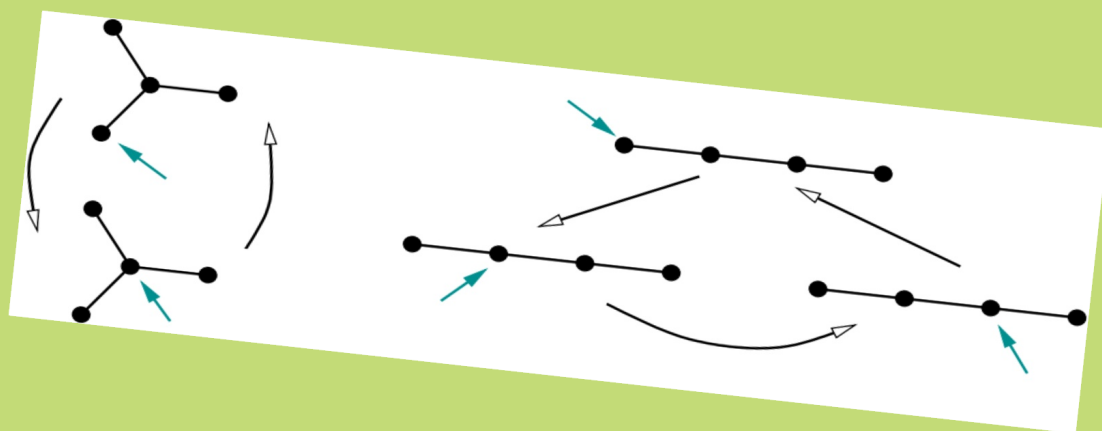
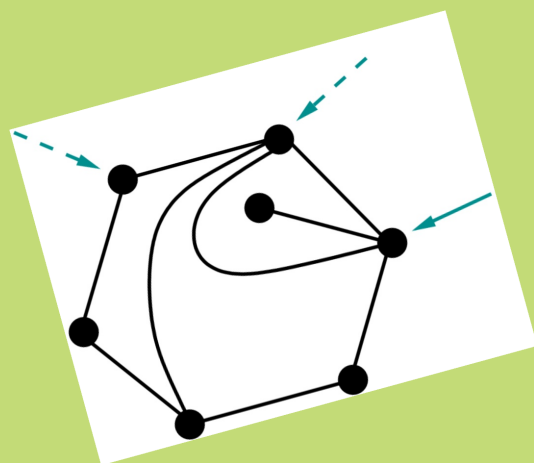


Counting trees and maps: cyclic sieving phenomena



Mireille Bousquet-Mélou
CNRS, LaBRI, Université de Bordeaux, France

Christian Krattenthaler
University of Vienna

arXiv:2512.18656

Marie Albenque
CNRS, Université Paris Cité

arXiv:???

Enumerative combinatorics

Enumerative combinatorics

One of Jean-Bernard's playgrounds

Matrix Integrals and the Generation and Counting
of Virtual Tangles and Links

P. Zinn-Justin
and

J.-B. Zuber

Quantum Field Theory Techniques in Graphical
Enumeration

D. BESSIS, C. ITZYKSON, AND J. B. ZUBER

*Commissariat à l'Énergie Atomique, Division de la Physique, Service de Physique
Théorique, Centre d'Études Nucléaires de Saclay, Boite Postale No. 2, 9119
Gif-Sur-Yvette, France*

Counting Partitions by Genus
I. Genus 0 to 2

Jean-Bernard Zuber

On the Counting of Fully Packed Loop Configurations:
Some new conjectures

J.-B. Zuber

C.E.A.-Saclay, Service de Physique Théorique de Saclay,

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how many objects of size n are there?

Sometimes there is a **nice formula** for this number.

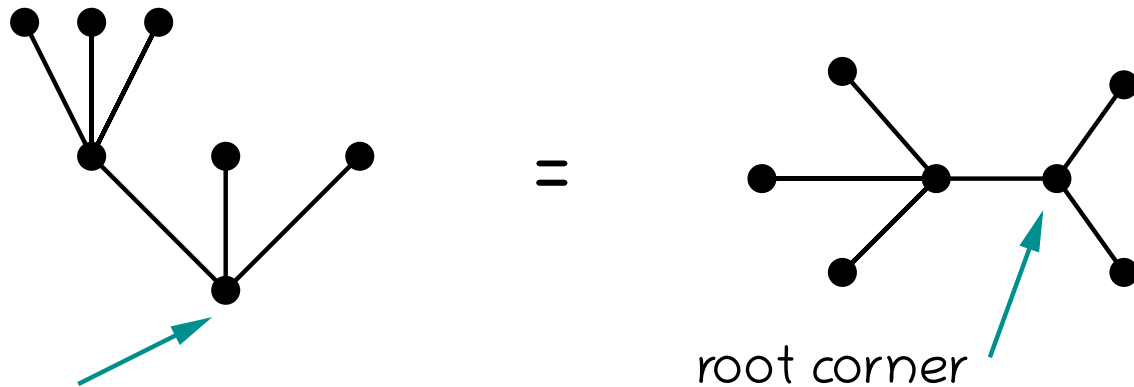
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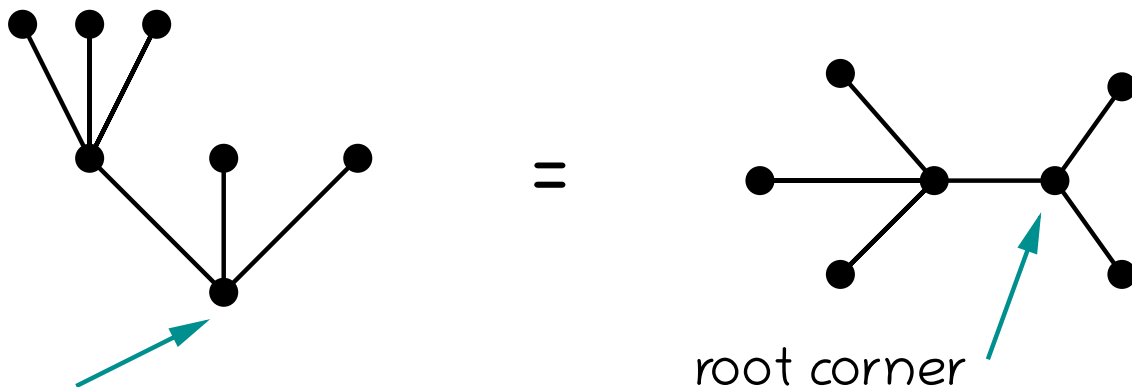
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The number of trees with n edges is the n -th Catalan number:

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Enumerative combinatorics: cyclic sieving

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Enumerative combinatorics: cyclic sieving

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Assume that a cyclic group G acts on objects of size n :
how many objects (of size n) are fixed by a given g in G ?

Sometimes there is a **nice polynomial** encoding these numbers.

This is what cyclic sieving is about.

The cyclic sieving phenomenon

Definition. Let S be a finite set of objects (e.g., objects of size n), on which a cyclic group G acts. Let $P(q)$ be a (nice) polynomial with non-negative integer coefficients.

The 3-tuple (S, G, P) exhibits the **cyclic sieving (CS) phenomenon** if for any $g \in G$ of order d , and ρ any primitive d -th root of unity,

$$\text{Fix}_g(S) = P(\rho).$$

[Reiner, Stanton, White 04]

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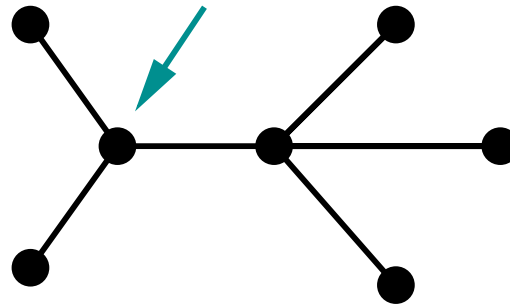
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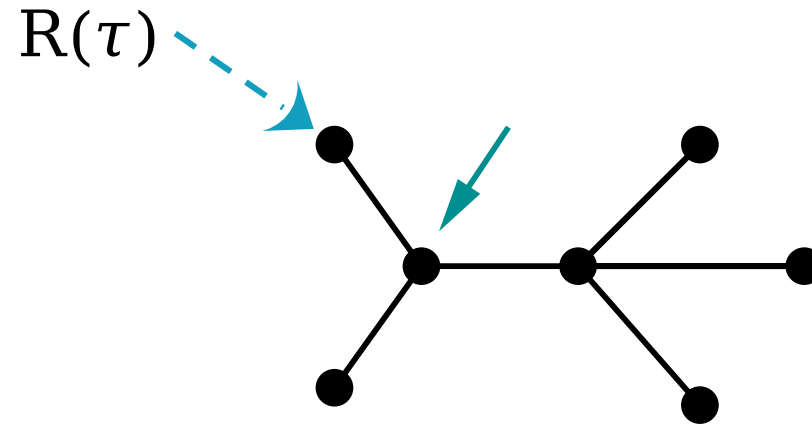
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Move the root to the next corner around the tree in cc order.



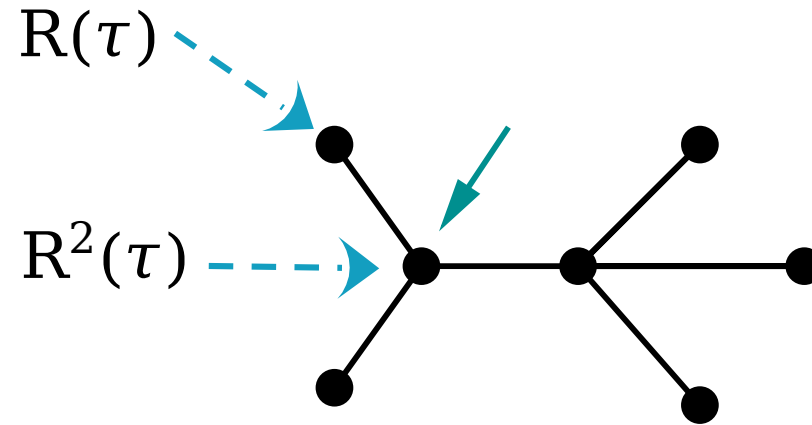
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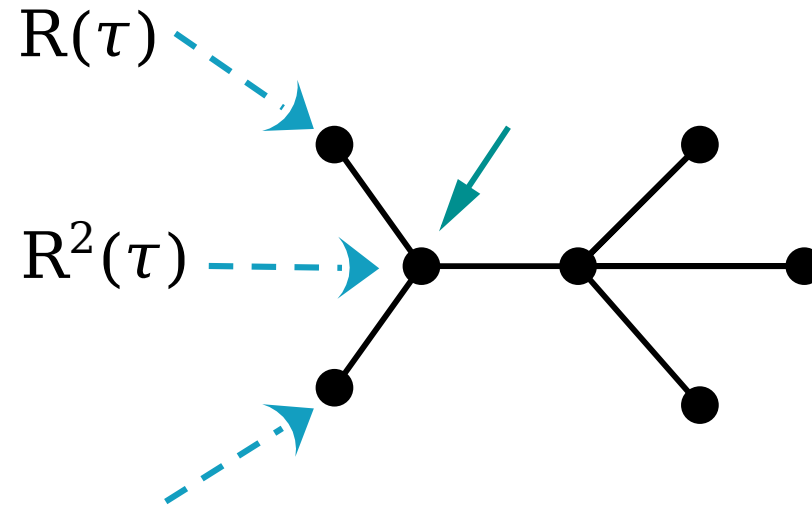
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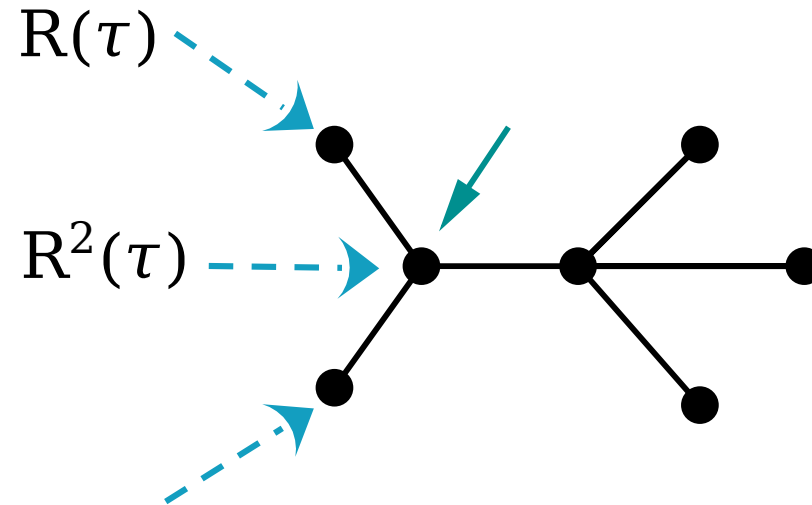
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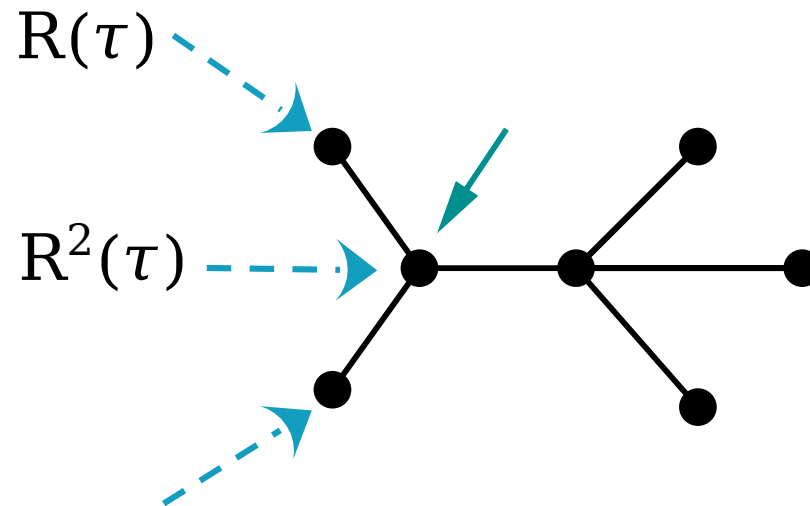
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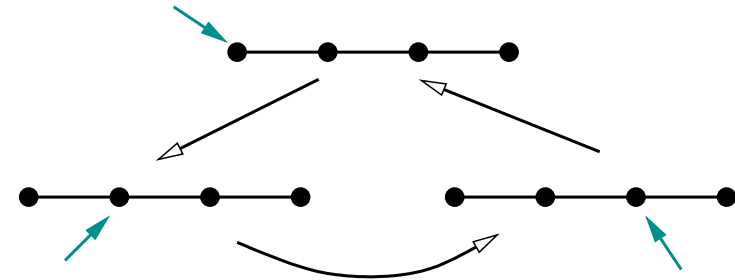
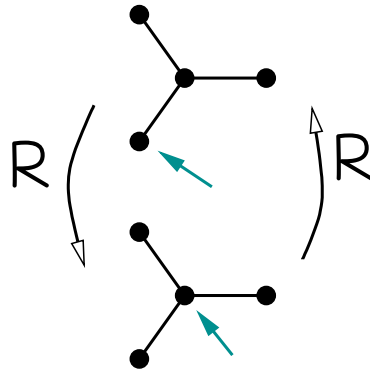


