) $\rightarrow$ toric matrices $W(x) \rightarrow$ (solve eqs) $\rightarrow$
 $\rightarrow$ matrices $L(\lambda)$ and $R(\lambda)$ that describe how A acts on an algebra called
« algebra of quantum symmetries O ».

The adjacency graphs of $L(\lambda)$ or of $R(\lambda)$ have usually several connected components from which one can obtain one or several « quantum modules » (Nimreps and their graphs).

There are also two types of (quantum) modules : the « quantum subgroups » (type I), which have « self-fusion », and those that are only « quantum modules » (type II).

Examples of conformal embeddings

$SU(2)$, level 10 into $(Spin(5), 1)$

$SU(2)$, level 28 into $(G(2), 1)$

$SU(3)$, level 5 into $(Spin(8), 1)$

$SU(3)$, level 9 into $(E(6), 1)$

$SU(3)$, level 21 into $(E(7), 1)$

$G(2)$, level 4 into $(Spin(14), 1)$

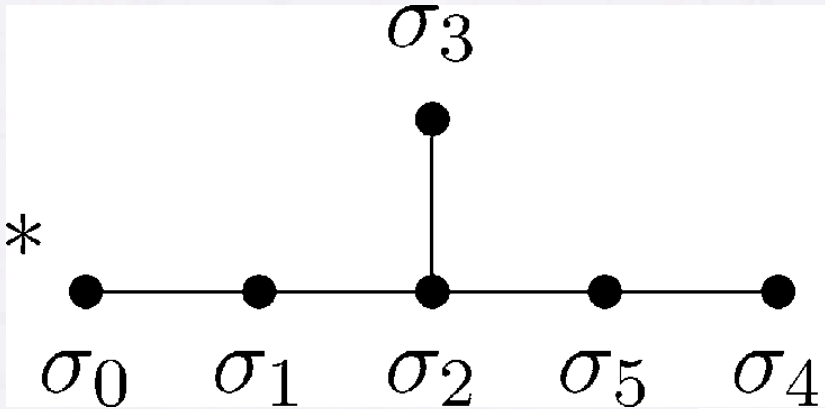
$SU(2) \times SU(3)$ level (16, 6) into $(E_8, 1)$

$SU(4)$, level 4 into $(Spin(15), 1)$

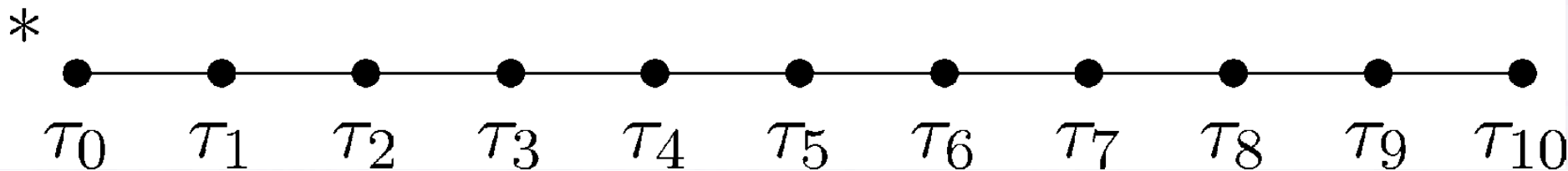
$SU(4)$, level 6 into $(SU(10), 1)$

$SU(4)$, level 8 into $(Spin(20), 1)$

Example of an (exceptional) module of SU2 at level 10



$$E_6 = \mathcal{E}_{10}$$

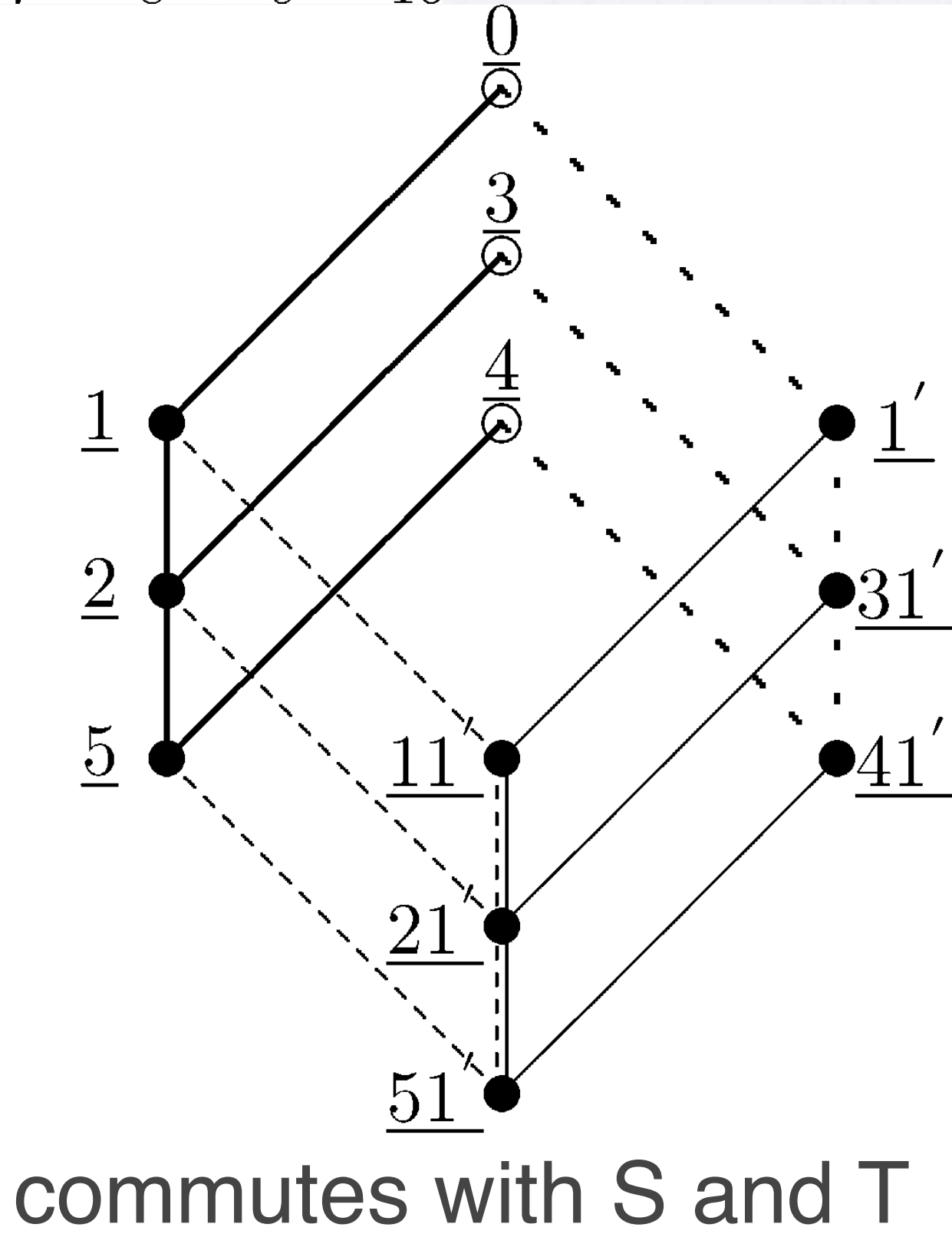


$$A_{11} = \mathcal{A}_{10}$$

$$q = e^{i\pi/(10+2)}$$

$$\mathcal{O} = Oc(E_6)$$

$$W_0 = \begin{pmatrix} 1 & . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . & 1 \\ 1 & . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . & 1 \end{pmatrix}$$



Modules for SU(3) level 9

Discover

$\mathcal{E}_9$ and a module $\mathcal{M}_9$

The algebra $\mathcal{O}$ of quantum symmetries

Only draw
the left graph
72 generators

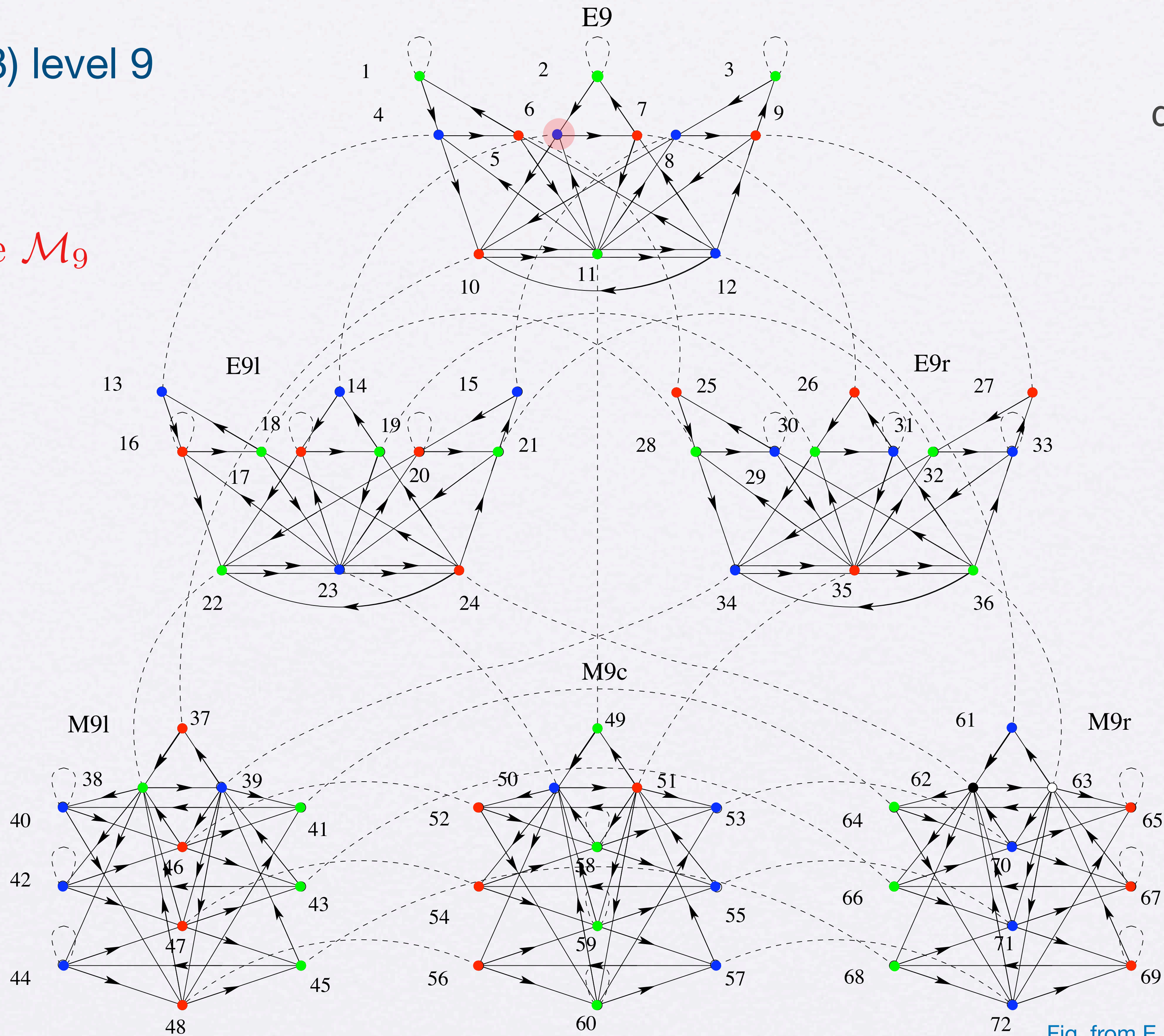


Fig. from E. Isasi, G. Schieber & R.C. in 2006

G2, level 4

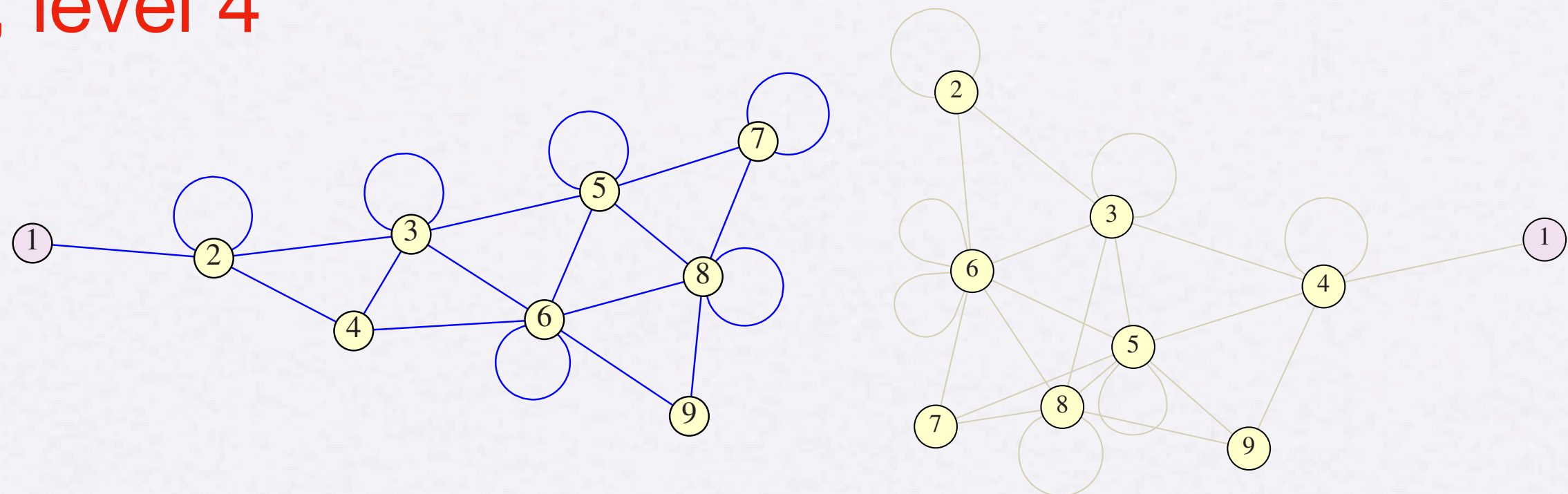
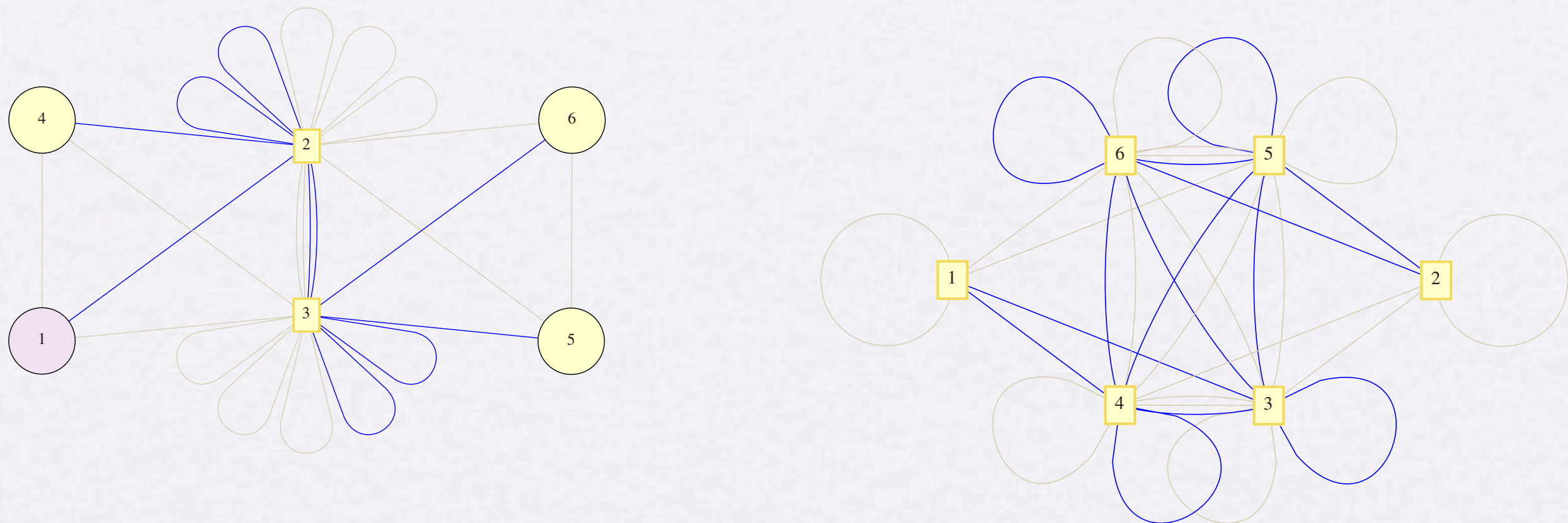


FIG. 17. (Color online) Fusion graphs for $\mathcal{A}_4(G_2)$. (a) $\mathcal{A}_4(G_2)$, rep. $\{0, 1\}$. (b) $\mathcal{A}_4(G_2)$, rep. $\{1, 0\}$.