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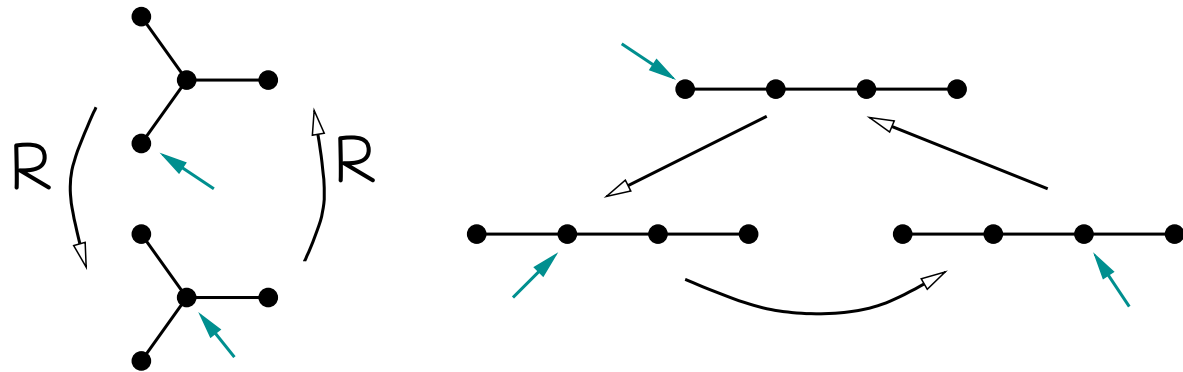
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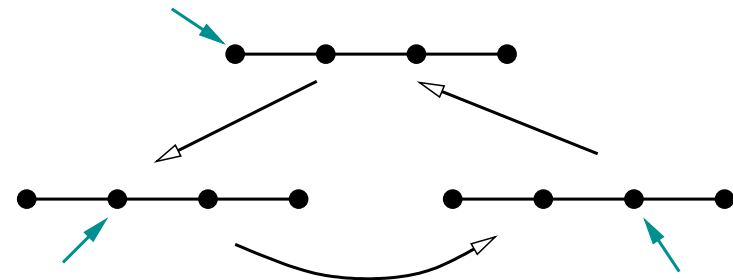
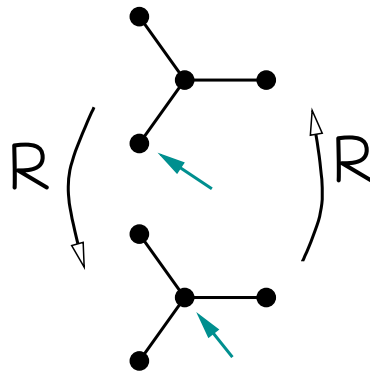


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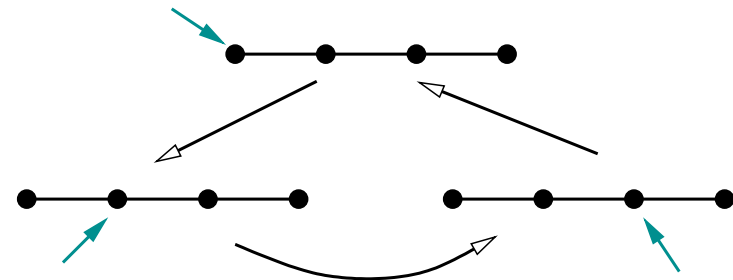
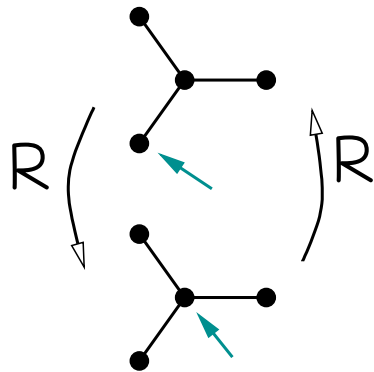


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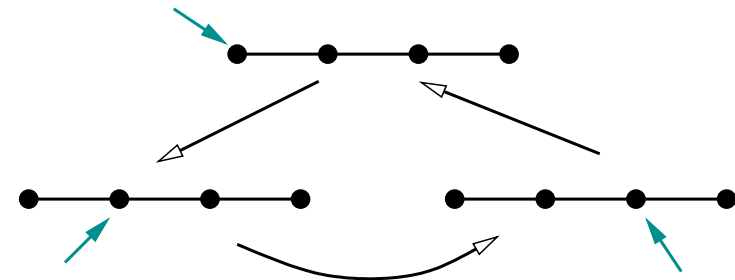
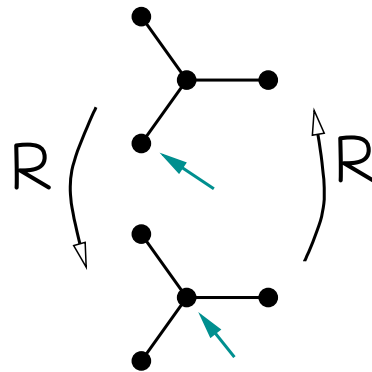
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$$P(\omega^1) = 0, \quad P(\omega^2) = 2 \quad \text{and} \quad P(\omega^3) = P(-1) = 3, \quad P(\omega^6) = P(1) = 5.$$

A cyclic sieving polynomial always exists

Proposition. For G of order m , take

$$\bar{P}(q) = \sum_{i=0}^{m-1} a_i q^i,$$

where a_i is the number of orbits O such that $|G|/|O|$ divides i .

This is a CS polynomial, and any CS polynomial equals this one modulo $q^m - 1$ (because we evaluate P at powers of $\omega = e^{2\pi i/m}$).

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$$= (1 - q + q^2)(1 + q + q^2 + q^3 + q^4) = \Phi_6 \Phi_5$$

q-Analogues: nice polynomials

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- **Cyclotomic polynomials:**

$$1 - q^n = \prod_{d|n} \Phi_d$$

I. Cyclic sieving for plane trees

with Christian Krattenthaler

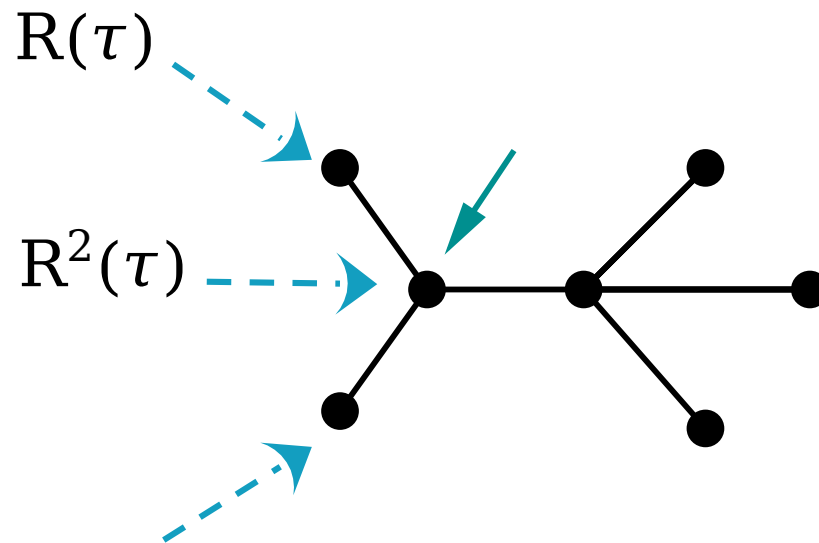
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Ordinary rotation on trees: basic result