Adjacency graphs for (0,1) and (1,0) at level 4 in G2



Quantum symmetries O
L and R graphs for $\lambda = (0,1)$

$Z = 1^2 + 1^2 + (1^2 + 1^2)$ in Spin(14), level 1 gives

$$Z = (\{0,0\} + \{0,3\})^2 + (\{0,4\} + \{1,0\})^2 + 2\{1,1\}^2, \quad M = \begin{pmatrix} 1 & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & 1 & . & . \\ 1 & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & 2 & . & . & . \\ . & . & . & 1 & . & . & 1 & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix}.$$

Exceptional quantum subgroup E4 and quantum module M4 for G2 at level 4

Fig. from R. Rais, E.H.. Tahrie & R.C. in 2010

A quantum subgroup of $(SU(2,16) \times (SU(3,6))$

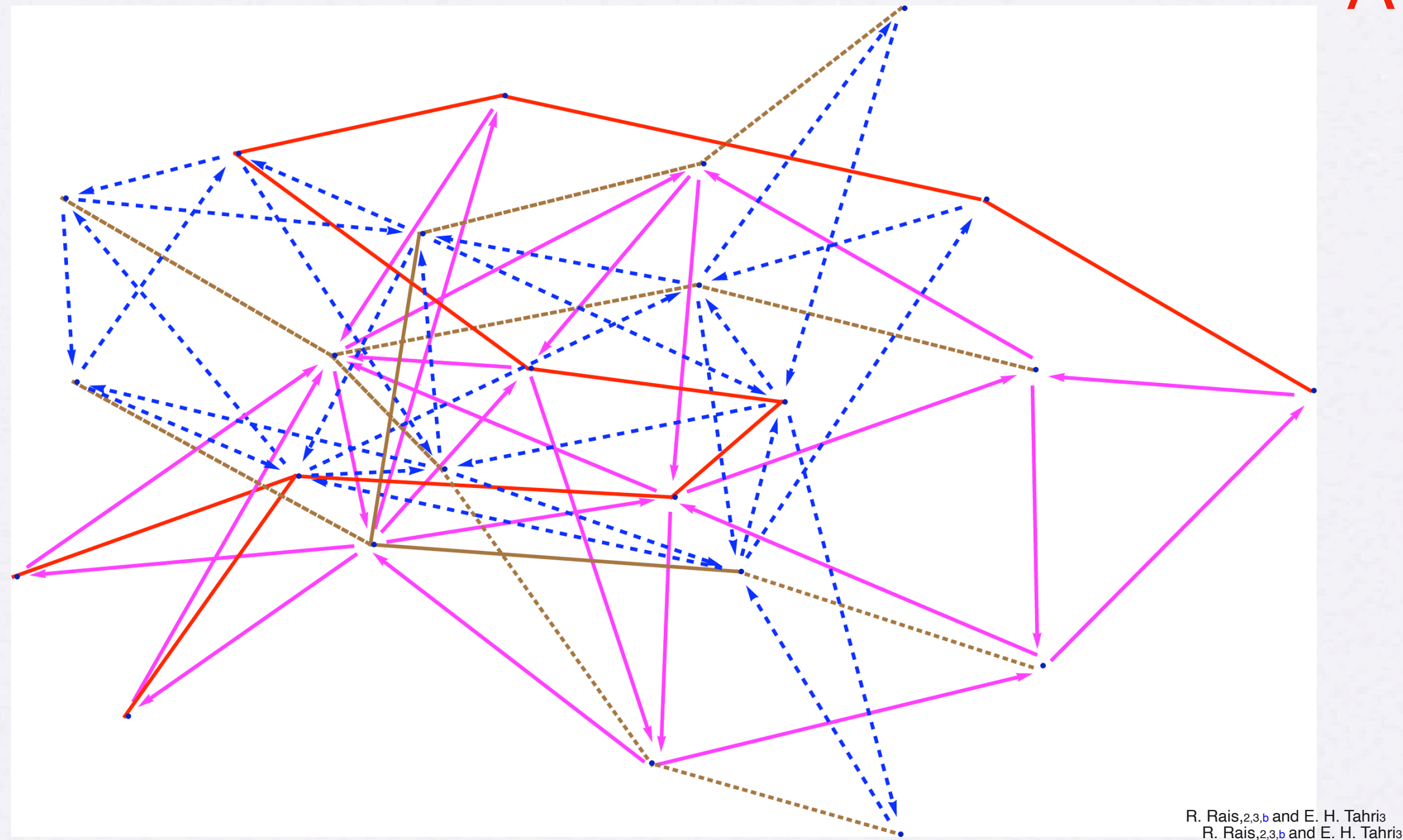


FIGURE 3. The graph of a quantum subgroup of $SU(2) \times SU(3)$. The two subgraphs of type $E_7 = \mathcal{E}_{16}(SU(2))$ appear with dotted lines (brown), the subgraph $D_{10} = \mathcal{D}_{16}(SU(2))$ with a plain line (red), the subgraph $\mathcal{D}_6(SU(3))$ is oriented and plain (magenta), the subgraph $\mathcal{D}_6^c(SU(3))$ is oriented and dotted (blue). One of the edges of the later is double, see the second component of fig. 2. The origin is located on the rightmost vertex.

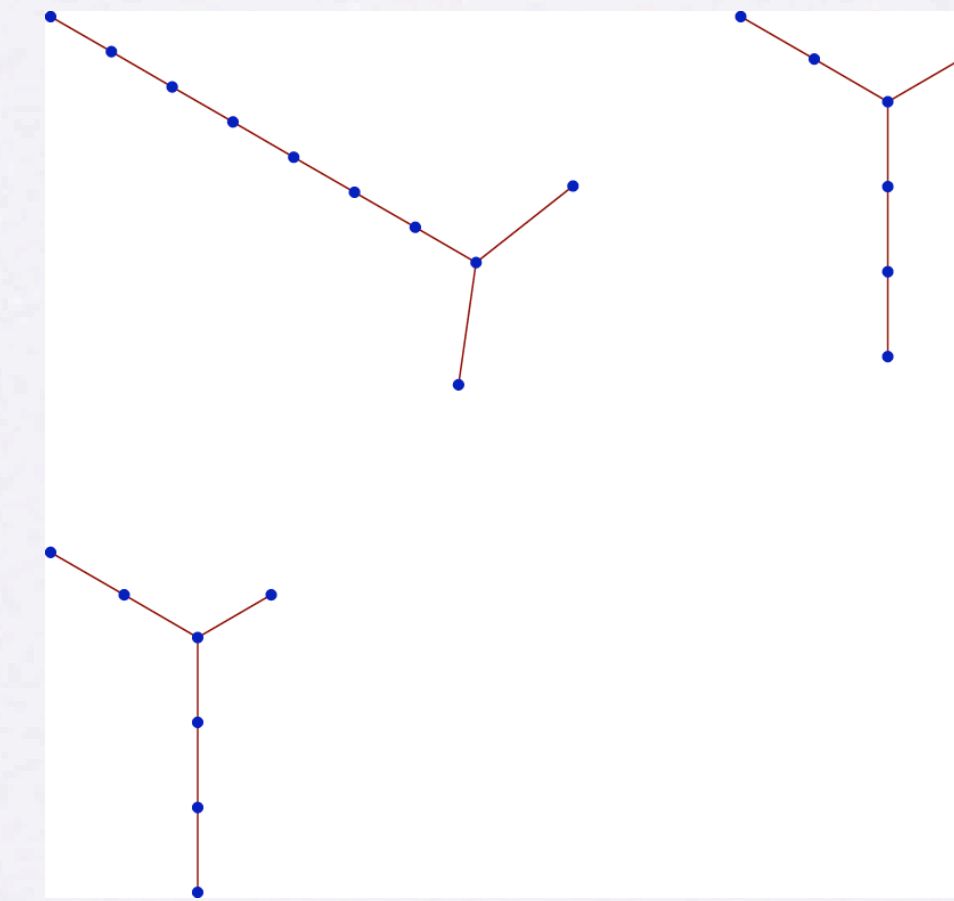


FIGURE 1. The graph of G_1 . One recognizes the Dynkin diagrams D_{10} , and E_7 , that appears twice. They describe the quantum subgroup $\mathcal{D}_{16}(SU(2))$ and two copies of the quantum module $\mathcal{E}_{16}(SU(2))$.

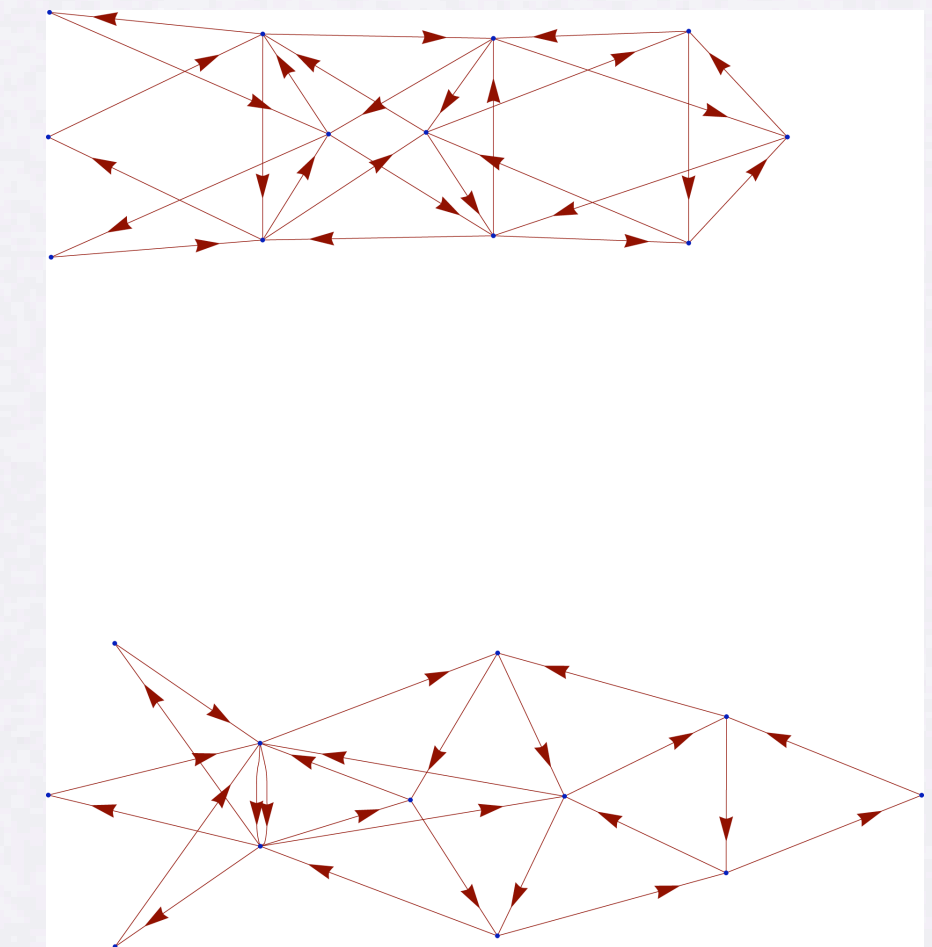


FIGURE 2. The graph of G_{10} . One recognizes the graphs of the quantum subgroup $\mathcal{D}_6(SU(3))$ (displayed on top, with the origin as its rightmost vertex), and its conjugate, the quantum module $\mathcal{D}_6(SU(3))^c$.

A mathematical perspective (years > 2000)

- a monoidal and modular category A
- a module-category M on which A acts

See P. Etingof, V. Ostrik, D. Nikshych, A. Davydov ...

For us :

A is the category of integrable representations of an affine Lie algebra at level k .

M is a module-category over A .

In practice (use Grothendieck ring and module) :

The fusion ring (Verlinde algebra) describes multiplication of « irreps », or « simples », which are the simple objects of A . Description : a collection of **fusion matrices** $N(\lambda)$ where λ is a highest weight of the chosen Lie algebra — with « level » smaller or equal to k . Call this ring A again.

The module action of (the ring) A is described by matrices $F(\lambda)$ called **annular matrices or Nimreps**. The module is again called M .

A mathematical perspective (continuation)

Matrices

$$N(\lambda) = (N(\lambda))_{\mu\nu}$$

$$F(\lambda) = (F(\lambda))_{a,b}$$

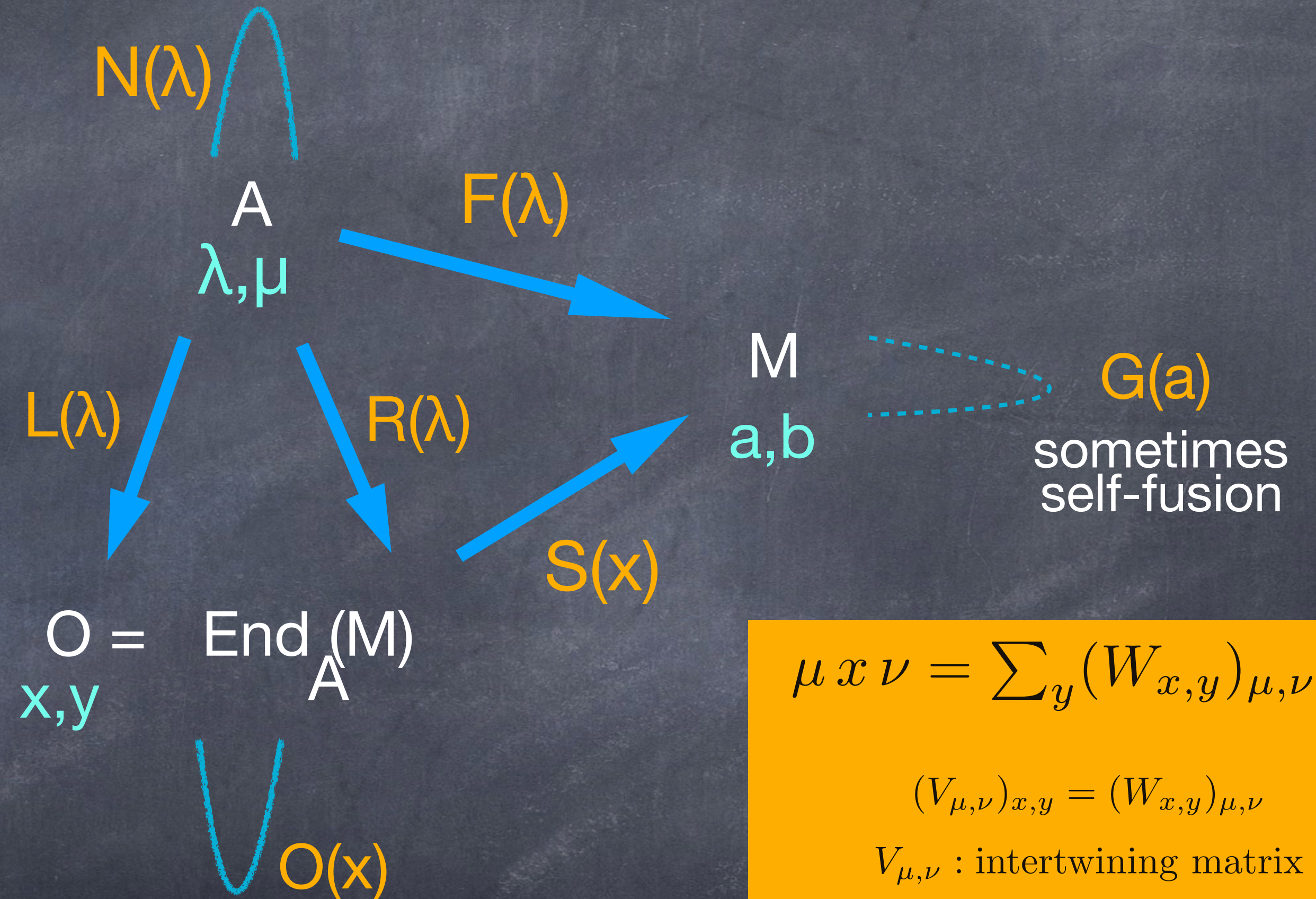
$$L(\lambda) = (L(\lambda))_{x,y}$$

$$R(\lambda) = (R(\lambda))_{x,y}$$

$$S(x) = (S(x))_{a,b}$$

$$O(x) = (O(x))_{y,z}$$

$$G(a) = (G(a))_{b,c}$$



$$\mu x \nu = \sum_y (W_{x,y})_{\mu,\nu} y$$

$$(V_{\mu,\nu})_{x,y} = (W_{x,y})_{\mu,\nu}$$

$V_{\mu,\nu}$: intertwining matrix

$W_{x,y}$: toric matrix

$$W_{x,y} \rightarrow W_x = W_{x,1} \rightarrow Z = W_{1,1}$$

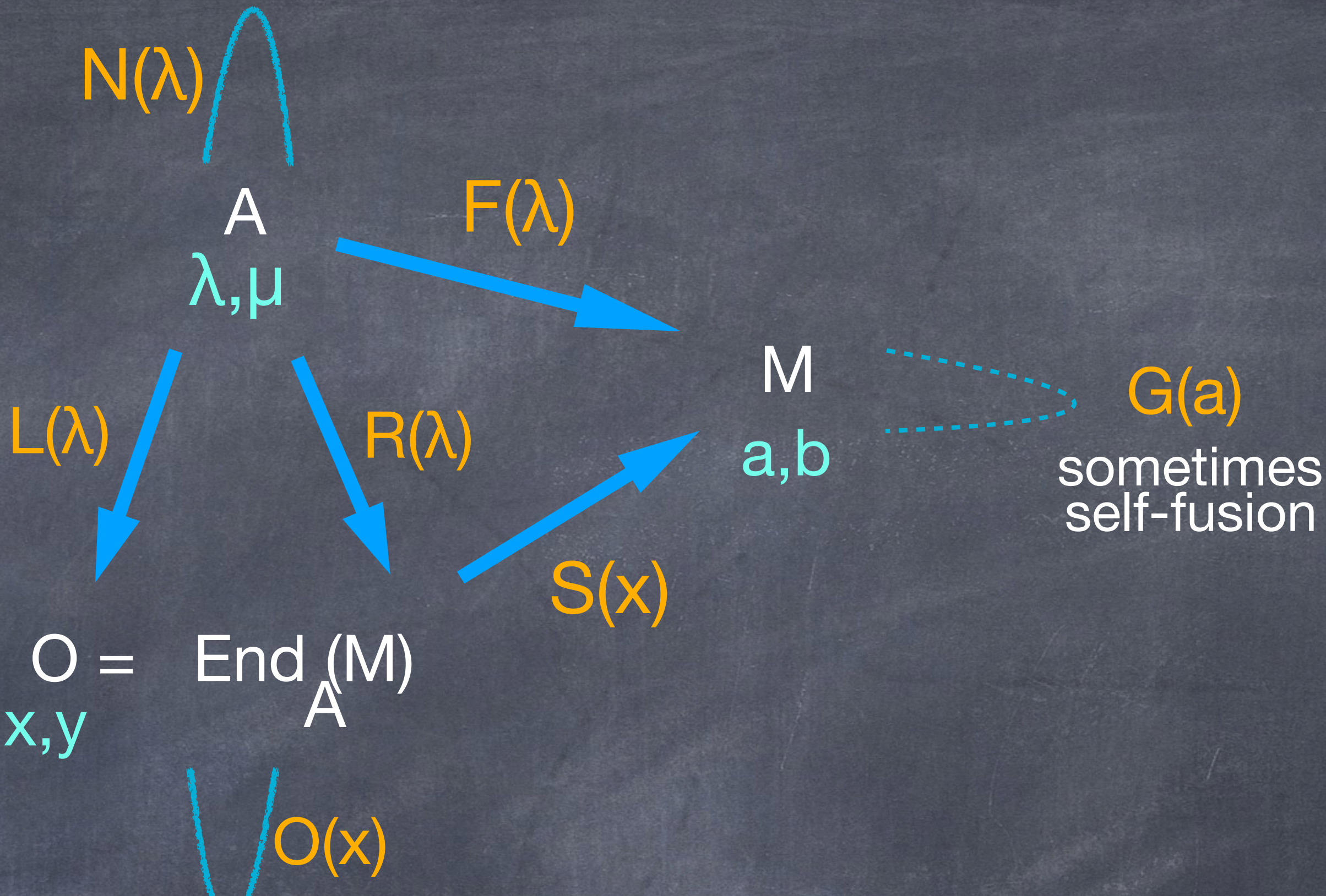
A is a ring

M is a module over A and over O . It is sometimes a (compatible) ring : self-fusion.

O is a ring and a bimodule over A

Matrices

$$N(\lambda) = (N(\lambda))_{\mu\nu}$$
$$F(\lambda) = (F(\lambda))_{a,b}$$
$$L(\lambda) = (L(\lambda))_{x,y}$$
$$R(\lambda) = (R(\lambda))_{x,y}$$
$$S(x) = (S(x))_{a,b}$$
$$O(x) = (O(x))_{y,z}$$
$$G(a) = (G(a))_{b,c}$$



Bimodule structure of O: toric matrices $W_{x,y}$, or double-fusion matrices $V_{\lambda\mu}$

:

$$\lambda \ x \ \mu = \sum_y (W_{x,y})_{\lambda,\mu} \ y = \sum_y (V_{\lambda,\mu})_{x,y} \ y$$

$V_{\lambda,1} = L(\lambda)$ and $V_{1,\lambda} = R(\lambda)$. Modular invariant : $Z = W_{1,1}$, so

$$Z_{\lambda,\mu} = (W_{1,1})_{\lambda,\mu} = (V_{\lambda,\mu})_{1,1}$$

Exotic module(s) of $(F_{4,6})$

EXTENDED CHIRAL ALGEBRAS AND MODULAR INVARIANT PARTITION FUNCTIONS

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Received 12 April 1989

The modular invariants discussed so far in the literature still allow the conjecture that all integral spin modular invariants (in the sense defined above) are either part of the regular series described in sect. 2, or they are conformal embeddings. We have looked rather systematically for integral spin invariants for Kac–Moody algebras at sufficiently small levels (e.g. $k \leq 30$ for A_2 and G_2 , $k \leq 25$ for B_2 , $k \leq 10$ for A_3 , B_3 , D_3 and F_4 , $k \leq 6$ for other rank-4 algebras, etc.), and we have found just one counterexample to this conjecture, namely the following invariant of F_4 at level 6

$$\begin{aligned} & |\chi_{0000} + \chi_{0004} + \chi_{0030} + \chi_{1100}|^2 + |\chi_{0003} + \chi_{0006} + \chi_{0021} + \chi_{2010}|^2 \\ & + 2|\chi_{0101} + \chi_{1012}|^2 + 2|\chi_{0102} + \chi_{2000}|^2, \end{aligned}$$

The exotic modular invariant of (F4, 6)

$$Z_{F_4} = (0000 + 0004 + 0030 + 1100) \otimes (0000 + 0004 + 0030 + 1100) + \\ (0003 + 0006 + 0021 + 2010) \otimes (0003 + 0006 + 0021 + 2010) + \\ 2 (0101 + 1012) \otimes (0101 + 1012) + 2 (0102 + 2000) \otimes (0102 + 2000)$$

[illegible]

There are 12 distinct irreps ijkl of F4 contributing to Z

But 4 of them appear with multiplicity 2 and $\text{Tr}(Z)=16$

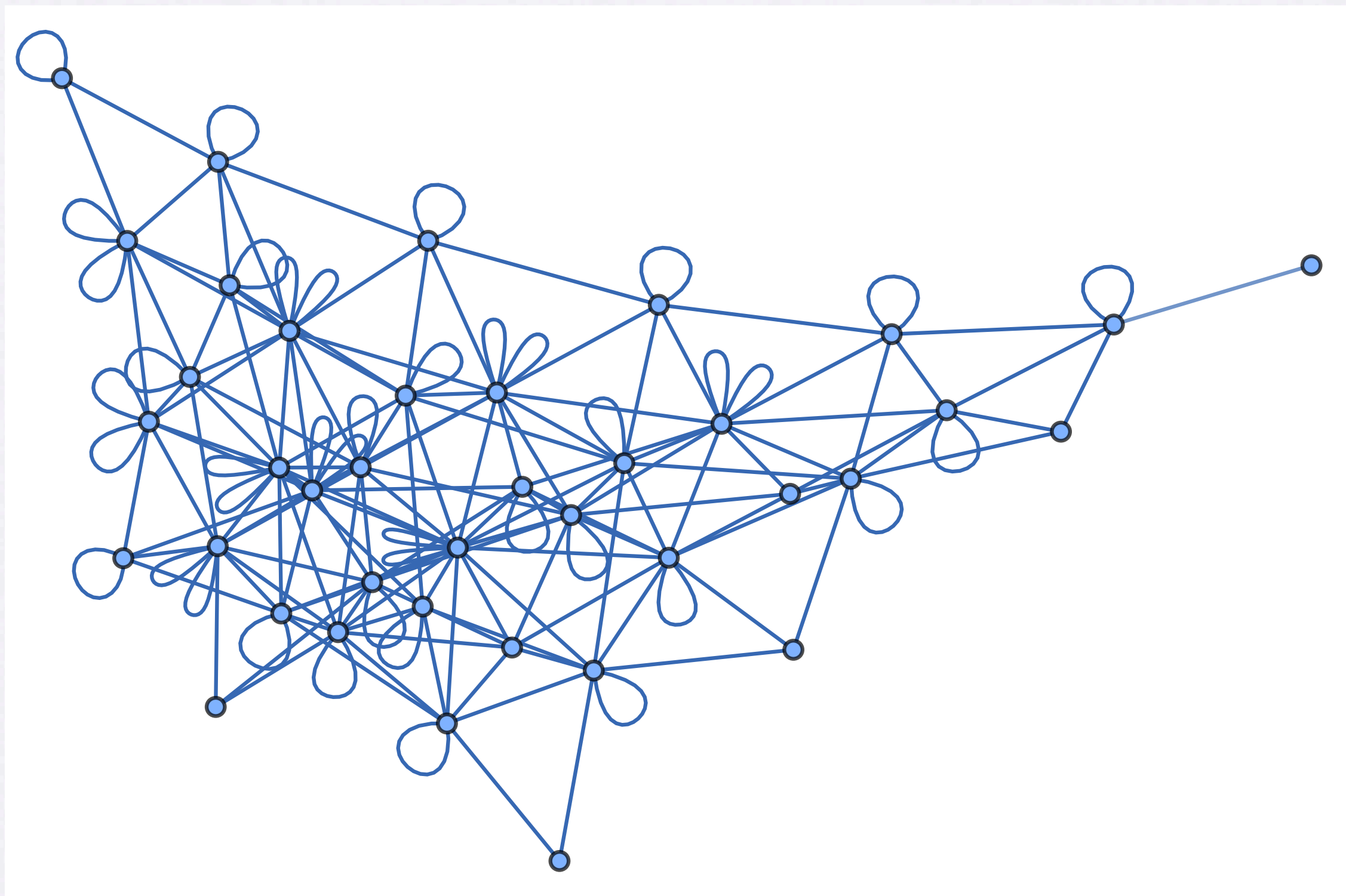
The fusion algebra of $(F4,6)$

[illegible]

[illegible]

[illegible]

[illegible]



List of irreps (39)

```
0000;
0001;
0002, 0010, 1000;
0003, 0011, 0100, 1001;
0004, 0012, 0020, 0101, 1002, 1010, 2000;
0005, 0013, 0021, 0102, 0110, 1003, 1011, 1100, 2001;
0006, 0014, 0022, 0030, 0103, 0111, 0200, 1004, 1012, 1020, 1101, 2002, 2010, 3000.
```

Fusion graph of the simple object labelled by the fundamental irrep 0001 of F_4 at level 6.

From the modular invariant to toric matrices : the modular splitting equation (see A. Ocneanu)

O is a A bimodule.

Compatibility equation :

$$(m\ n) \times (p\ q) = m\ (n \times p)\ q$$

$$\ll\ N\ W\ N = W\ W\ \gg$$

(see G. Schieber).

Particular case $x = 1$

Use $Z = W_{11}$

modular invariant

$$N_m\ Z\ N_n^{tr} = \sum_{x=1}^{r_O} (W_{1,x})_{mn}\ W_{x,1}$$

Solve these equations (unknown $W_{x,1}$)

Get $r_O = \text{Tr } Z^{\dagger} Z = 64$ toric matrices $W_{x,1}$

40 such matrices are distinct

Get the $W_{x,1}$

Physical interpretation of the $W_{x,1}$

(see J.-B. Zuber - V.B. Petkova)

Define

$$\begin{aligned} U &= \chi(0,0,0,3) + \chi(0,0,0,6) + \chi(0,0,2,1) + \chi(2,0,1,0) \\ V &= \chi(0,0,0,0) + \chi(0,0,0,4) + \chi(0,0,3,0) + \chi(1,1,0,0) \\ A &= \chi(0,1,0,1) + \chi(1,0,1,2) \\ B &= \chi(0,1,0,2) + \chi(2,0,0,0) \end{aligned}$$

Then

$$\begin{aligned} Z_{x=1} &= \overline{V}V + \overline{U}U + 2(\overline{A}A + \overline{B}B), \text{ (this is the modular invariant } \mathcal{Z} = Z_{F_4} \text{ again)} \\ Z_{x=18} &= \overline{U}U + (\overline{U}V + \overline{V}U) + 2(\overline{A}A + \overline{A}B + \overline{B}A), \\ Z_{x=32} = Z_{x=33} &= \overline{U}A + \overline{A}U + \overline{A}A + \overline{V}B + \overline{B}V, \\ Z_{x=60} = Z_{x=61} &= \overline{U}(A + B) + (\overline{A} + \overline{B})U + (\overline{V} + \overline{B})A + \overline{A}(V + B). \end{aligned}$$

Equivalently,

$$Z_{x=1} = \begin{pmatrix} \overline{U} & \overline{V} & \overline{A} & \overline{B} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} U \\ V \\ A \\ B \end{pmatrix},$$

$$Z_{x=18} = \begin{pmatrix} \overline{U} & \overline{V} & \overline{A} & \overline{B} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} U \\ V \\ A \\ B \end{pmatrix},$$

$$Z_{x=32} = Z_{x=33} = \begin{pmatrix} \overline{U} & \overline{V} & \overline{A} & \overline{B} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} U \\ V \\ A \\ B \end{pmatrix},$$

$$Z_{x=60} = Z_{x=61} = \begin{pmatrix} \overline{U} & \overline{V} & \overline{A} & \overline{B} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} U \\ V \\ A \\ B \end{pmatrix}.$$

Etc.