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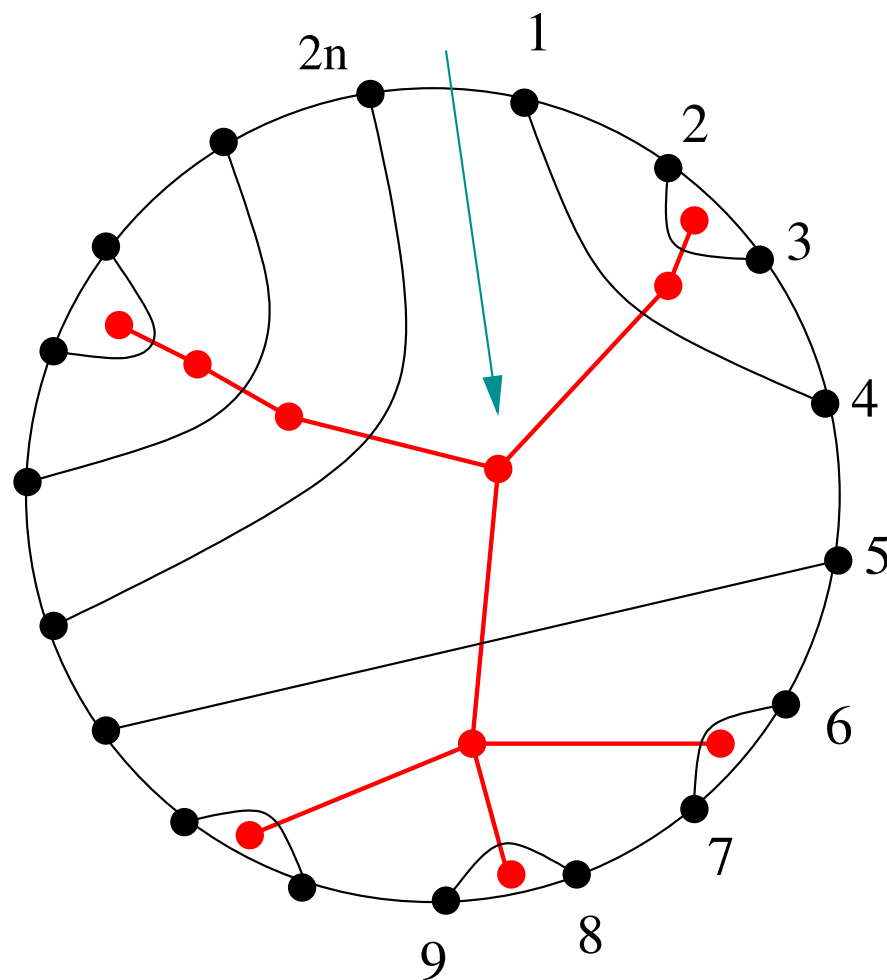
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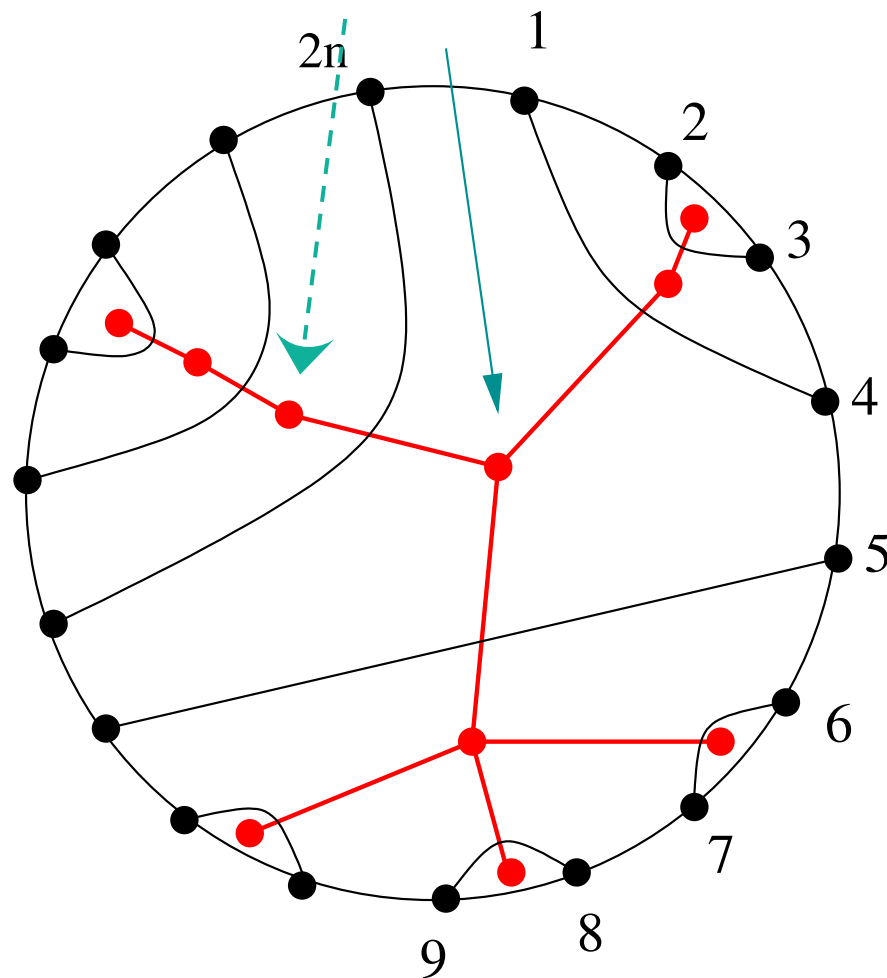
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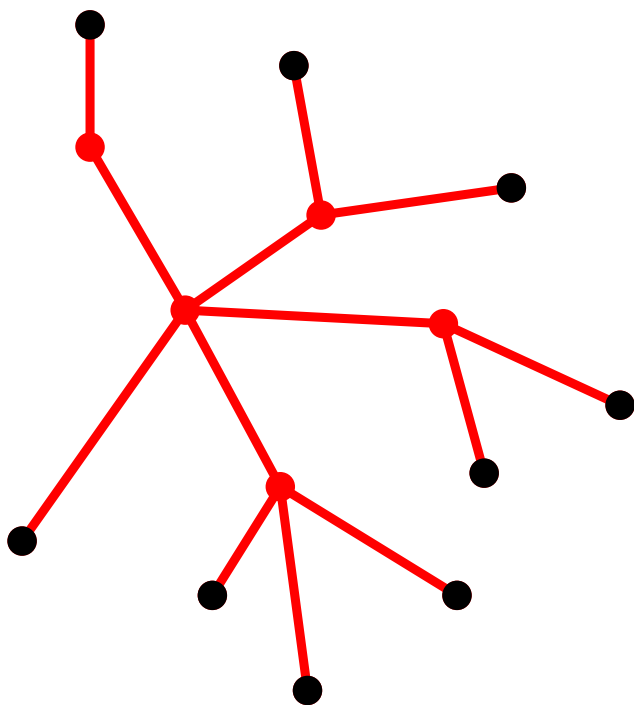
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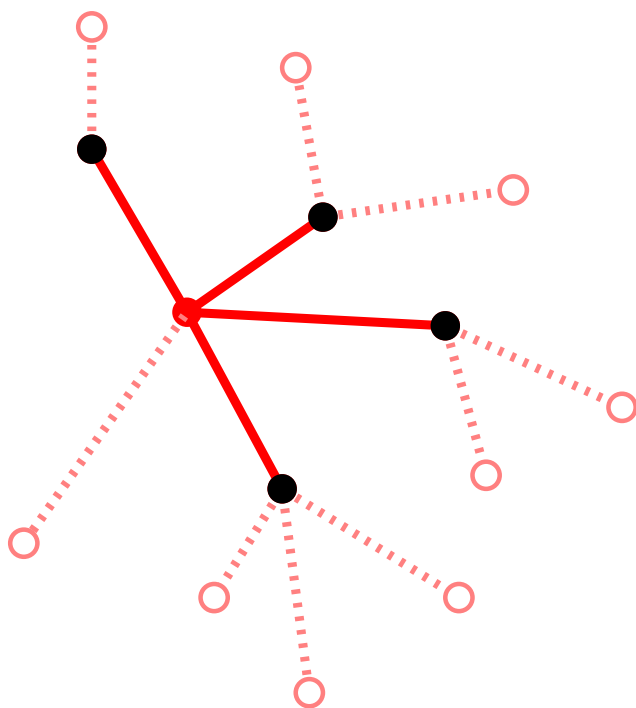
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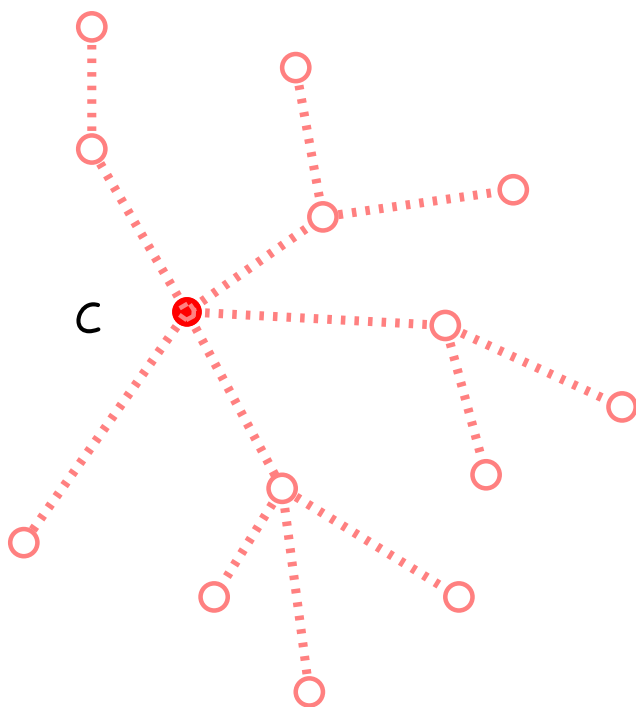
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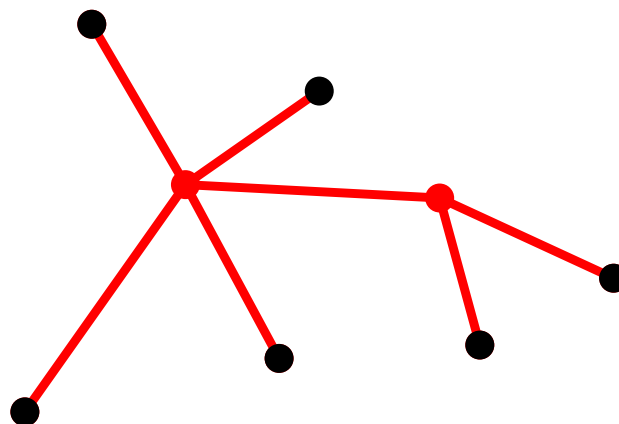
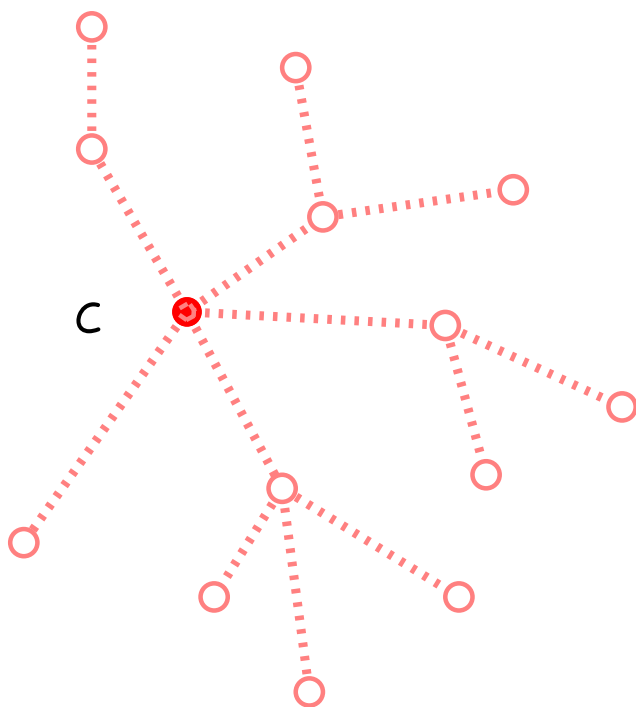
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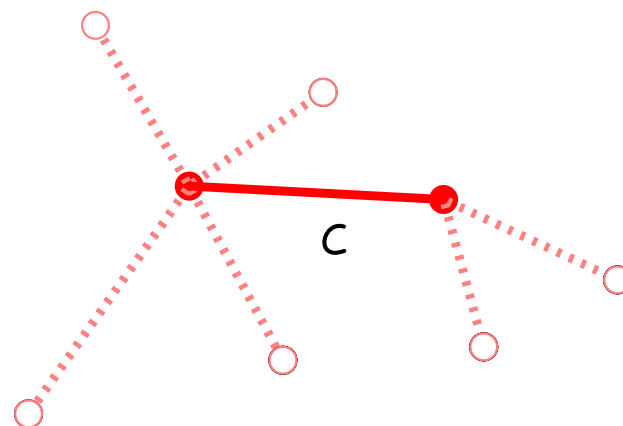
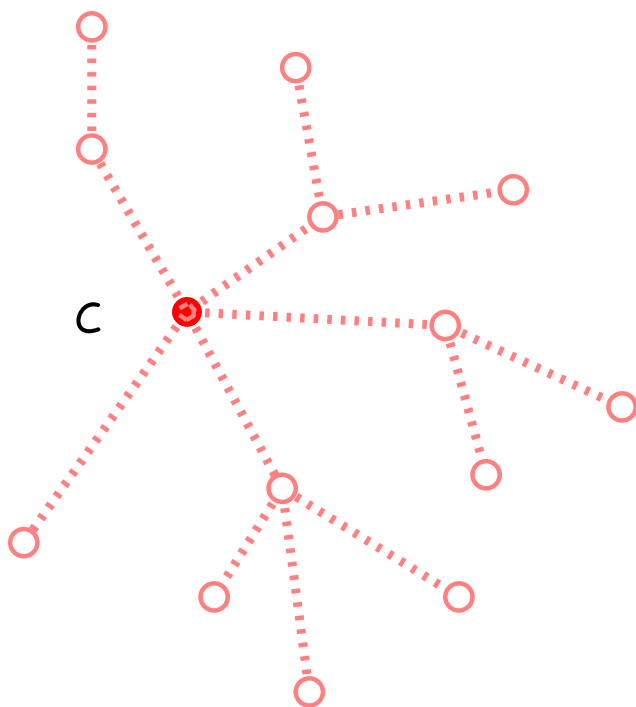
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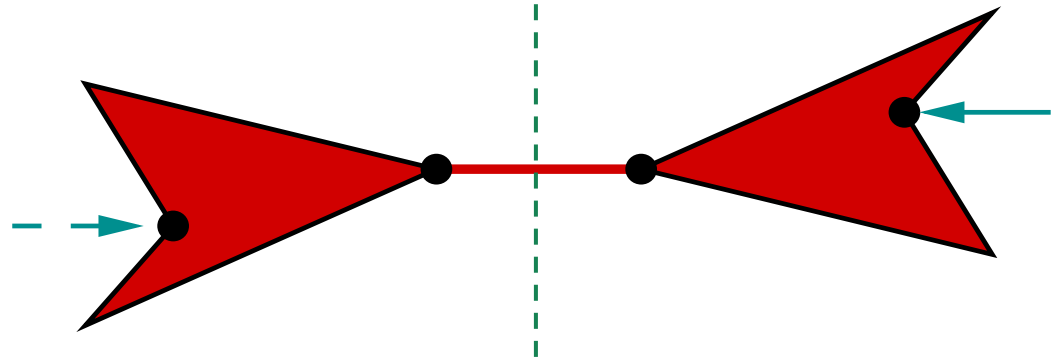
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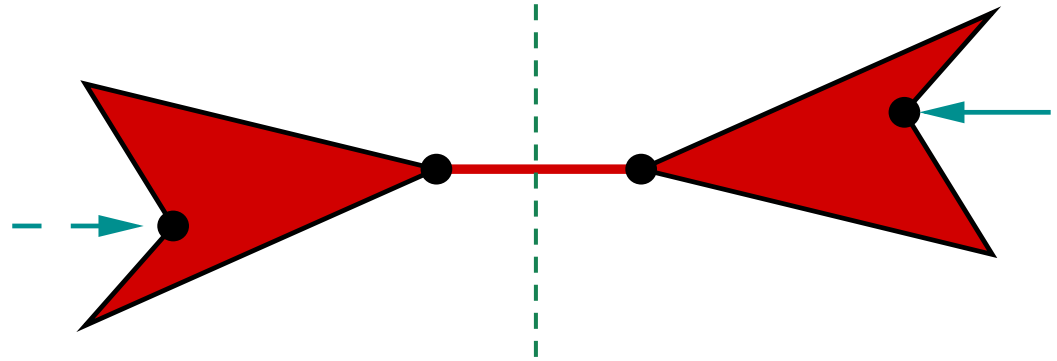
Trees invariant by some re-rooting?