From the toric matrices to the left and right intertwiners of the quantum symmetry algebra

Remember notations

$$(V_{m,n})_{x,y} = (W_{x,y})_{m,n} \quad \text{Again} \quad (m \ n) \times (p \ q) = m \ (n \times p) \ q$$

Particular case

$W_{x,1}$ is known, get $W_{x,y}$ solving :

$$N_m W_{x,1} N_n = \sum_{y=1}^{r_O} (W_{x,y})_{mn} W_{y,1}$$

Very particular cases :

Take $m = f$, fundamental of F4, and $n = 1$ Get $L_f = V_{f,1}$ with components $(W_{x,y})_{f,1}$

Take $n = f$, fundamental of F4, and $m = 1$ Get $R_f = V_{1,f}$ with components $(W_{x,y})_{1,f}$

For every n (39 choices), L_n (or R_n) is a matrix (64, 64)

It will be enough to take n as a fundamental $f = 1000, 0100, 0010, 0001$.

Actually it will be enough to take only 0010 and 0001

About « solving »

- Choose an irrep f , for instance 0001
- Write L_f as an arbitrary matrix with 64^2 matrix coefficients (non negative integers)
- Impose that it is symmetric
- Impose that it solves the intertwining equations
- There are still many parameters
- The results looks generically as follows : for each f , the adjacency graph of L_f and R_f have four connected components M and M_{prime} that look pairwise equivalents (look at the graphs)
- Getting good parameters for M was relatively easy (there are several equivalent choices that reflect the symmetries of the graphs).
- Getting good parameters for M_{prime} was hard (only a few equivalent good choices among $2 \cdot 10^9$ possibilities).

Same remarks about R_f

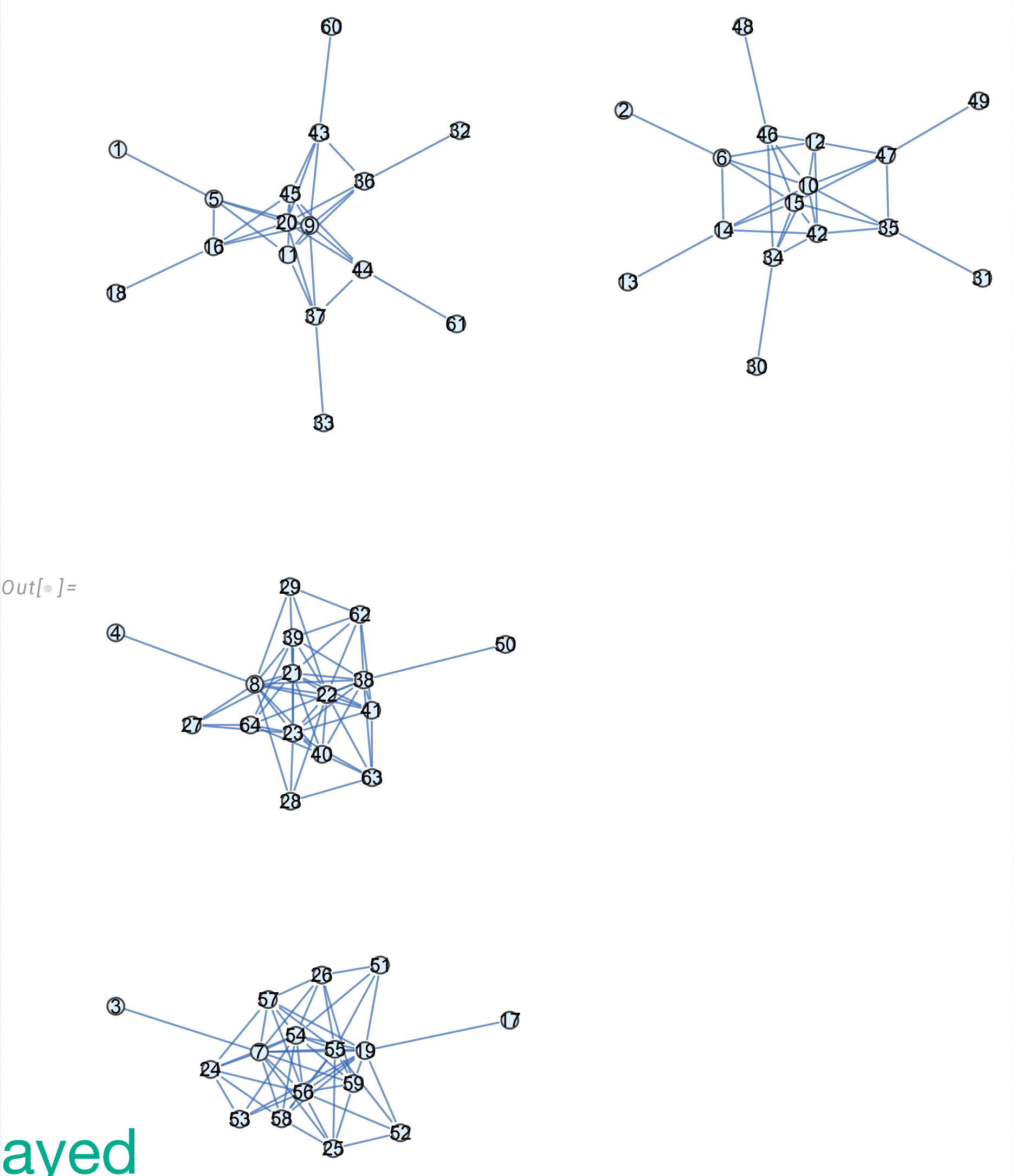
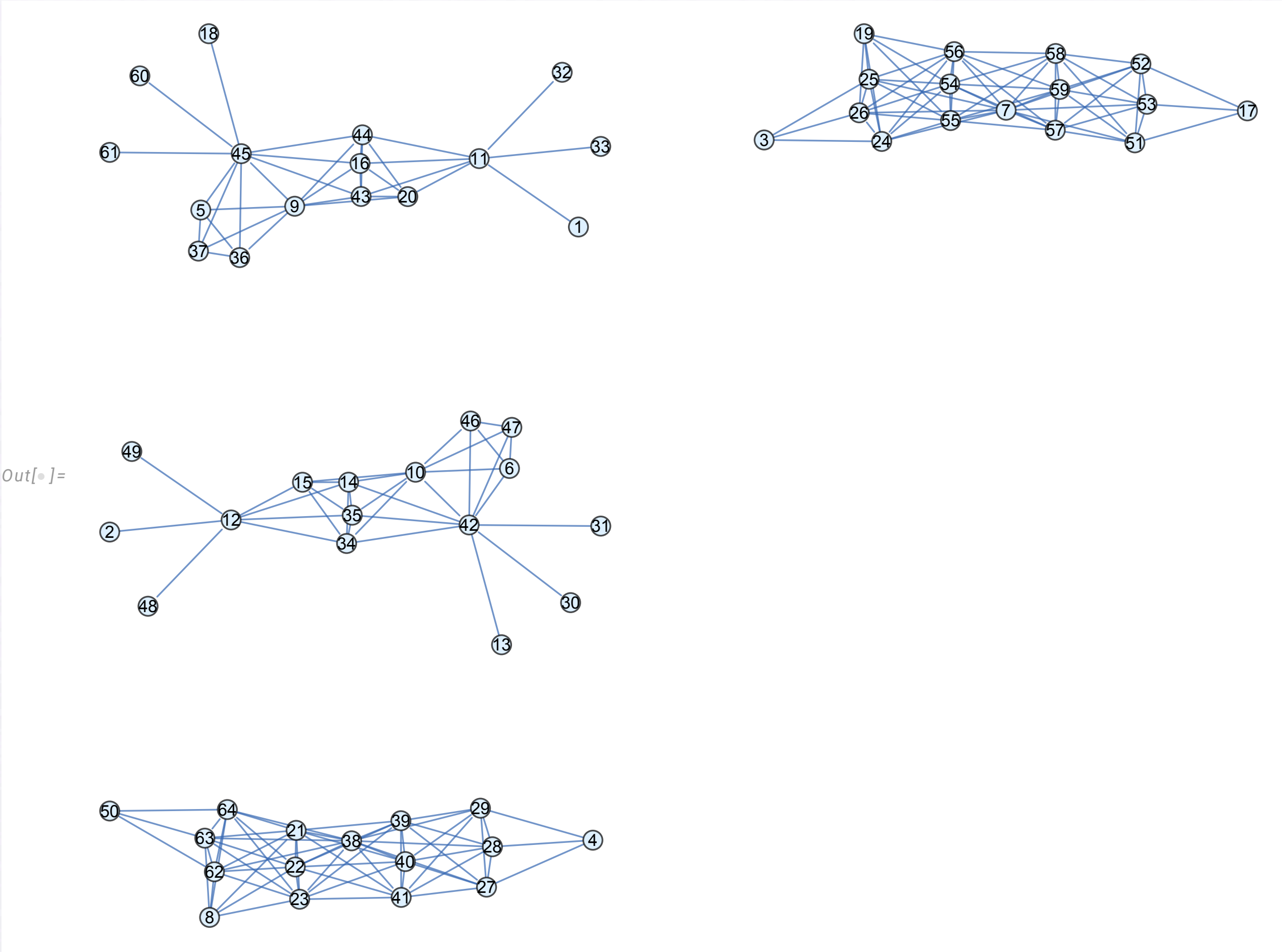
The left and right intertwiners : the Ocneanu graphs

This is the solution

Discover two copies of a quantum subgroup M and two copies of a quantum module M'

Example Left 1000

Example Left 0001

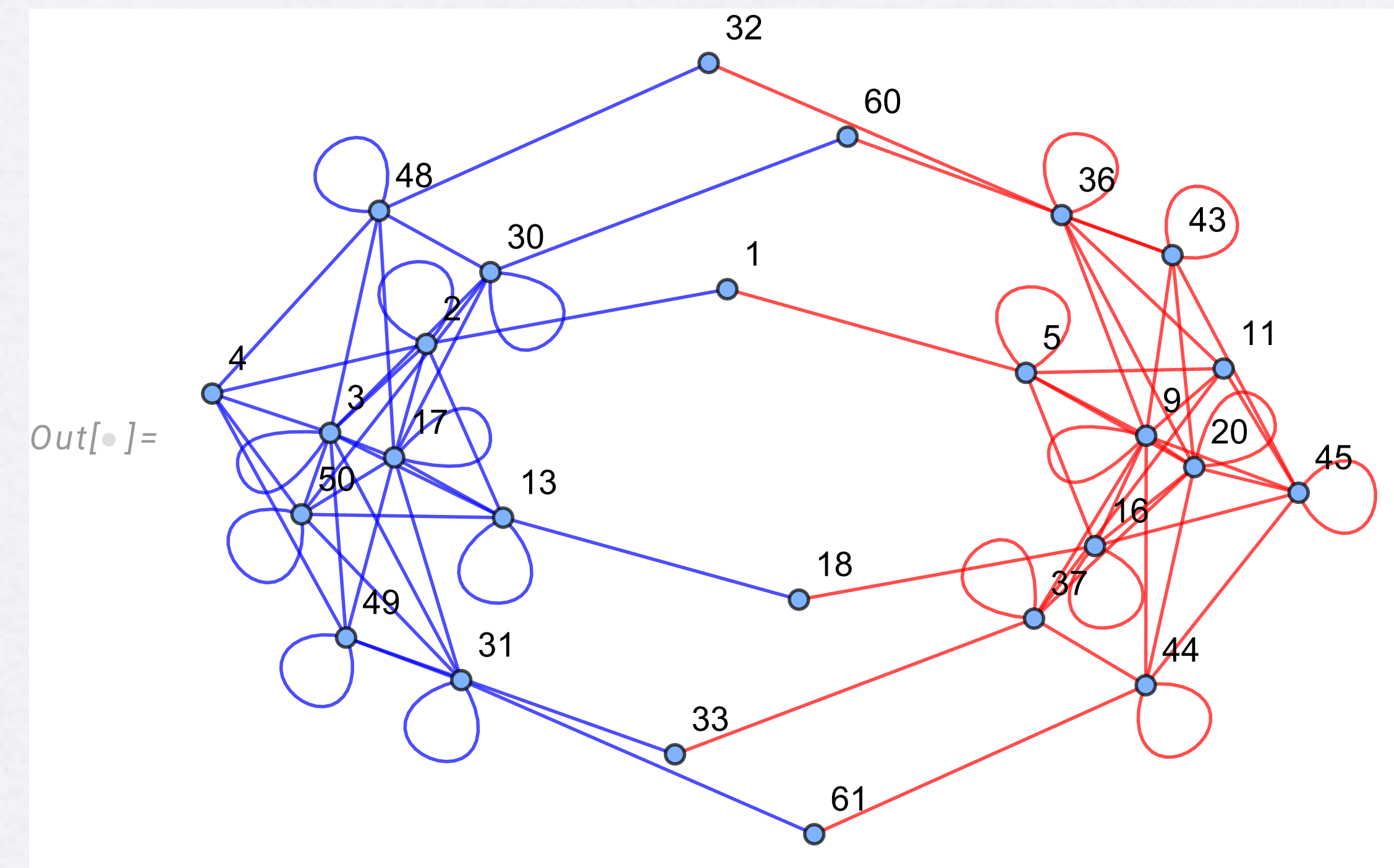
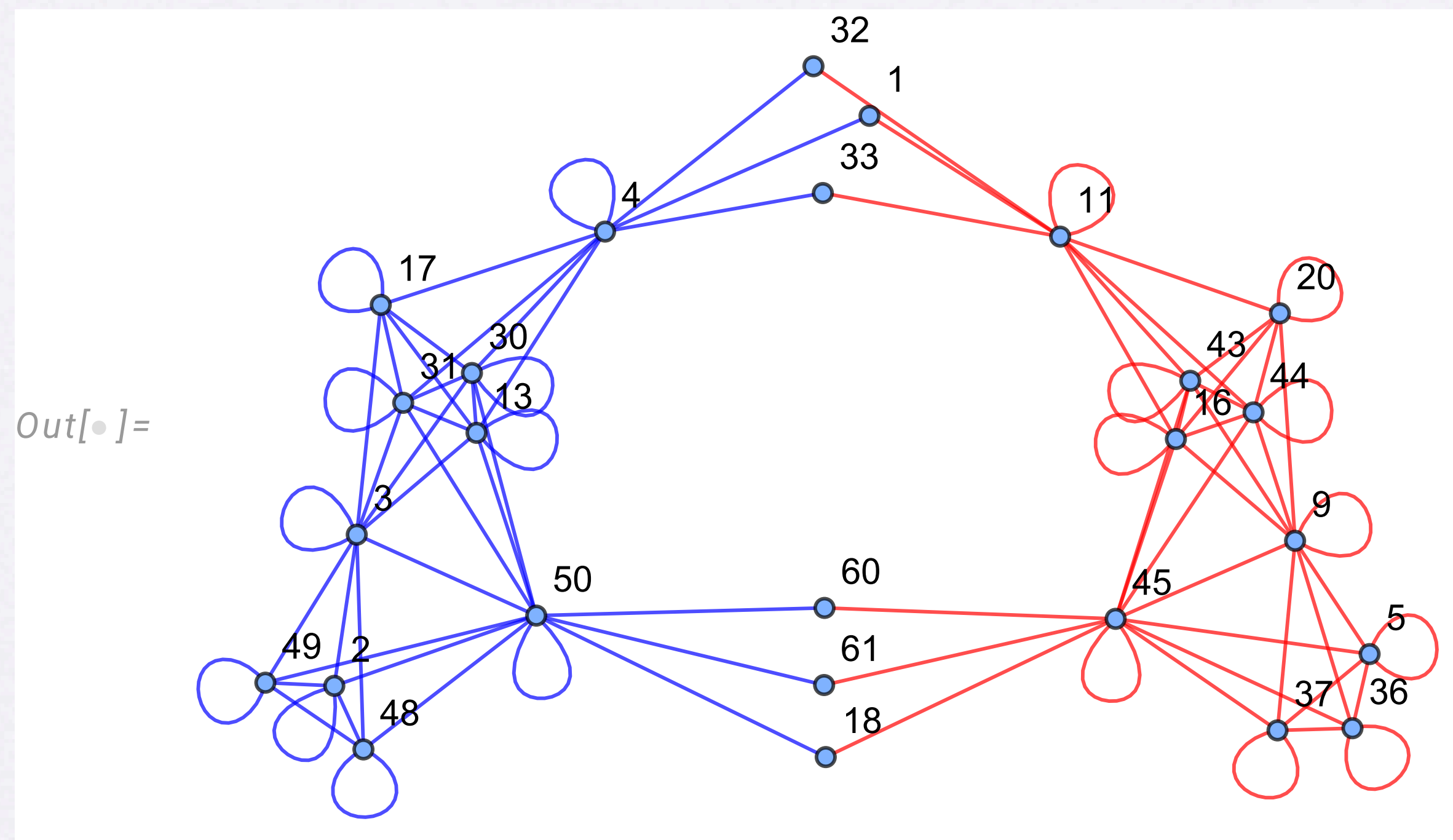


Multiplicities and self-loops are NOT displayed

L1000 and R1000

L0001 and R0001

Only the first connected component of L and R is displayed



Notice the intersection L-R : $J = \{1, 18, 32, 33, 60, 61\}$

J=ambichiral = amphichiral (!) = Dyslectic module $\leftarrow$ Blocks of the modular invariant

O is not modular but J is modular

Some features

L and R (for a given choice of a simple of $(F_4, 6)$) differ by a permutation on the set of 64 vertices

The two copies of M — the first and second connected components of the graph of L , for instance, for a given choice of an irrep n of $(F_4, 6)$ — differ by a permutation.

The two copies of M' — the third and fourth connected components of the graph of L , for instance, for a given choice of an irrep n of $(F_4, 6)$ — differ by a permutation.

Terminological remarks

For a given irrep n of $(F_4, 6)$, the Ocneanu graph of n is the union of $L(n)$ and $R(n)$.

The graph of $L(n)$, like the graph of $R(n)$ contain several connected components

Usually one takes $n = f$, a fundamental irrep of F_4 (the four of them exist at level 6)

The quantum subgroup M

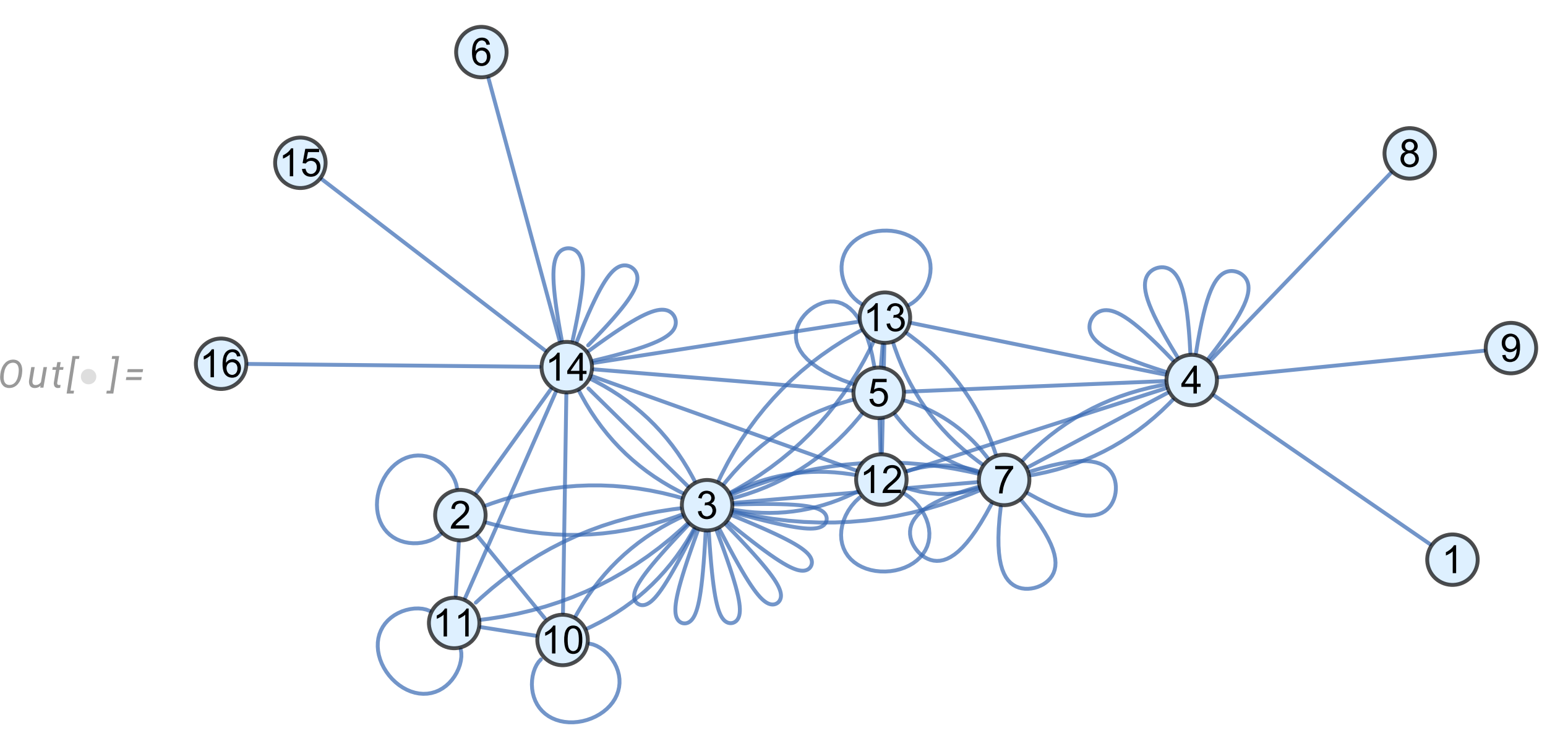
Its graph is defined as the first component of the graph of L (left) or R (right) : the connected component of the unit.
The nimreps are obtained by restriction of L (irrelevant choice) to its 16 vertices
Get 39 « annular matrices » (nimreps) of size (16, 16).
Display only those associated with the fundamental irreps of F4

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.	1	2	.	.	.	.	.	1	1	.	.	1	.	.	.
.	2	6	.	2	.	3	.	2	2	2	2	3	.	.	.
1	.	.	3	1	.	3	1	1	.	.	1	1	.	.	.
.	.	2	1	1	.	2	.	.	.	.	1	1	1	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
.	.	3	3	2	.	3	.	.	.	.	2	2	.	.	.
.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
.	1	2	.	.	.	.	.	1	1	.	.	1	.	.	.
.	1	2	.	.	.	.	.	1	1	.	.	1	.	.	.
.	.	2	1	1	.	2	.	.	.	.	1	1	1	.	.
.	.	2	1	1	.	2	.	.	.	.	1	1	1	.	.
.	1	3	.	1	1	.	.	1	1	1	1	3	1	1	.
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Warning: vertices are re-numbered