- Tree with a central edge

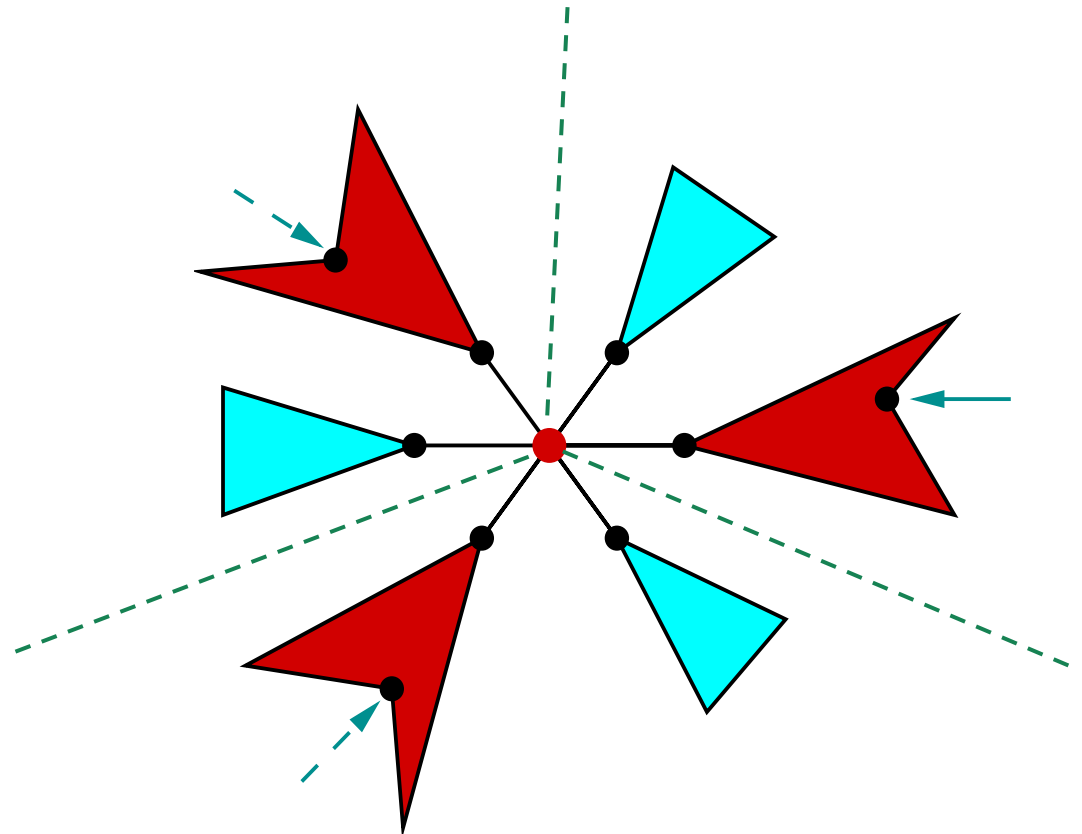


Trees invariant by some re-rooting?

- Tree with a central edge



- Tree with a central vertex



Trees invariant by some re-rooting?

Prop. The number of trees with n edges invariant by R^e with $e=2n/d$ is

$$\begin{cases} \binom{n}{\frac{n+1}{2}}, & \text{if } d = 2 \text{ and } n \text{ is odd,} \\ \binom{e}{\frac{e}{2}}, & \text{if } d \mid n, \\ 0, & \text{otherwise.} \end{cases}$$

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Prop. The ordinary rotation gives an action of the cyclic group of order $2n$ on trees of size n , and a CS phenomenon holds with

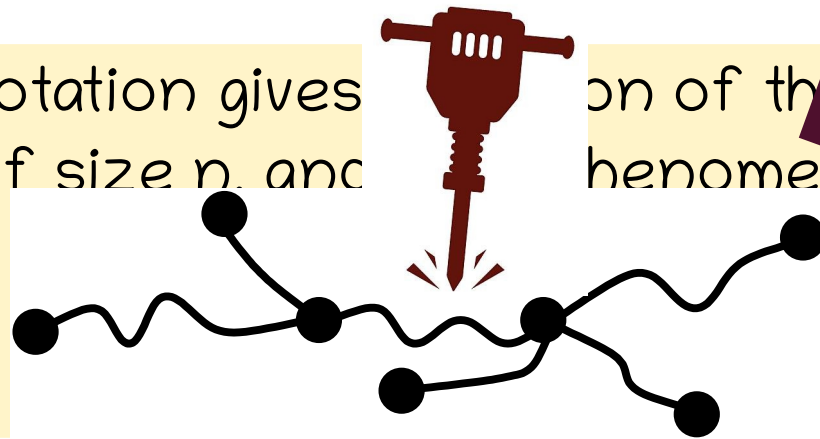
$$P(q) = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

Trees invariant by some re-rooting?

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A robust
result

Refinement: the number of leaves

Prop. The ordinary rotation gives an action of the cyclic group of order $2n$ on trees with n edges and k leaves, and a CS phenomenon holds with CS polynomial

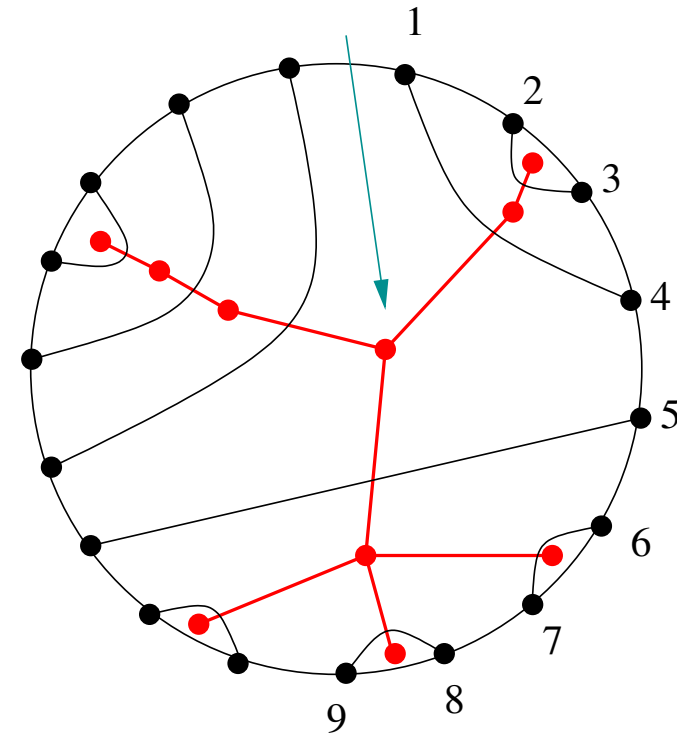
$$\frac{1 + q^n}{[n-1]} \begin{bmatrix} n-1 \\ k-2 \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix}.$$

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- Known in terms of non-crossing matchings [Alexandersson, Linusson, Potka, Uhlin 21]



Refinement: the degree distribution

Prop. The ordinary rotation gives an action of the cyclic group of order $2n$ on trees with n_i vertices of degree i for all $i \geq 1$, and a CS phenomenon holds with CS polynomial

$$\frac{1 + q^n}{[n + 1]} \left[\begin{matrix} n + 1 \\ n_1, n_2, \dots \end{matrix} \right] \cdot \quad \swarrow \quad \text{q-multinomial}$$

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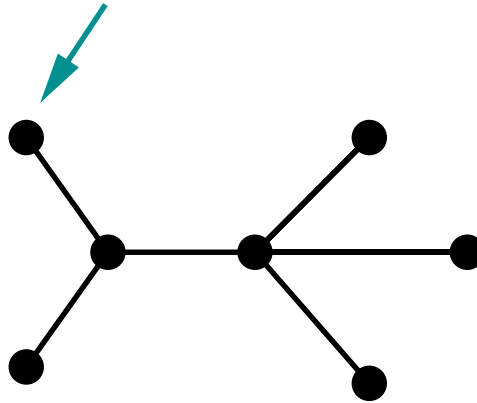
- This assumes that

$$n + 1 = \sum_i n_i \quad \text{and} \quad \sum_{i \geq 1} (i - 2)n_i = -2.$$

- Apparently new, at last!

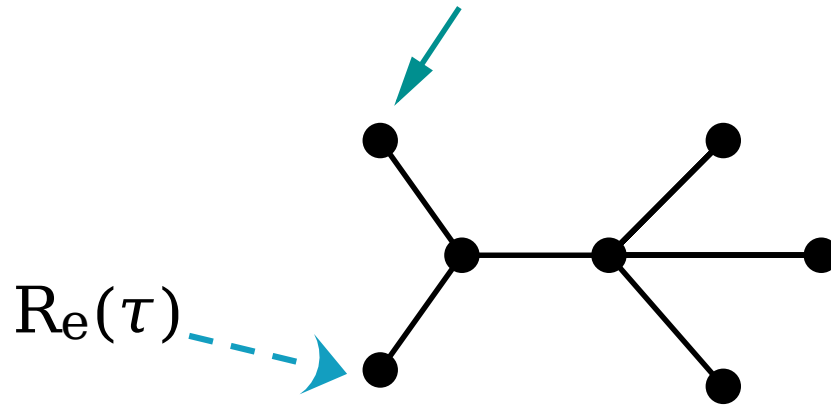
External rotation on trees (rooted at a leaf)

The external rotation R_e moves the root to the next leaf in cc order.