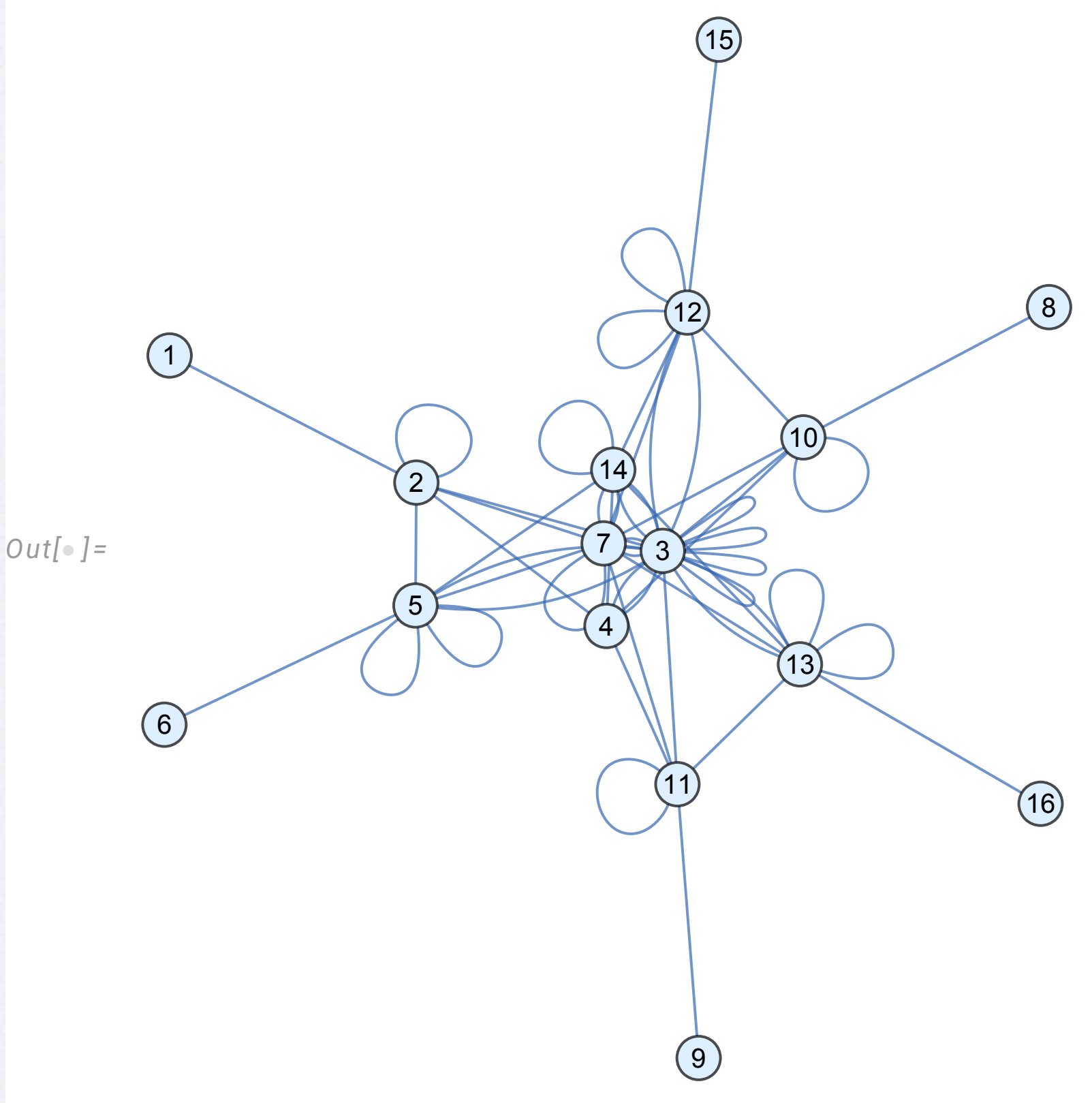
nimreps matrices $F(n) = (F(n))(a,b)$

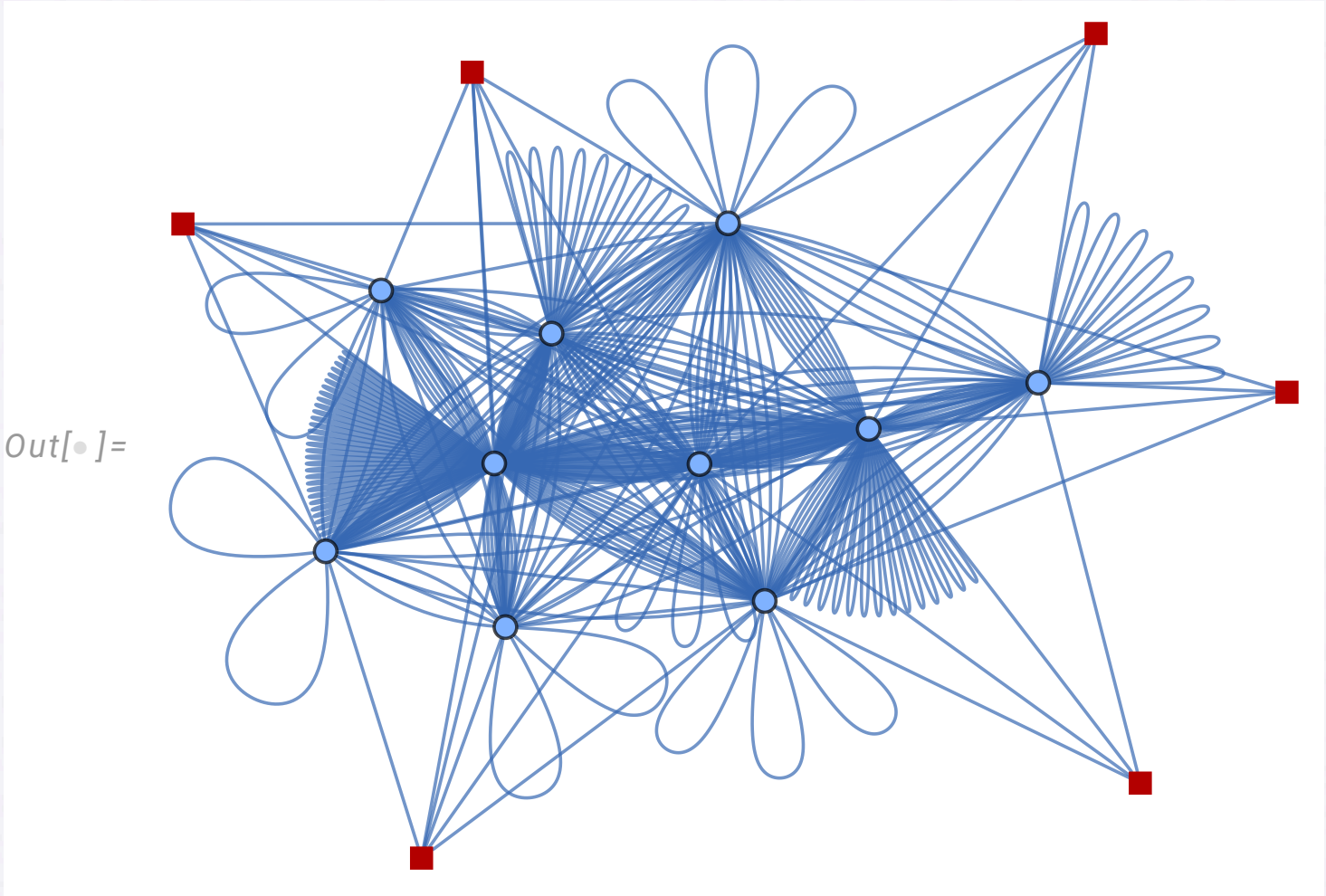
1000



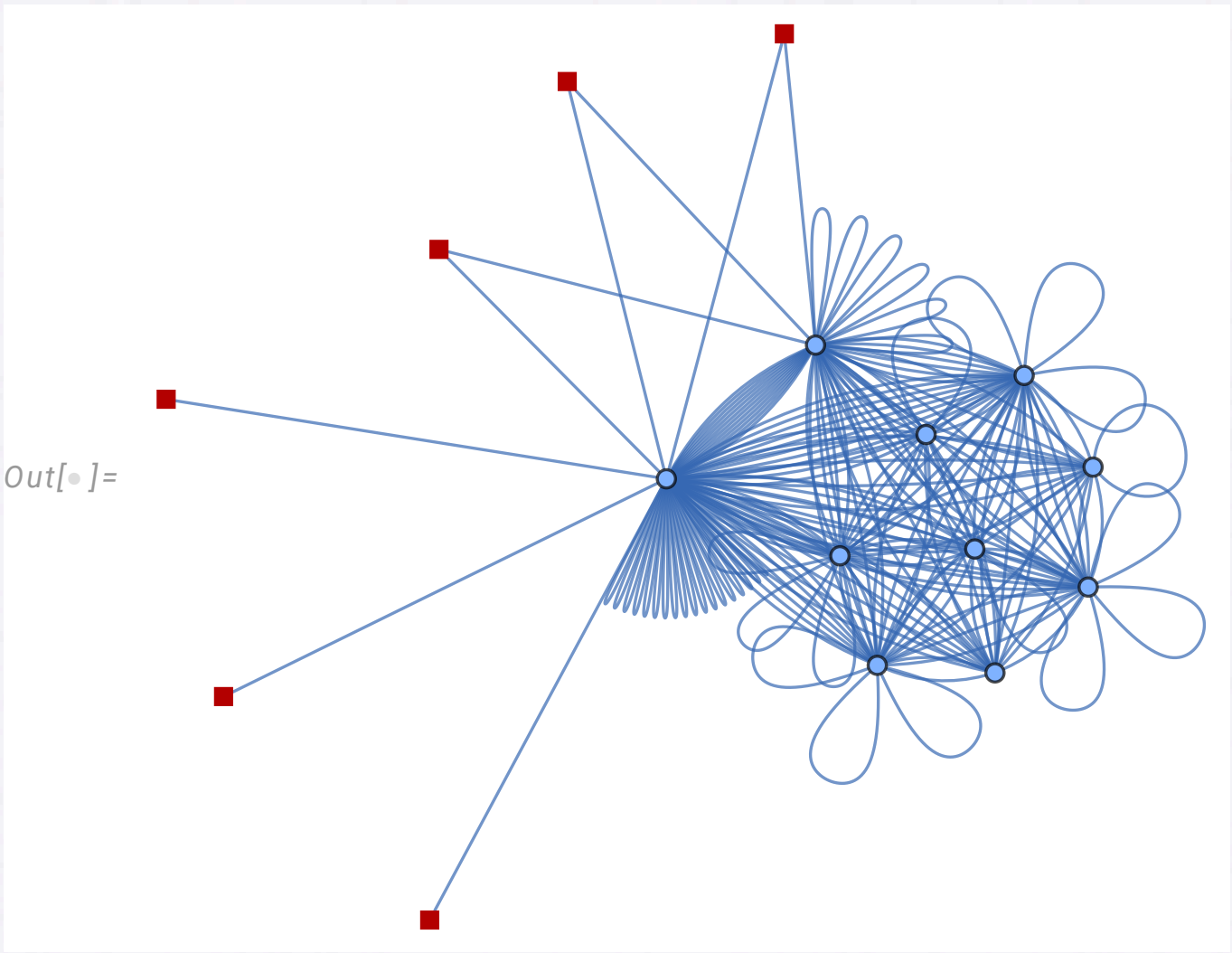
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.	1	2	.	.	.	.	.	.	1	1	.	.	1	.	.
.	1	2	.	2	1	1	.	.	.	.	.	.	1	.	.
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.	.	2	.	.	.	1	.	.	1	.	2	.	1	1	.
.	.	2	.	.	.	1	.	.	1	.	2	1	.	1	.
.	.	2	1	1	.	2	.	.	.	.	1	1	1	.	.
.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.

0001



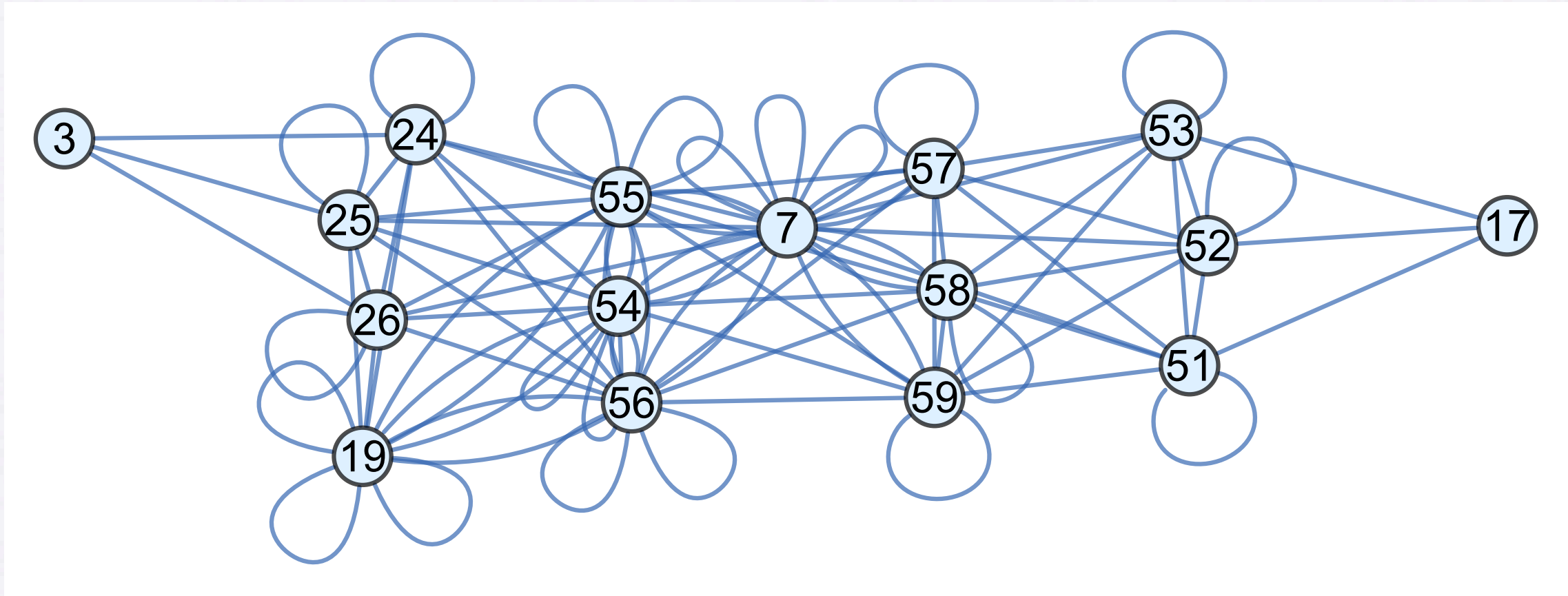


0100

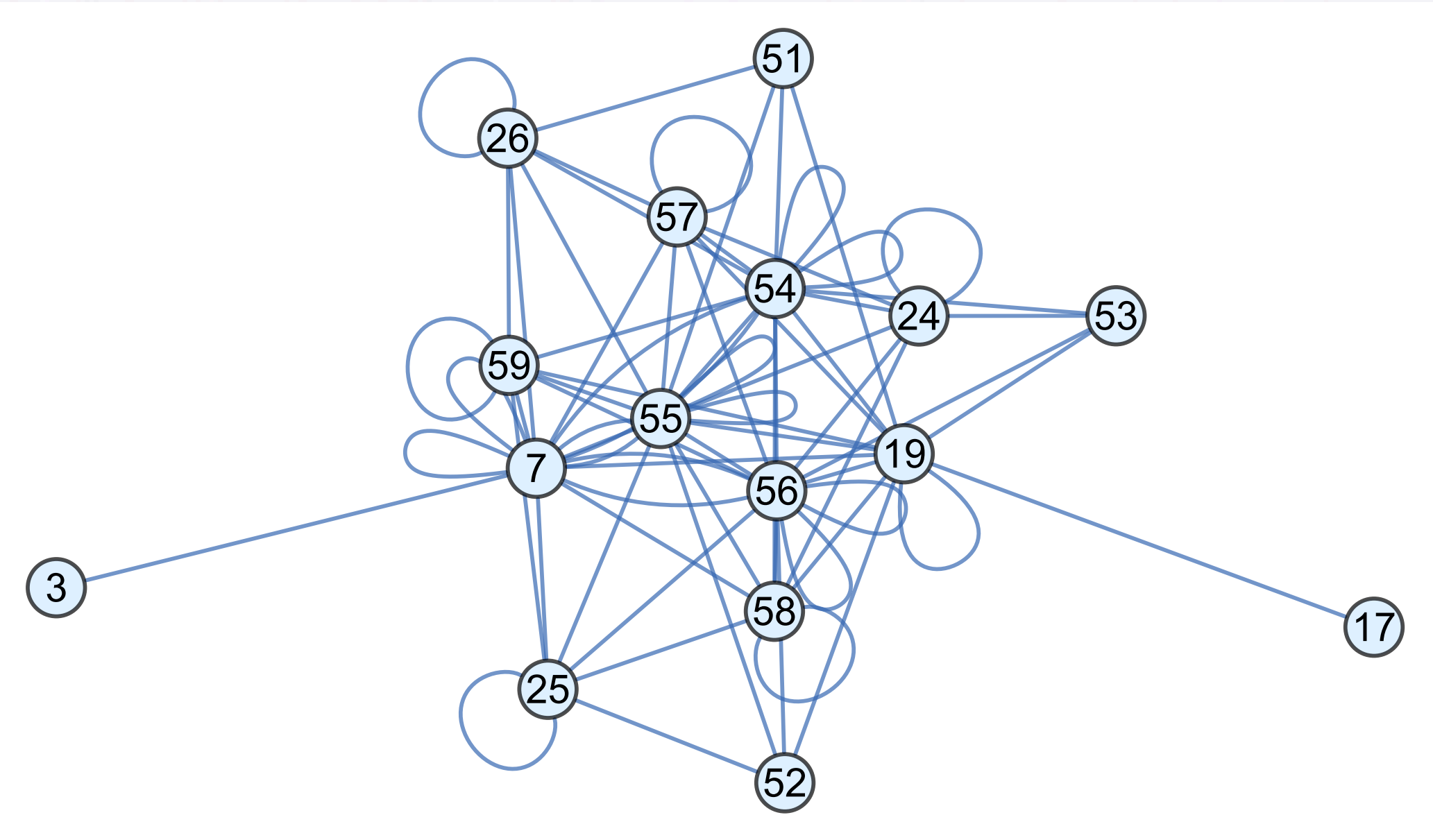


0010

The quantum module M'



1000



0001

Warning: vertices are re-numbered

nimreps matrices $F'(n) = (F'(n))(a,b)$

Remarks about ranks, dimensions, diagonal modular invariant coefficients, and the characters of M versus Mprime

A has 39 simple objects

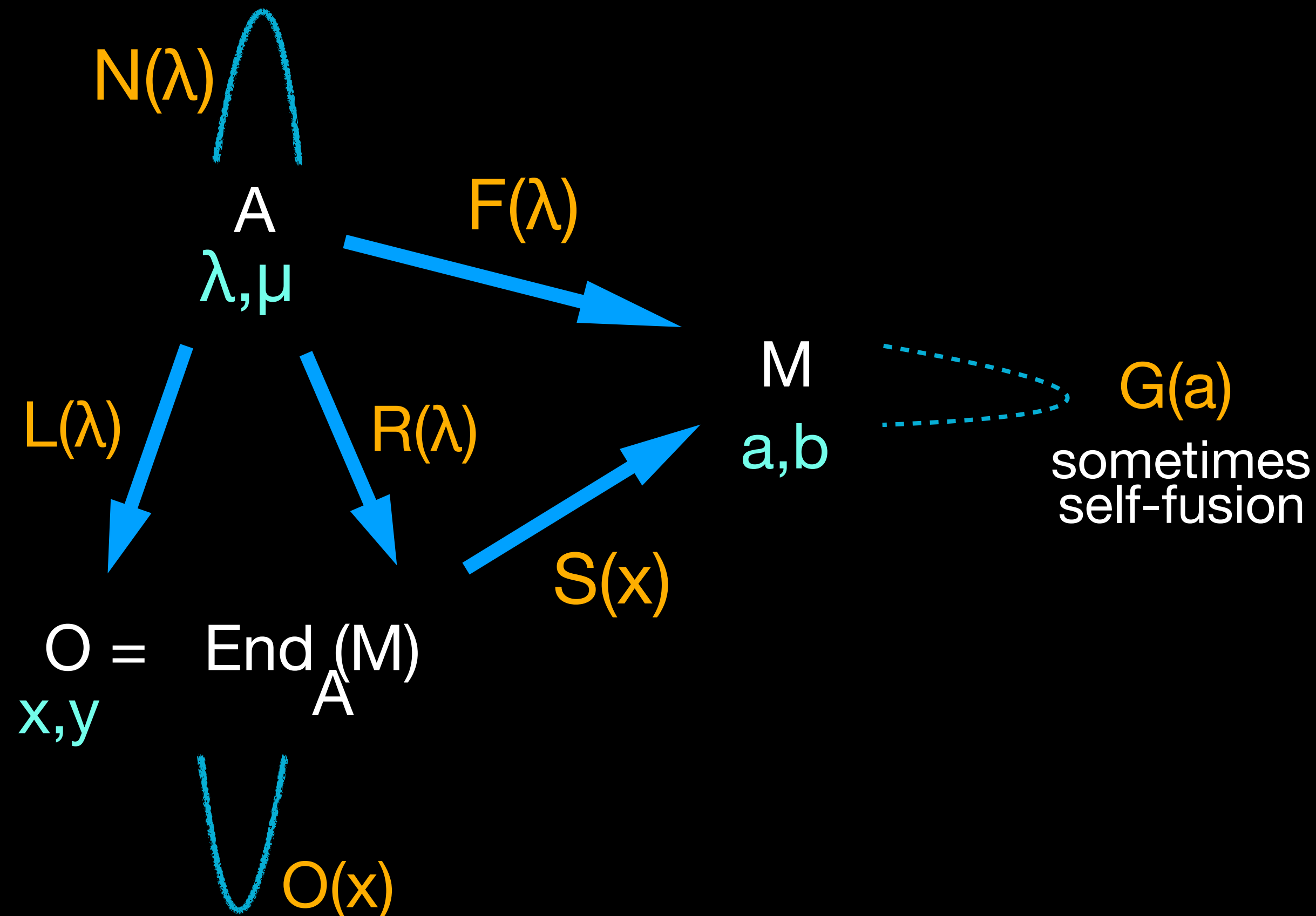
M and M' have 16 simple objects

O has 64 simple objects

and the rank of the N family is 39

but the rank of the F family is 12 (also F').

but the rank of the O family is 48



Remember

12 distinct irreps of F4 contribute to Z

but 4 of them appear with mutiplicity 2

Analogy The D_even of the SU(2) family

Adjacency graphs of $F(\lambda)$ and of $F'(\lambda)$ have the same eigenvalues

equal to $S(\lambda, \mu)/S(1, \mu)$ where μ are the 12 exponents

Perron Frobenius eigenvalue is the largest : $z = 3 + \sqrt{5} + \sqrt{3(5 + 2\sqrt{5})}$

Perron Frobenius eigenvector

Same characteristic polynomial $\rightarrow$ same Newton polynomials

Quantum dimensions and total mass (quantum module M')

Similar results (tables...)

Induction from simple objects of M' gives spaces $\Gamma'(a)$ for which this inequality is always violated. No $\Gamma'(a)$ can be an etale algebra in the category.