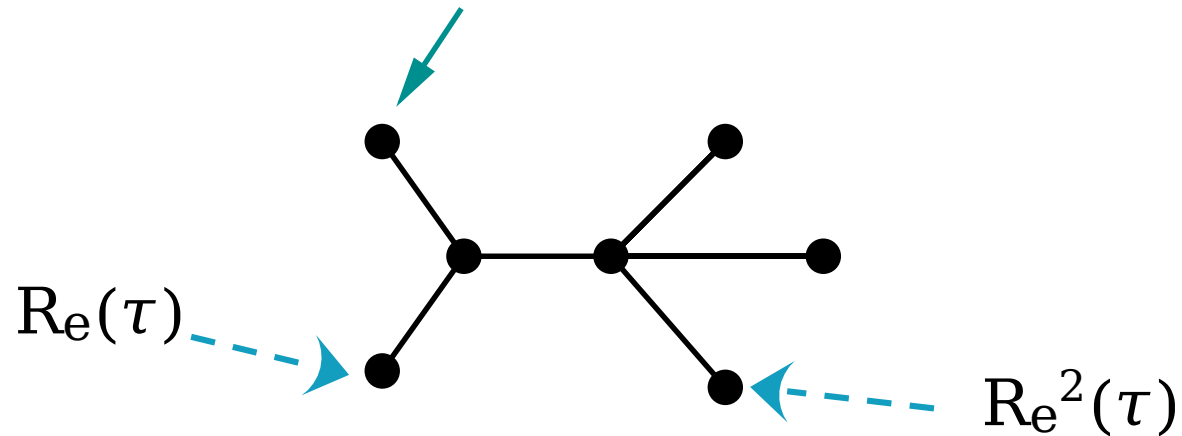
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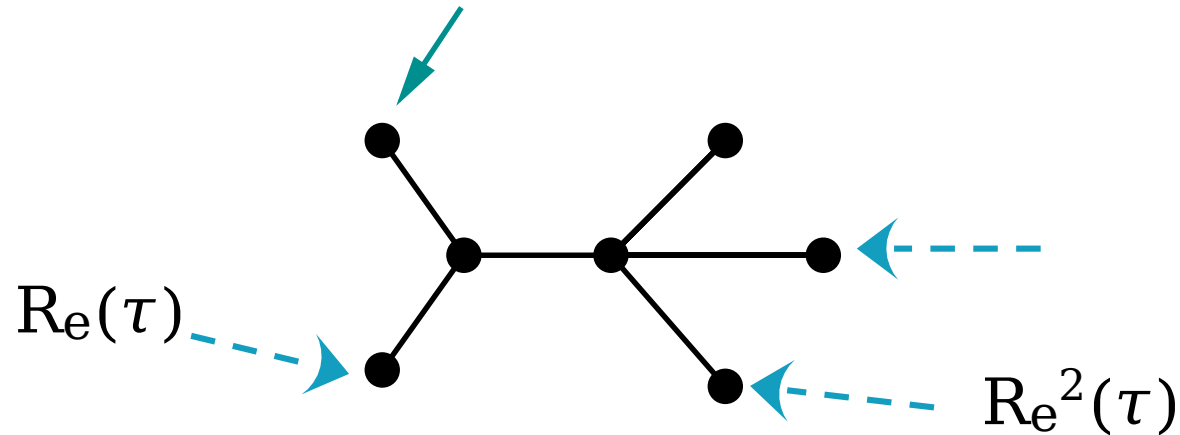
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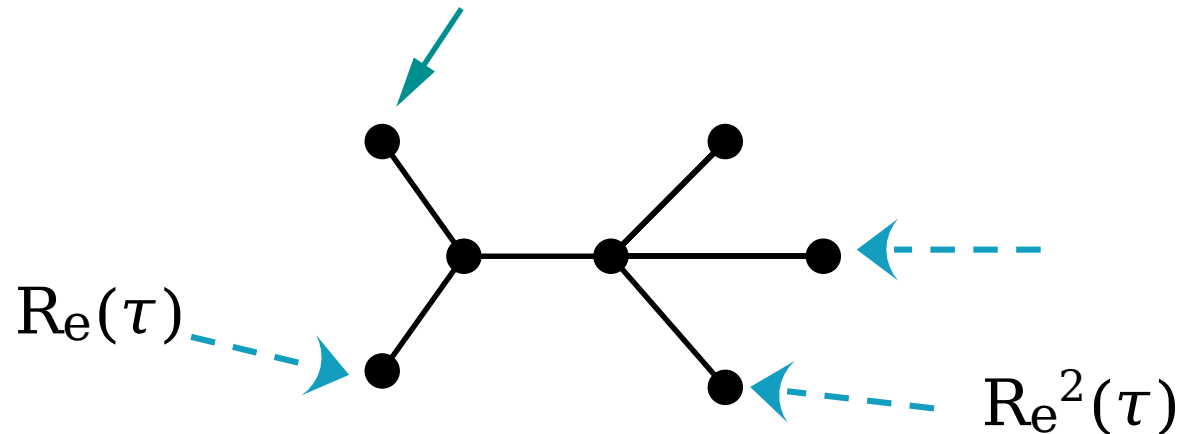
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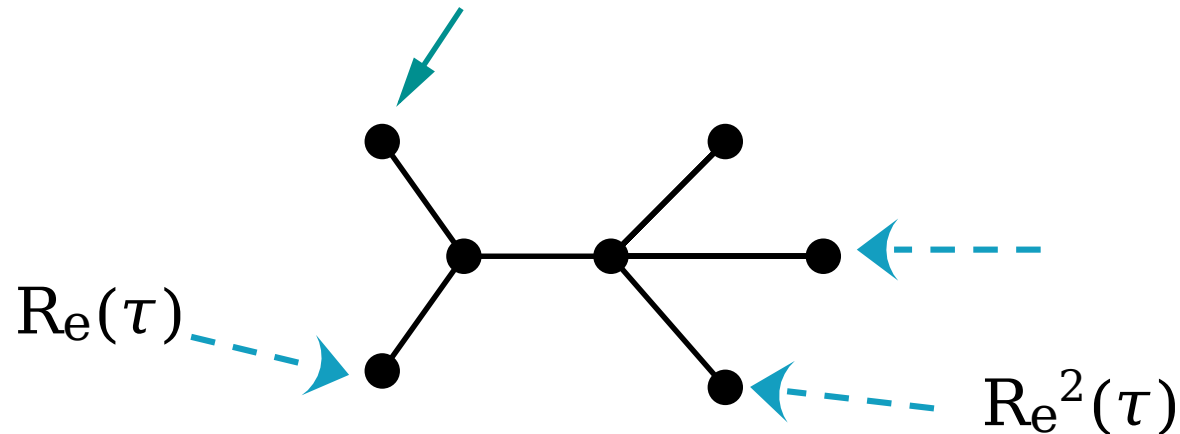


Prop. The external rotation gives an action of the cyclic group of order k on trees having n edges and k leaves, and a CS phenomenon holds with

$$P(q) = \frac{1}{[n-1]} \begin{bmatrix} n-1 \\ k-2 \end{bmatrix} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

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Refinement. CS for trees with n_i vertices of degree i for all $i \geq 1$.

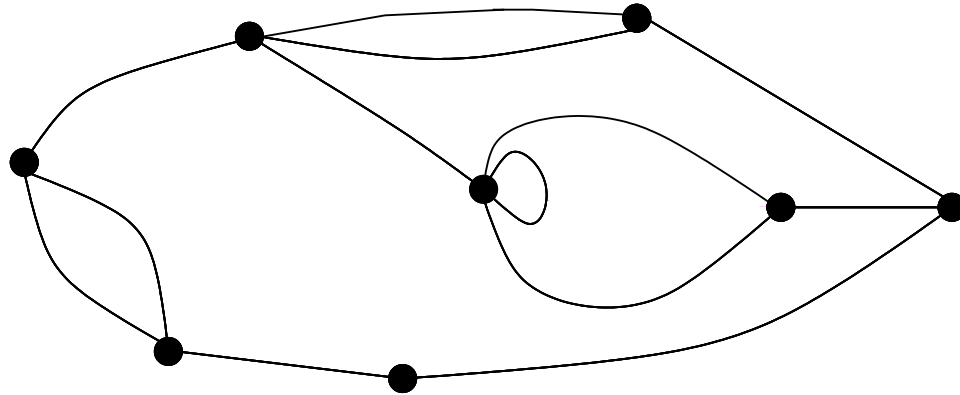
**II. Between trees and
maps:
Tree-rooted maps**

Planar maps

Def. A connected planar (multi)graph, given with a proper embedding in the **sphere**, taken up to continuous deformation.

Components:

- vertices
- edges
- faces

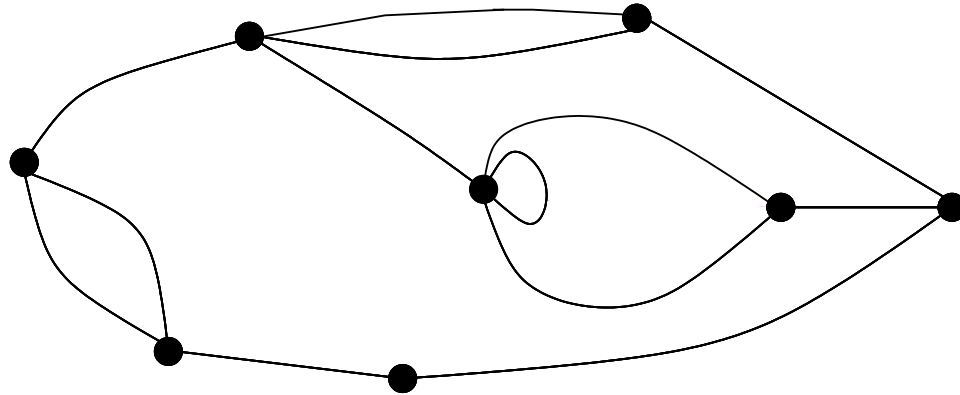


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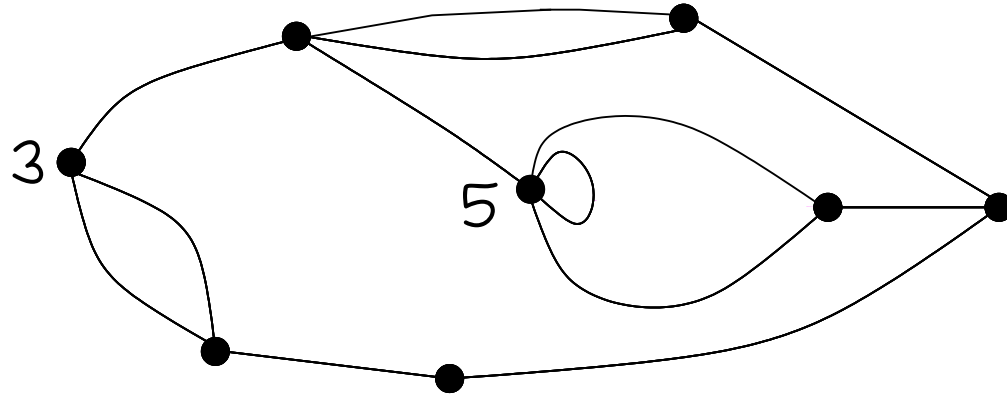


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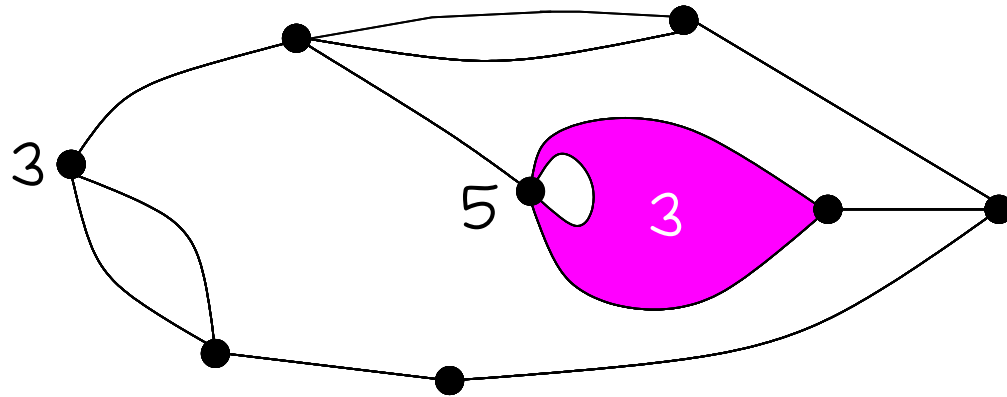


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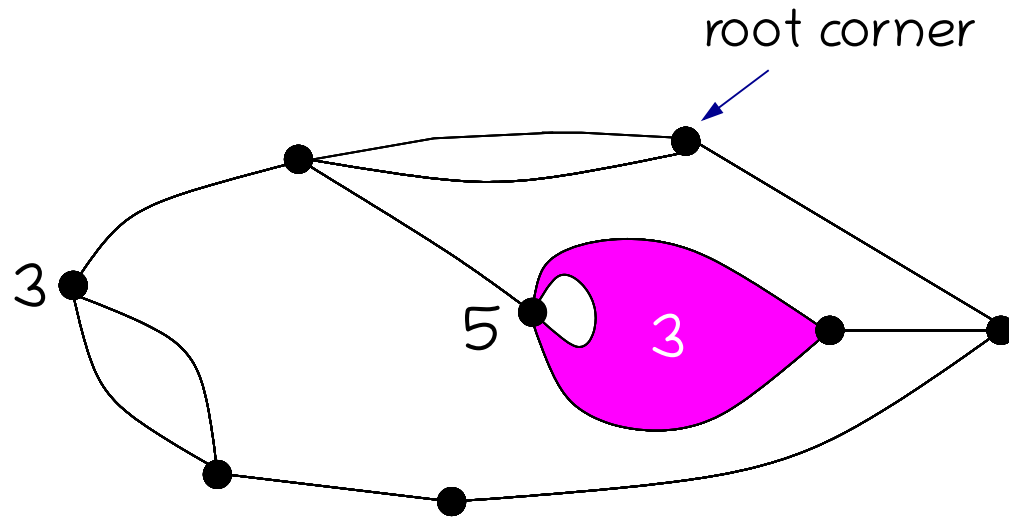


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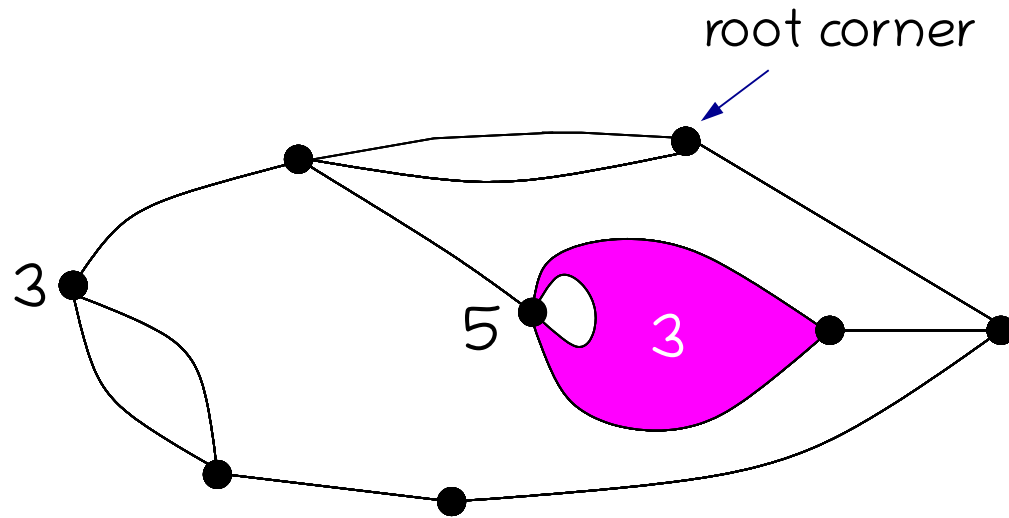
Rooted map: a distinguished corner

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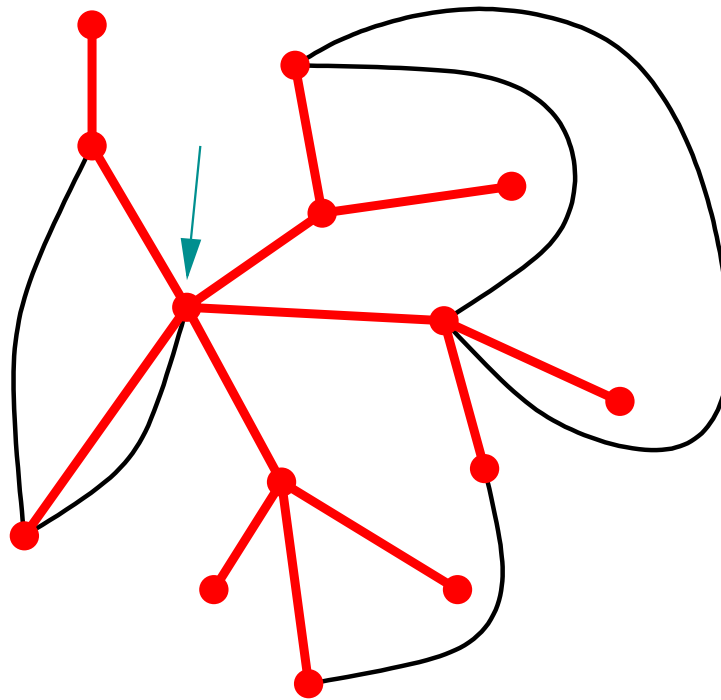
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Rooted map: a distinguished corner

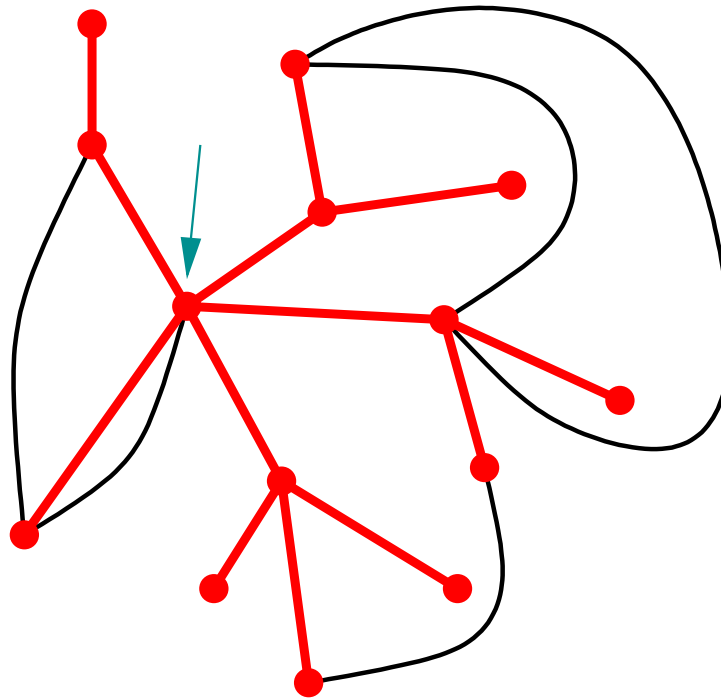
A map with n edges has $2n$ corners.

Def. A rooted planar map with a distinguished spanning tree.



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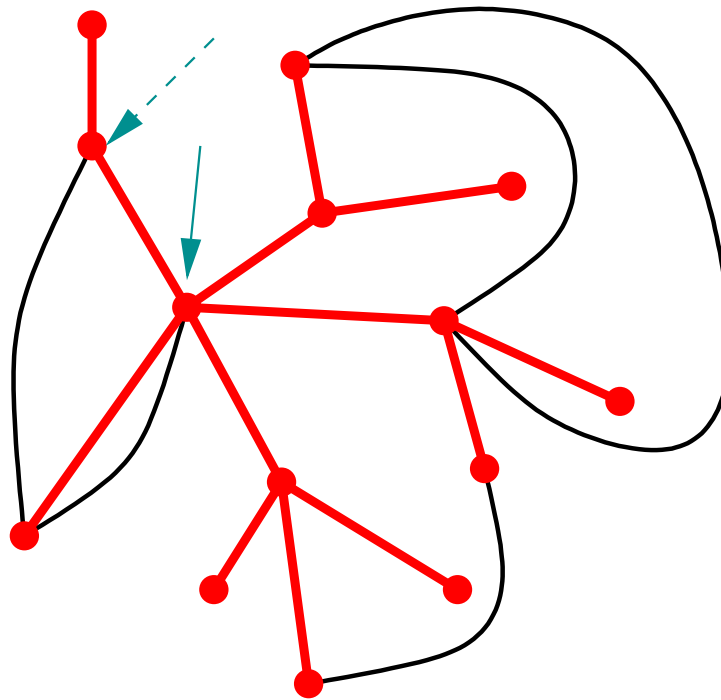
Rotation: move the root corner to the next corner of the map around the tree, in cc order.



Tree-rooted planar maps

Def. A rooted planar map with a distinguished spanning tree.

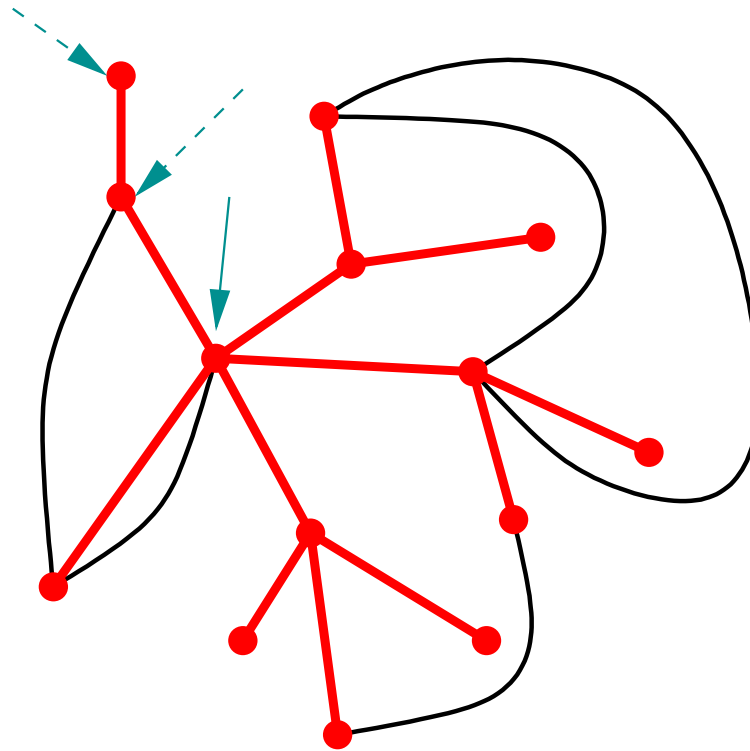
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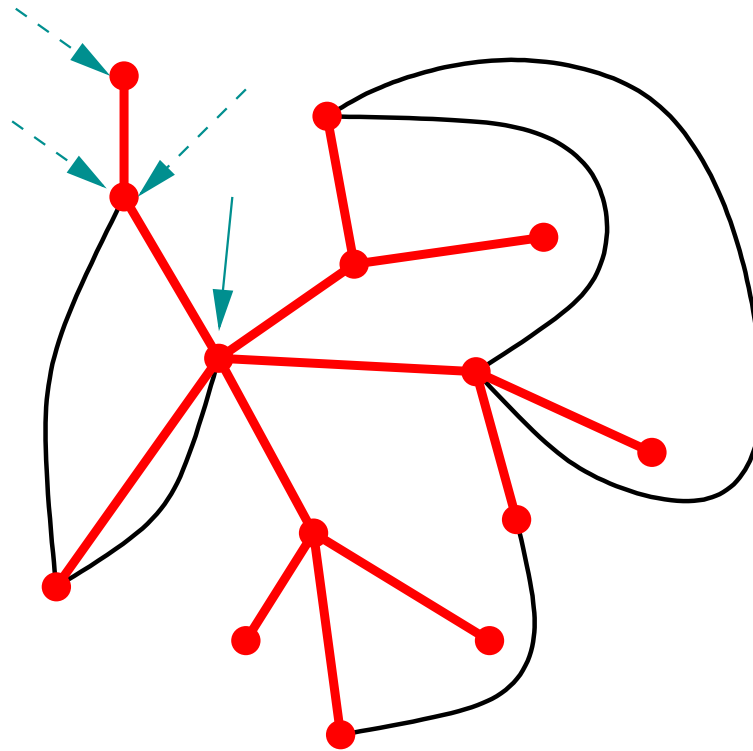
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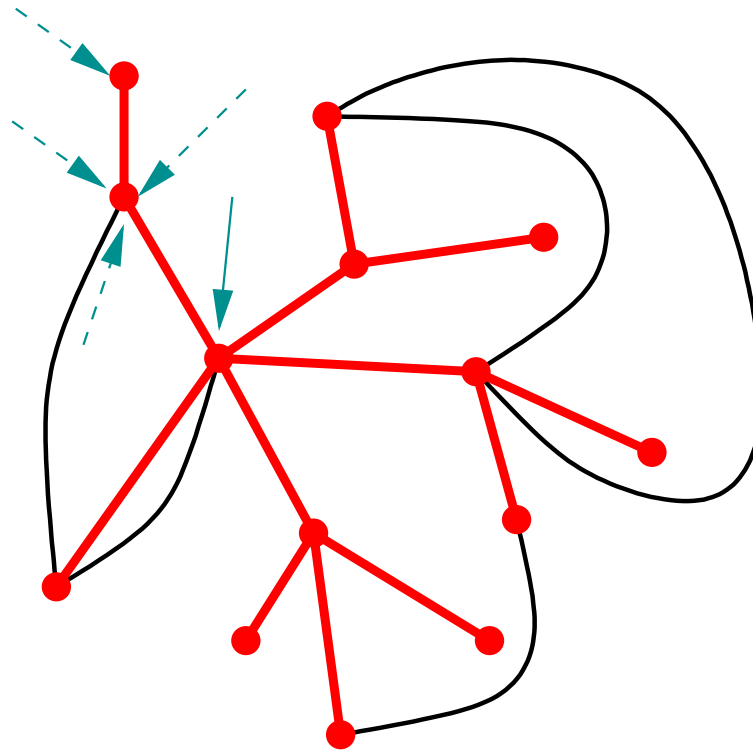
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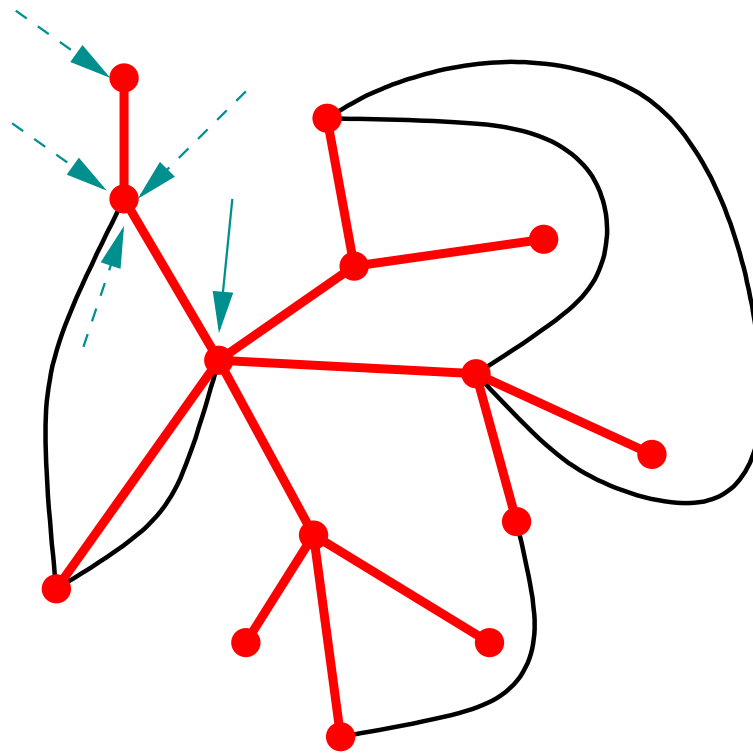