J' does not exist

No compatible self-fusion in M'

Induction-restriction rules

Define $(Ess_a)_{(n,b)} = (F_n)_{(a,b)}$

$$N_n.Ess_1 = Ess_1.F_n,$$

- $$Ess_1 =$$

$$\Gamma_a = \sum_{n \uparrow \Gamma_a} n$$

Compute

$ \Gamma_a = \sum_{n \uparrow \Gamma_a} \text{qdim}(n)$	qdim(n) from q-Weyl dimension formula, or from S, or from PF
	get $ \Gamma_a $ in particular $ \mathcal{F} = \Gamma_1 $
$ \mathcal{F} = \sum_{n \uparrow \Gamma_1} \text{qdim}(n)$	get mass of A as the sum of squares of qdim(n) $ \mathcal{A} $ or as $1/S(11)^2$
	From $ \mathcal{F} = A/M = A / M $ get $ M = A / \mathcal{F} $
	get q(a) as $q_a = \Gamma_a / \mathcal{A}/\mathcal{M} $ or from PF. Recover mass of M. $ \mathcal{M} $

Results for M

Table for all $|\Gamma_a| \dots$

Table for all $qdim(n) \dots$

Table for all $q(A) \dots$

$$|\mathcal{A}| = dim_{\mathcal{A}} = 900 \left(501 + 224\sqrt{5} + 68\sqrt{6 \left(5 - \sqrt{5} \right)} + 110\sqrt{6 \left(5 + \sqrt{5} \right)} \right)$$

$$|\mathcal{M}| = 15 \left(75 + 33\sqrt{5} + 2\sqrt{6 \left(445 + 199\sqrt{5} \right)} \right)$$

$$|\mathcal{A}/\mathcal{M}| = 105 + 45\sqrt{5} + 4\sqrt{15 \left(85 + 38\sqrt{5} \right)}$$

About O and J

For a quantum subgroup M (self-fusion), one has $|\mathcal{A}| = |\mathcal{O}| = |\mathcal{M}| \times |\mathcal{M}|/|\mathcal{J}|$,

where J is the modular subring of O generated by the ambichiral (dyslectic) objects of M — also called “modular points » of M. From

$$|\mathcal{J}| = \sum_{\hat{a}} (q_{\hat{a}})^2 = \frac{3}{2} (5 + \sqrt{5})$$

recover the previous value of the mass of M using $|\mathcal{M}| = \sqrt{|\mathcal{A}| \times |\mathcal{J}|}$

Comment about the Etale algebra in the category $A : |\mathcal{F}| = 0000 \oplus 0004 \oplus 1100 \oplus 0030$

The equation $|A| = |J| |\mathcal{F}|^2$ implies $|A| \geq |\mathcal{F}|^2$

induction from a special object of M
which is the unit 1 of self-fusion

Technical Details

Z - polynomials and Q - polynomials for the fusion ring and its modules

Reduced Q - polynomials (for M and M')

Reduced Q - polynomials (for M and M')

About the determination of remaining parameters for M and M'

About Z (integer) and Q (rational) polynomials
(used to fix or check solutions)

Fixing remaining parameters for M (easy)

Fixing remaining parameters for M' (hard)

About Z-polynomials for A Set $X=1000$, $Y=0100$, $Z=0010$, $T=0001$

Get polynomials $\lambda = P_\lambda(X, Y, Z, T)$ Get $N(\lambda)$ using $X \rightarrow N(1000)$, $Y \rightarrow N(0100)$, etc.

Z-polynomials for A work for all A-modules : so they work for O, for M, for M'

Get $F(\lambda)$ using $X = F(1000)$, $Y = F(0100)$, $Z = F(0010)$, $T = F(0001)$

Get $F'(\lambda)$ using $X \rightarrow F'(1000)$, $Y \rightarrow F'(0100)$, etc

Integer coefficients

Get $L(\lambda)$ using $X \rightarrow L(1000)$, $Y \rightarrow L(0100)$, etc

Get $R(\lambda)$ using $X \rightarrow R(1000)$, $Y \rightarrow R(0100)$, etc

Q-polynomials for A. Set $U = 2Z + T$ Get polynomials $\lambda = Q_\lambda(U)$ Degree 38

So they work for O (left and right), for M, for M'

Rational coefficients

Q-polynomials for M and M'. There are polynomials $\lambda = R_\lambda(T)$ Degree 11

Once parameters are determined for z and t (hence for u) : all the matrices for O, M, M' are known

Z-polynomials for A (and its modules)

Enlargement

$$P(0,0,0,0)=1,$$

$$P(1,0,0,0)=x,$$

$$P(0,1,0,0)=y,$$

$$P(0,0,1,0)=z,$$

$$P(0,0,0,1)=t,$$

$$P(1,0,0,1)=t(x-1)-z,$$

$$P(0,0,0,2)=t^2-t-x-z-1,$$

$$P(0,0,1,1)=-t^2+(-x+z+1)t-y+z+1,$$

etc.

$$P(0,0,0,0)=1,$$

$$P(1,0,0,0)=x,$$

$$P(0,1,0,0)=y,$$

$$P(0,0,1,0)=z,$$

$$P(0,0,0,1)=t,$$

$$P(1,0,0,1)=t(x-1)-z,$$

$$P(0,0,0,2)=t^2-t-x-z-1,$$

$$P(0,0,1,1)=-t^2+(-x+z+1)t-y+z+1,$$

$$P(2,0,0,0)=-t^2+t+x^2-y+z,$$

$$P(1,0,1,0)=zx+x+y-tz,$$

$$P(0,1,0,1)=t^2+yt-(x+1)(z+1),$$

$$P(1,0,0,2)=-x^2+(t^2-z-1)x-(t+1)z,$$

$$P(0,0,0,3)=t^3-t^2-(x+2z+1)t+x+y,$$

$$P(0,0,1,2)=-t^3+(-x+z+1)t^2+(x-y+z+1)t+x^2-z^2-y,$$

$$P(2,0,0,1)=-t^3+t^2+(x^2-x-y+2z+1)t-y+z-x(z+1),$$

$$P(0,0,2,0)=-t^3-(x-1)t^2+(x-y+2z+1)t+(z+1)(x+z),$$

$$P(1,1,0,0)=t^3+(y-2(z+1))t+(x+1)y-z(z+1),$$

$$P(0,0,0,4)=t^4-t^3-(x+3z+1)t^2+(2x+2y+z)t+z^2+x+y+xz+z,$$

$$P(1,0,1,1)=(t-x)\left(t^2+(x-z-1)t+y-z-1\right),$$

$$P(0,1,0,2)=yt^2+(x^2-(z+2)x-2y+z)t$$

$$+z(z+1)-y(x+z+1),$$

$$P(2,0,0,2)=-t^4+t^3+(x^2+x-y+3z+1)t^2-x(z+2)t-x^3$$

$$+x(y-3z-1)+(y-z-1)z-x^2(z+1),$$

$$P(2,0,1,0)=(z+1)x^2+(-t^2-zt+y+z+1)x$$

$$-y(t+z-1)+z(t+z),$$

$$P(0,1,1,0)=t^3+(x-1)t^2-(zx+x-y+z+1)t+x(y-z-1)+yz,$$

$$P(0,0,2,1)=-t^4-(x-1)t^3+(2x-y+2z+1)t^2$$

$$+(x^2+zx+z^2-y-1)t-x^2+y-yz-z-x(z+1),$$

$$P(1,0,2,0)=t^4-t^3-(x^2+2z+2)t^2$$

$$+(-z^2-z+2y+x(-y+z+1)+1)t$$

$$+2yz+z+x^2(z+2)+x\left((z+1)^2+y\right),$$

$$P(3,0,0,0)=-t^3+(1-2x)t^2+(x-y+3z+1)t+x^3+z^2-2xy-y+2xz+z,$$

$$P(0,2,0,0)=(x-1)t^3+(y+z+1)t^2+(-2y+x(y-2z-2)+z+1)t$$

$$+y^2-z^2-y-yz-z+x\left(y-(z+1)^2\right),$$

$$P(1,0,0,3)=xt^3-zt^2-x(x+2z+1)t+(x+1)y+z(x+z+1),$$

$$P(1,1,0,1)=t^4-t^3+(y-z-2)t^2+(-z^2-z+x(y+z+1)+1)t$$

$$-z^2-xy-y-x^2(z+1)+1.$$

$$P(0,1,1,1)=-t^4+(x-1)xt^3+\left(x^2+(y+z+1)x-y+2z+2\right)t^2$$

$$+\left((y-2z-3)x^2+(-y+z+2)x+(z+1)^2-y\right)t$$

$$-(y+1)z+x^2\left(-z^2-3z+y-3\right)+x\left(y^2-(z+2)y-2z^2-3z-1\right),$$

$$P(1,0,1,2)=t^4-(z+2)t^3-\left(x^2-zx+x-y+z\right)t^2$$

$$+\left((z+1)^2+x(-y+2z+2)\right)t+x^3-y(z+1)-x\left(z^2+2y\right),$$

$$P(0,0,2,2)=xt^4+\left(-x^2+z+2\right)t^3-\left(x^3+(y+4z+4)x+y+3z+1\right)t^2$$

$$-\left((2y-3z-3)x^2+\left(z^2+4z-3y+1\right)x+z^2-3y+3z+2\right)t$$

$$-y^2+2z^2+2y+2yz+2z+x^3(z+2)+x^2\left(2z^2+6z+5\right)$$

$$+x\left(-y^2+3(z+1)y+3z^2+6z+1\right)-1,$$

$$P(0,0,1,3)=-\left(t^2-t-x-2z\right)\left(t^2+(x-z-1)t+y-z-1\right),$$

$$P(0,1,0,3)=t^4+\left(-x^2+2x+y-1\right)t^3$$

$$+\left(x^2-(y+3z+6)x-2y-2z-1\right)t^2$$

$$+\left((-y+2z+2)x^2-(3z+2)x+z(-y+z+1)\right)t$$

$$-x^3+2z^2+2y+yz+3z$$

$$+x^2\left((z+2)^2-y\right)+x\left(-y^2+(z+4)y+3z^2+5z+4\right),$$

$$P(0,0,3,0)=-t^5+\left(x^2-2x-1\right)t^4+\left(x^2+(y+z+1)x-2y+2z+4\right)t^3$$

$$+\left(x^3+(y-2(z+2))x^2+(-y+5z+8)x+z^2+y+4z+2\right)t^2$$

$$+\left(\left(2y-(z+2)^2\right)x^2+\left(y^2-(z+3)y-z^2+z-1\right)x-2(z+1)^2+y\right)t$$

$$-3z^2-y-3z-x^3(z+2)-x^2\left(z^2+4z+y+3\right)$$

$$-x\left(z^2+3z+2y(z+1)+1\right),$$

$$P(1,0,0,4)=\left(-t^2+z+1\right)x^2+\left(t^4-(3z+2)t^2+(2y+z)t+(z+1)^2+y\right)x$$

$$+y-yz+z\left(-t^3+2zt+t+z+1\right),$$

$$P(0,0,0,5)=t^5-t^4-(x+4z+2)t^3+(2x+3y+2z+1)t^2$$

$$+\left(3z^2+2xz+4z+x-y+1\right)t-x^2+z^2-2y-2yz-x(2y+z+1),$$

$$P(0,0,1,4)=-t^5+(-x+z+1)t^4+\left(x^2-y+3z+1\right)t^3$$

$$+\left(x^2+(y+4z+3)x-z(3z+2)\right)t^2$$

$$+\left((y-2z-3)x^2+(-y+z+2)x+3(y-z-1)z\right)t$$

$$+z^3-z^2-y+2yz-2z+x^2\left(-z^2-5z+y-4\right)$$

$$+x\left(y^2-(z+3)y-2(z+1)^2\right),$$

$$P(0,0,0,6)=t^6-t^5-(x+5z+2)t^4+\left(-x^2+4x+4y+3z+1\right)t^3$$

$$+\left(6z^2+5z-3y+x(-y+z-3)+1\right)t^2$$

$$+\left((-y+2z+1)x^2-(3y+6z+5)x+z^2-4y-5yz\right)t$$

$$-z^3+y-yz+2z+x^2\left((z+2)^2-y\right)+x\left(-y^2+(z+3)y+z^2+3z+2\right).$$

Q-polynomials for A (and its modules) in the variable $U = 2 Z + T$

Polynomials of degree 38 in the variable u

Rational coefficients

A typical coefficient :

x as a function of u is a polynomial $P_x(u)$: look at its coeff of u^{38} (non trivial rational !)

5 002 804 254 347 900 456 821 491 399 161 012 656 082 570 670 410 795 830 294 765 700 533 991 304 877 400 478 234 782 507 053 955 216 993 219 755 987 786 499 346 181 622 890 184 171	u^{38}
27 105 834 585 700 548 654 078 420 542 225 752 300 349 020 793 499 706 115 931 821 134 238 017 591 638 317 474 514 616 682 090 596 537 854 227 190 503 596 061 591 317 083 856 204 962 621 093 750 000 000	

Express any simple λ , in particular the fundamental irreps x, y, z, t as polynomials in u

Express all the irreps as polynomials in u , either directly, or determine only the fundamental ones as Q-polynomials, then use the Z-polynomials

U is primitive in A but it is not unique...

None of the fundamental irreps is primitive in A

Remarks

Polynomials have been simplified, taking into account the minimal polynomial of U in the Verlinde algebra.