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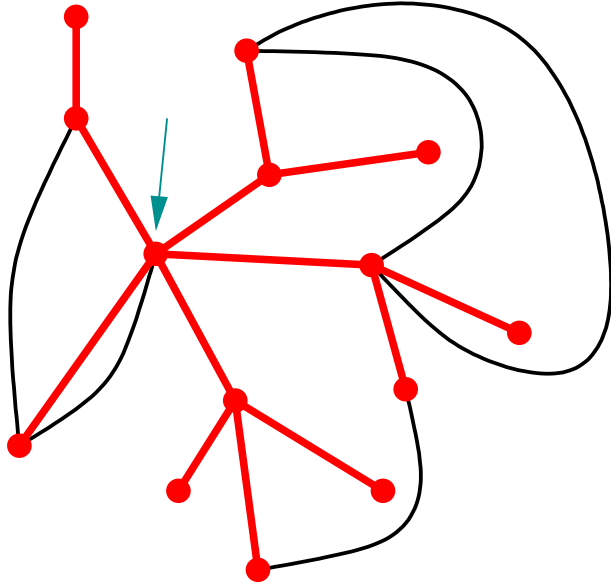
⇒ Action of the cyclic group of order $2n$ on tree-rooted maps having n edges



Tree-rooted planar maps: structure

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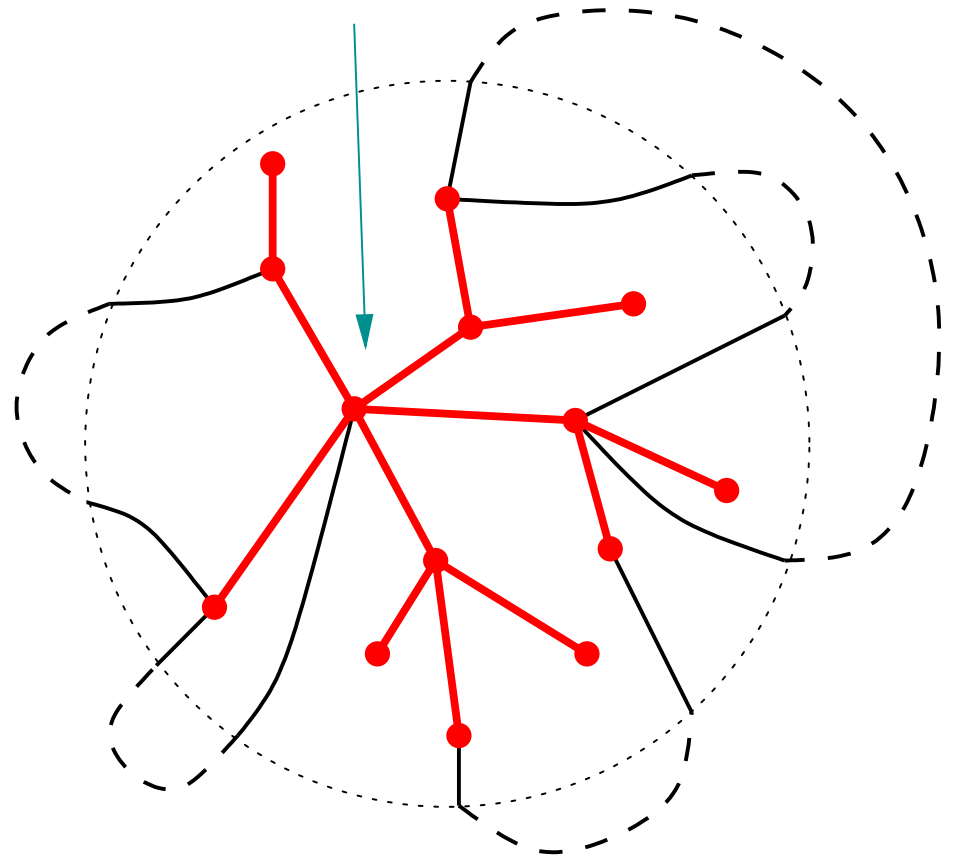
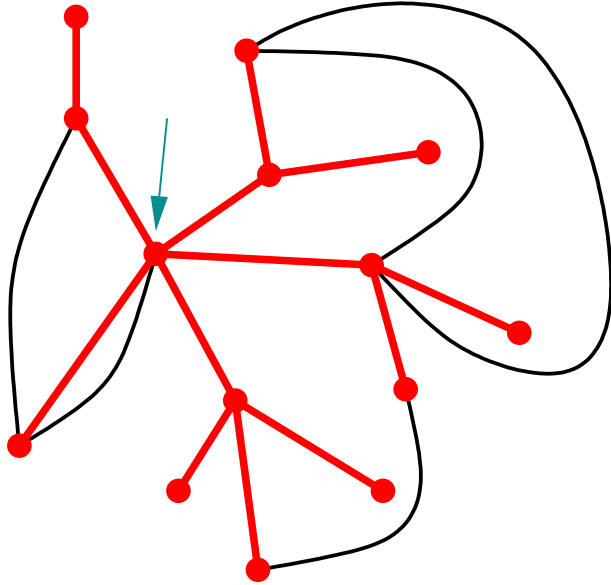
i inner edges, j outer edges $\Leftrightarrow i+1$ vertices, $j+1$ faces $(i+j=n)$



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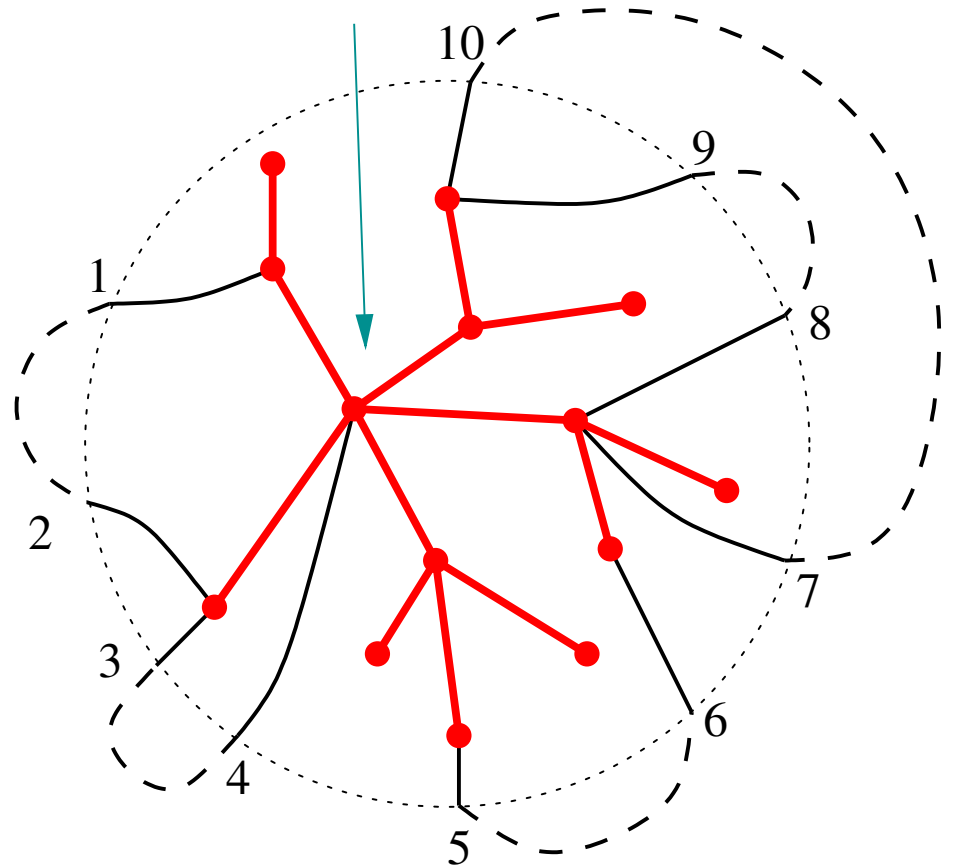
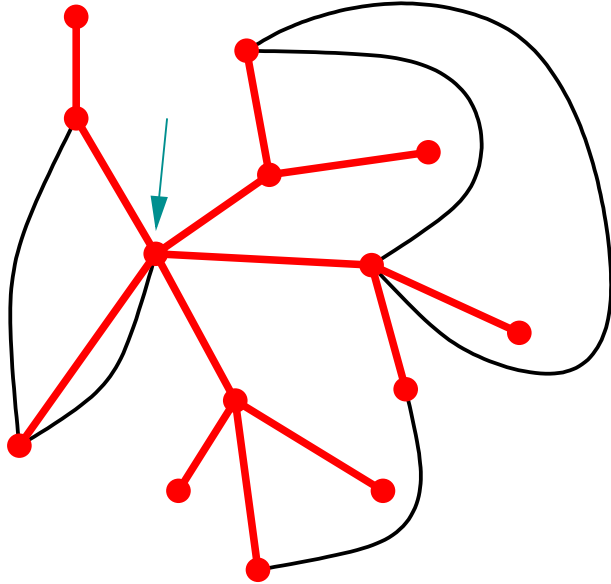
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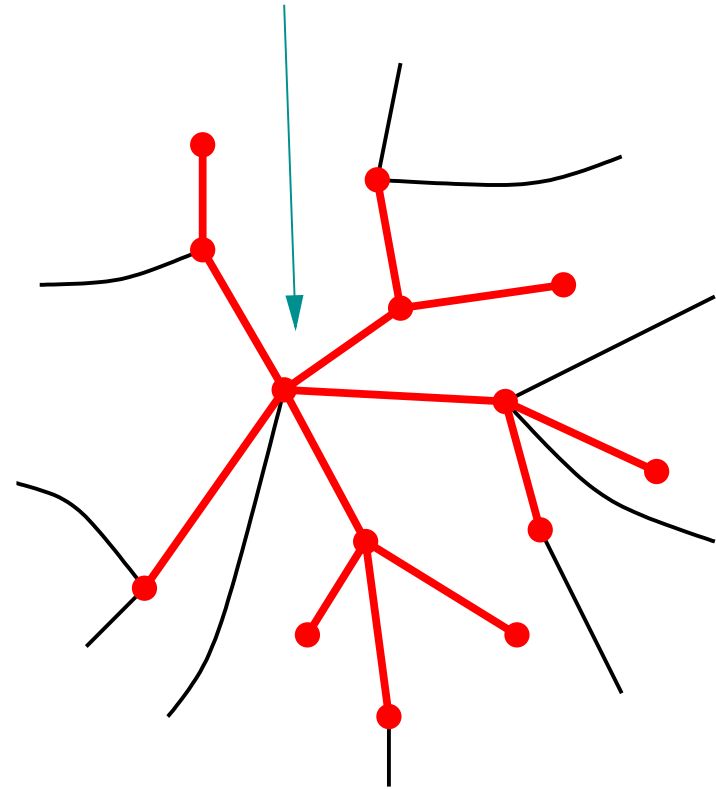
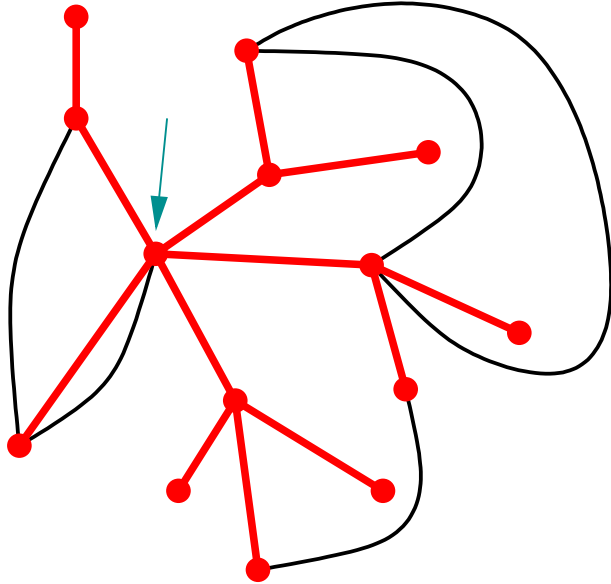
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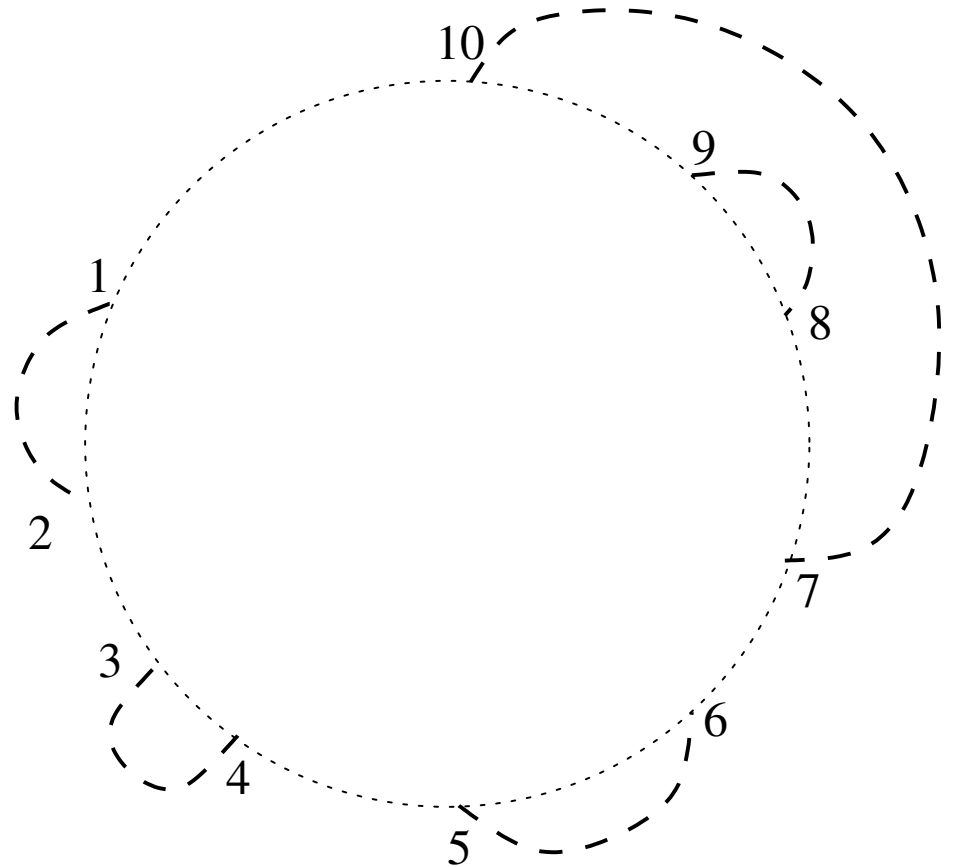
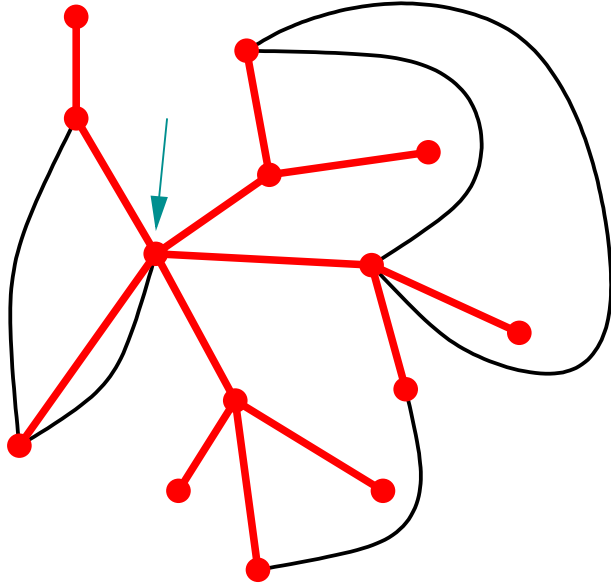