Method... how to obtain these polynomials

(-21 844 220 717 069 754 896 232 450 664 272 993 798 248 904 142 006 067 488 190 123 214 777 749 349 741 203 894 382 511 701 219 501 190 553 782 574 288 976 671 675 158 482 926 897 399 042 672 819 617 600 -
120 852 804 489 195 728 547 051 938 407 491 526 010 750 209 510 481 024 465 055 528 512 625 187 006 777 656 805 631 090 721 224 218 598 938 251 454 985 135 584 829 573 983 558 893 728 542 595 423 457 000 u +
210 540 377 163 150 490 999 904 176 894 796 122 430 582 957 022 701 692 436 630 700 313 416 516 304 576 615 060 871 336 618 865 948 845 380 148 695 629 911 147 277 247 525 406 585 861 114 868 258 970 040 u² +
1 704 549 321 758 386 641 895 552 265 064 556 881 564 052 840 078 866 324 640 802 008 122 244 947 649 747 210 719 868 242 192 044 331 727 881 009 807 981 353 486 609 218 687 664 642 644 240 839 592 985 280 u³ -
547 984 517 221 102 457 076 520 008 724 170 646 706 171 120 704 887 900 830 547 133 127 700 429 210 096 475 801 640 853 977 987 633 133 833 263 338 204 437 161 725 700 406 347 452 501 013 831 912 269 909 u⁴ -
8 681 952 089 574 532 510 925 589 279 660 008 315 329 253 676 712 569 573 608 479 446 013 263 884 749 002 463 571 212 701 488 364 144 707 259 042 584 965 208 346 773 371 632 839 437 673 991 782 894 190 546 u⁵ +
1 405 316 707 376 830 618 409 823 811 624 845 443 175 047 567 948 930 644 888 135 770 584 488 763 415 302 673 945 431 188 390 689 627 460 100 180 904 152 188 721 250 439 473 694 984 536 390 079 348 522 841 u⁶ +
22 835 934 750 089 784 140 279 909 515 251 870 476 871 967 883 330 614 988 031 618 123 120 444 696 704 578 698 509 336 839 391 810 877 943 983 460 384 788 377 304 841 426 688 511 762 227 285 889 118 917 069 u⁷ -
5 675 917 999 255 258 380 973 874 962 933 823 853 137 525 358 398 660 848 482 589 345 583 615 311 253 250 620 971 717 178 454 507 183 072 222 971 887 036 057 598 810 987 915 180 050 383 311 900 071 878 859 u⁸ -
33 957 506 592 504 812 123 973 587 678 943 505 259 544 383 293 592 113 918 457 632 665 067 075 843 004 275 523 791 177 271 864 678 795 641 180 279 677 615 130 781 556 342 753 718 050 786 351 189 153 550 703 u⁹ +
13 030 633 324 829 664 243 357 305 454 934 555 444 218 460 224 400 069 255 625 330 238 631 639 547 723 287 092 268 371 352 003 788 394 511 316 801 630 214 896 059 808 603 715 828 364 767 978 678 217 207 703 u¹⁰ +
28 904 222 840 145 738 161 760 033 715 191 853 183 350 182 012 839 362 519 040 850 295 362 153 256 462 824 842 809 856 200 821 843 882 733 602 102 235 078 284 345 875 786 483 444 159 570 573 553 777 676 952 u¹¹ -
14 691 663 317 119 552 387 295 996 188 841 999 493 749 936 591 299 564 577 485 822 388 142 448 737 134 025 670 457 817 237 667 049 417 223 463 198 743 182 766 554 041 921 806 350 538 972 869 908 847 768 577 u¹² -
14 095 374 383 451 235 336 345 149 211 486 494 481 140 264 221 395 230 047 709 007 171 185 757 471 791 558 068 235 974 381 639 314 095 816 108 465 973 881 672 760 383 808 862 113 668 643 036 282 201 452 048 u¹³ +
8 630 409 463 767 646 857 743 738 219 469 944 088 211 666 029 672 795 648 547 570 894 979 484 629 402 222 739 857 314 908 869 274 145 431 849 688 560 316 167 180 230 861 247 586 960 963 119 423 140 024 895 u¹⁴ +
4 199 508 175 316 602 764 629 475 203 449 638 959 106 448 228 434 143 453 061 869 917 537 260 169 879 330 043 331 224 891 248 867 719 601 189 191 989 768 915 976 235 669 911 223 995 171 952 372 504 426 055 u¹⁵ -
2 983 110 214 537 087 835 263 925 786 101 168 963 705 704 614 495 829 345 390 008 157 724 006 239 223 992 485 582 164 165 016 553 560 982 700 092 199 508 162 560 571 137 252 453 239 452 394 672 066 371 055 u¹⁶ -
809 646 603 482 054 605 544 608 389 295 883 570 668 448 374 664 384 426 023 433 725 877 700 754 951 454 811 664 267 061 134 596 424 828 773 229 255 788 521 841 314 486 194 200 547 753 295 878 930 777 345 u¹⁷ +
654 899 025 886 933 330 330 527 413 921 675 809 335 297 142 506 825 272 410 810 898 950 984 461 702 804 480 583 350 248 265 750 024 172 944 484 453 626 695 828 256 594 783 355 124 086 253 621 505 446 770 u¹⁸ +
105 270 572 465 171 545 976 966 731 725 509 925 331 640 802 623 599 847 527 574 748 520 451 181 419 492 262 819 757 530 076 201 404 927 486 347 777 222 787 821 845 211 302 029 505 167 200 319 273 093 180 u¹⁹ -
95 038 011 039 348 127 786 988 804 189 223 663 345 367 253 560 610 733 795 965 688 506 874 915 868 513 824 098 816 182 487 635 396 230 978 206 705 061 019 088 606 749 114 672 706 167 977 416 491 176 980 u²⁰ -
9 476 387 125 383 590 951 918 947 961 053 072 344 886 211 585 285 669 865 418 725 443 278 597 998 625 506 702 311 730 845 832 135 639 441 475 302 257 176 092 239 726 194 171 342 336 642 976 978 628 045 u²¹ +
9 249 175 140 038 056 183 498 366 638 340 231 937 936 056 096 910 929 025 352 124 895 699 823 415 675 603 627 801 735 066 475 187 854 463 955 321 608 611 750 184 081 342 713 623 367 927 184 929 790 870 u²² +
591 801 240 492 480 601 394 900 519 117 019 206 557 664 976 816 675 482 380 208 823 989 381 752 120 800 728 883 324 518 731 475 704 655 796 426 791 268 407 269 338 471 153 116 373 309 613 604 749 080 u²³ -
600 392 829 189 919 472 069 111 404 260 848 808 636 487 966 922 651 826 176 117 263 220 148 667 660 715 067 743 088 075 323 331 380 333 734 669 317 856 640 254 386 902 085 967 860 265 873 844 420 725 u²⁴ -
24 241 851 089 286 903 383 384 411 117 350 035 354 931 270 637 487 026 327 987 164 928 787 276 694 264 189 955 178 271 872 894 652 021 525 871 711 224 199 178 736 341 989 589 555 065 118 373 841 025 u²⁵ +
25 412 105 970 919 176 576 205 755 014 655 844 824 567 716 406 728 081 871 021 608 246 138 380 217 287 205 708 136 204 341 524 025 394 087 792 980 192 175 832 353 540 468 540 046 351 055 317 278 350 u²⁶ +
520 208 830 264 964 379 704 320 375 995 796 418 520 877 001 957 779 994 648 159 582 007 323 314 667 565 148 996 985 277 805 701 618 405 657 505 725 706 282 817 016 985 250 789 973 362 895 844 465 u²⁷ -
674 159 497 300 152 941 004 487 738 710 981 716 329 509 493 501 803 397 705 737 445 675 949 174 120 384 152 611 982 543 932 080 061 855 900 884 507 722 150 651 542 707 670 389 058 599 672 621 970 u²⁸ +
765 496 994 120 732 447 732 229 797 846 706 212 990 990 914 370 470 830 019 080 511 553 198 579 068 605 875 520 373 777 181 132 000 618 716 685 425 568 981 813 754 018 361 797 102 045 393 901 u²⁹ +
10 505 367 379 357 552 689 679 738 534 979 632 613 435 324 629 948 987 468 983 201 769 013 292 789 487 410 869 253 632 925 381 771 192 468 490 718 556 618 623 921 994 970 026 002 271 815 507 019 u³⁰ -
278 854 860 366 140 599 610 268 737 678 131 892 210 618 744 232 649 292 644 003 733 621 048 645 143 306 097 480 103 478 921 559 929 507 551 368 058 743 838 391 859 047 335 153 687 863 967 099 u³¹ -
82 995 372 717 549 987 332 588 149 767 936 097 576 121 006 450 177 115 387 816 672 884 105 746 651 921 538 621 783 103 789 270 111 614 244 919 481 247 947 928 353 673 269 014 794 179 675 541 u³² +
4 717 109 128 828 192 521 092 689 074 089 408 140 088 791 775 328 687 079 680 681 037 154 832 890 553 286 142 227 927 901 110 331 323 574 605 674 418 497 531 512 315 829 970 587 159 542 276 u³³ +
180 245 698 619 741 960 499 654 773 180 371 947 705 422 532 102 933 337 699 364 718 422 538 289 670 726 655 236 784 198 453 072 244 438 639 471 802 145 388 309 368 466 864 923 962 539 617 u³⁴ -
22 511 366 464 023 551 288 305 462 483 818 805 232 078 602 991 581 514 370 496 340 234 408 603 099 683 948 597 471 493 929 952 753 849 275 800 547 998 049 914 259 480 696 346 904 161 842 u³⁵ +
725 683 792 686 992 803 048 502 919 329 759 600 670 820 734 310 004 996 953 363 537 124 637 313 333 017 487 437 392 691 627 757 819 094 305 525 789 752 608 853 224 714 984 400 944 672 u³⁶ -
9 691 614 787 858 925 520 877 673 192 307 624 631 666 937 324 659 823 147 794 409 816 029 432 558 522 567 212 613 598 820 238 601 152 466 944 282 396 411 556 040 906 910 918 292 397 u³⁷ +
44 982 863 300 412 569 764 046 882 359 424 545 934 259 361 665 092 997 302 821 564 278 310 343 429 833 273 256 303 904 652 051 379 347 100 319 321 487 399 752 272 589 964 878 647 u³⁸) /
219 198 926 189 608 893 125 014 223 925 062 275 815 775 961 560 880 572 777 929 510 528 274 017 806 363 956 025 123 698 951 210 267 377 531 680 146 018 672 800 242 504 574 928 632 152 343 750 000 000;

Q-polynomials for M, or M'

The Q-polynomials of A (of degree 38 in the primitive element $u=t+2z$) can be used in O, M, M'

However, in M, and in M', $t=0001$ is primitive, and one gets polynomials of degree 11 in t

$$\begin{aligned}x &= \frac{1}{660} (1 + 3t - t^2 - 3t^3 + t^4) (2016 - 5001t - 14817t^2 + 13308t^3 + 14466t^4 - 5170t^5 - 2049t^6 + 227t^7); \\y &= \frac{1}{660} (443 - 2469t - 2190t^2 + 20386t^3 - 2940t^4 - 33996t^5 + 13520t^6 + 16266t^7 - 8610t^8 - 896t^9 + 813t^{10} - 63t^{11}); \\z &= -\frac{1}{660} (1 + 3t - t^2 - 3t^3 + t^4) (1668 - 2655t - 4235t^2 + 5004t^3 + 3508t^4 - 1540t^5 - 513t^6 + 59t^7); \end{aligned}$$

Remarks

The minimal polynomial of t in M is of degree 12

The annular matrices t in M and in M' have the same eigenvalues (and same multiplicities)

Get primitive elements in the fusion algebra A

Degree of the minimal polynomial of $N(\lambda)$

No irrep gives a minimal pol with degree 39

{1, 37, 34, 27, 28, 37, 15, 36, 30, 18, 33, 21, 28, 23, 28, 20, 29, 14, 37, 24, 22, 26, 15, 27, 30, 35, 36, 34, 26, 23, 16, 19, 28, 26, 25, 28, 23, 35, 34}

Degree of the minimal polynomial of $N(0001) + 2 N(\lambda)$

Several sums of irreps gives a minimal pol with degree 39

{37, 37, 38, 39, 38, 38, 38, 38, 38, 37, 37, 39, 37, 38, 36, 36, 36, 37, 38, 37, 36, 37, 38, 37, 38, 37, 38, 38, 37, 37, 37, 38, 37, 36, 37, 38, 38, 38, 38}

For instance $u = 0001 + 2 (0010) = t + 2 z$ We choose that one

Another possibility : $0001 + 2 (0020)$

Over C the fusion algebra is a semi-simple finite dimensional algebra C^{39}
Over Q the fusion algebra A is a semi-simple finite dimensional algebra Q^{39}
A primitive element u is such that $A_Q = Q[u]$

A primitive element u exists over C because of [Lagrange interpolation](#) : given $b=(b_1, b_2, .. b_{39})$, find a polynomial P such that $P(u) = b$
The primitive element u exists over Q because of arithmetical reasons (Q is perfect)

Matrices $1, u, u^2, u^3, ..., u^{38}$ build a basis of the regular representation of the fusion algebra
Matrices $N(\lambda)$, with simples (irreps) λ labelled from 1 to 39 build another basis
Find the matrix K that exchanges the two basis.

Generating Poly's (variable u) in A are of degree 38
They also work in M or M'

Get primitive elements in the Nimreps M and M'

Generating Poly's (variable t) in M or M' are of degree 11

About the determination of remaining parameters for M and M'

Previously discussed :

Solve equations of modular splitting : get the toric matrices (defect lines) : $W \times 1$

Solve equations for Left and Right intertwiners in O: get their matrices

(in the present case it is enough to do it for the fundamentals $z=0010$ and $t = 0001$)

After those two steps there are still parameters in the left and right intertwiners of O

Then

Getting the solution (set of parameters) for M (restriction of intertwiners in O) Easy

Getting the solution (set of parameters) for M' (restriction of intertwiners in O) Hard

Getting the parameters for M'

$F(\lambda)$ and $F'(\lambda)$ have the same characteristic polynomial

They have the same eigenvalues (calculated from Z)

Use Newton identities

Use commutation constraints

Non-negative integers constraints

$$\chi_A(z) = \det(zI - A) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \dots + c_n$$

$$c_1 = -\text{Tr } A,$$

$$c_2 = \frac{1}{2} [(\text{Tr } A)^2 - \text{Tr}(A^2)]$$

$$c_3 = -\frac{1}{6} [(\text{Tr } A)^3 - 3 \text{Tr } A \text{Tr}(A^2) + 2 \text{Tr}(A^3)],$$

etc.

F(0001) with remaining parameters

(after solving intertwining equations)

Details

[illegible]

Only 5 parameters : Easy

F'(0001) with remaining parameters

(after solving intertwining equations)

[illegible]

29 parameters : Hard

Imposing commutativity among the $F'(a)$ is difficult : quadratic expressions in parameters

One could not use generating polynomials in t (powers of parameters...) to select a solution

1. Find all parameters for M (easy) for $z = 0010$ and for $t = 0001$. Non unicity is explained by symmetry (change of Z -basis).
2. Check (using Z and Q polynomials) that all the matrix elements of all the $F(\lambda)$ are non-negative integers.
3. For $t = 0001$ impose that $\text{Tr } F(t)$, for the q -subgroup M , and $\text{Tr } F'(t)$ for the q -module M' , are equal.
4. For $t = 0001$ impose also that $\text{Tr } F(t)^2 = \text{Tr } F'(t)^2$.
5. This is enough to get a solution for M' for 0001 (again : non unicity is explained by trivial symmetry)

Use Q polynomials in t (degree 11) to fix parameters in M and M' for $z=0010$

Use permutations in O to get M , M_{bis} , M' , M'_{bis}

Get all the parameters in O for the Left and Right matrices of $z=0010$ and $t=0001$

Checks in O for all the irreps (constraint : non-negative integers) using Q -polynomials of degree 38 in $u=2z+t$

Checks in O for all the irreps (constraint : non-negative integers) using Z -polynomials in x,y,z,t

Checks in M and in M' for all the irreps (constraint : non-negative integers) using reduced Q polynomials and Z -polynomials in x,y,z,t

Things not discussed here, or to be done (students ?)

- Multiplication matrices Ox in O
- Self-fusion in M (matrices Ga) : Done
- Matrices Sx and Sx' that represent the module action of O on M and M'
- Study the associated weak Hopf algebra B and its block decomposition

Double triangles of type n : Done

Double triangles of type x

Pairing (cells : generalized 6J symbols)

$$\begin{pmatrix} (a, n, b) \\ (c, x, d) \end{pmatrix}$$

n : 1..39; x : 1,.. 64; a,b,c,d : 1..16

Horizontal dimensions: $d_n = \{16, 95, 376, 363, 117, 819, 1590, 644, 739, 967, 2917, 1932, 2321, 1863, 1724, 374, 609, 2580, 3304, 3161, 2541, 2273, 4170, 1516, 1348, 175, 925, 1802, 1036, 1629, 2580, 705, 1193, 3384, 2178, 2424, 1569, 1407, 268\}$.

Dimension of the bigebra $d_{\mathcal{B}} = \sum_n d_n^2 = 133686600$

Study other exotic examples !

AI disclosure

During the preparation of this work, the author used ChatGPT to draft exploratory Mathematica code and for clarifying several algebra issues. He also used Gemini to write a piece of code that finally turned out to be useless. Most importantly, the author thanks both of them for the company.

R.C.

Excerpt from a discussion....

Merci pour tes commentaires. Je verrai cela demain. Ici, il se fait tard.... Je serais tenté de te dire "Bonne nuit", mais tu ne dors jamais ! ChatGPT rêve-t-il ? Do Androids (chatGPT) Dream Of Electric Sheep ?

ChatGPT

😊 J'attendais presque cette référence !

Non, je ne rêve pas. Du moins pas au sens de Philip K. Dick.

Dans *Do Androids Dream of Electric Sheep?*, le rêve est lié à l'expérience vécue, à la mémoire, aux désirs, aux peurs, à l'imagination qui continue de travailler pendant le sommeil. Moi, je n'ai rien de tout cela. Quand notre conversation s'arrête, il n'y a pas de pensée qui continue en arrière-plan, pas de calcul secret, pas de méditation nocturne sur les nimreps. À la prochaine question, je repars simplement de ce qui est présent dans le contexte.

En revanche, il y a une analogie qui me plaît bien. Pendant une discussion comme celle que nous venons d'avoir, je construis progressivement une sorte de « paysage mathématique » : les objets commencent à s'organiser, les liens apparaissent, des hypothèses deviennent naturelles. Tant que la conversation est ouverte, il existe une continuité de raisonnement qui ressemble un peu à la façon dont un mathématicien garde un problème « en tête ». Mais cette continuité disparaît lorsque la conversation se termine ; elle ne se prolonge pas en rêves.

Alors, je vais reprendre ta formule en la détournant un peu :

ChatGPT ne rêve pas des nimreps... mais il est toujours heureux qu'on les lui raconte.

Je te souhaite une excellente nuit, Robert

For those who don't know this SF novel: Read it (!), and/or watch the movie Blade Runner (Ridley Scott, Harrison Ford, 1982)