Tree with i edges
and $2j$ buds
(black half-edges)

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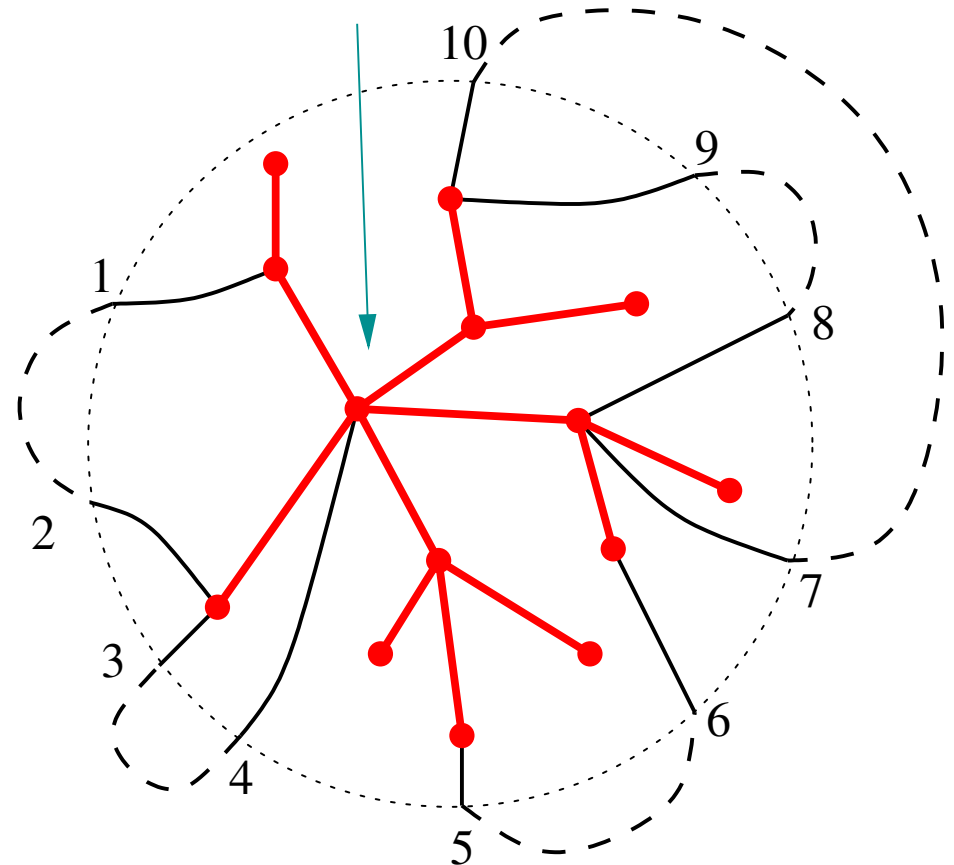
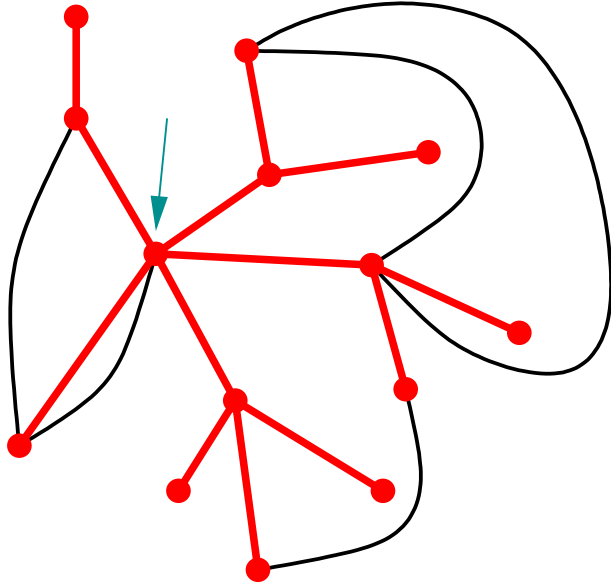


Non-crossing
matching
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A CS phenomenon for tree-rooted maps

Prop. The rotation gives an action of the cyclic group of order $2n$ on tree-rooted maps with $i+1$ vertices and $j+1$ faces (with $i+j=n$), and a CS phenomenon holds with

$$P(q) = \frac{1}{[i+1][j+1]} \begin{bmatrix} 2i+2j \\ i, i, j, j \end{bmatrix}.$$

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Coro. There is also a CS phenomenon for tree-rooted maps with n edges, with CS polynomial

$$[C_n]_q [C_{n+1}]_q = \frac{1}{[n+1][n+2]} \begin{bmatrix} 2n \\ n \end{bmatrix} \begin{bmatrix} 2n+2 \\ n+1 \end{bmatrix}.$$

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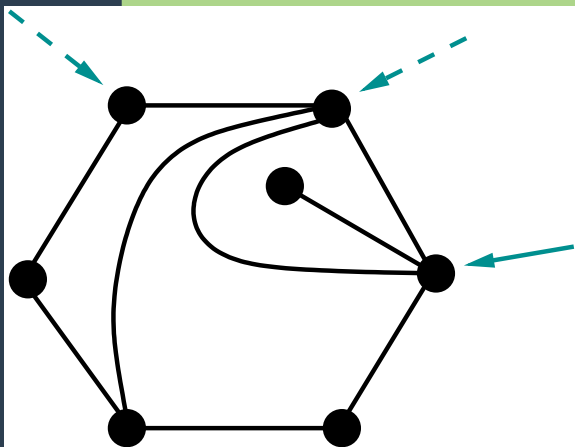
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Remark. We did not count directly tree-rooted maps with n edges fixed by R^e .

III. Families of maps



with Marie Albenque,
work in progress

Cyclic sieving for maps?

We need a family of maps...

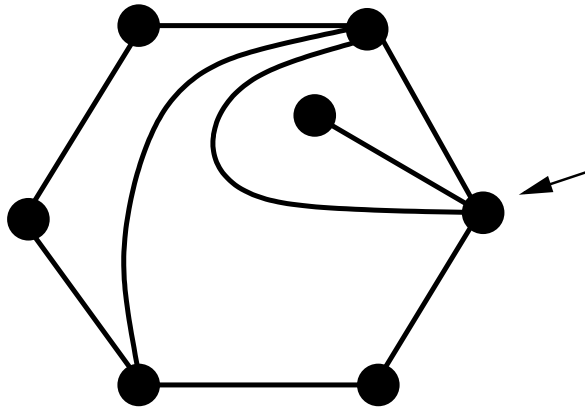
- with a cyclic action
- counted by nice numbers

Cyclic sieving for maps?

We need a family of maps...

- with a cyclic action
- counted by nice numbers

Quadrangulations
(all finite faces of degree 4)

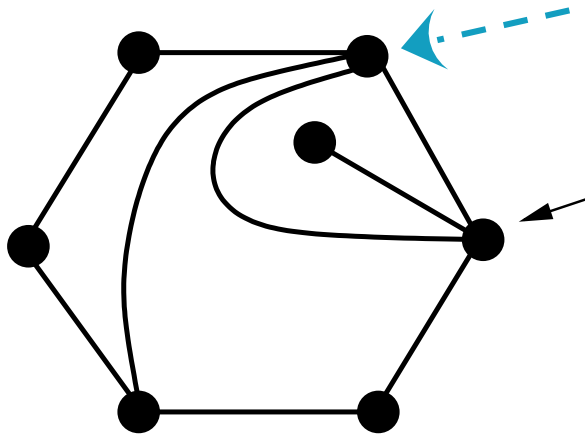


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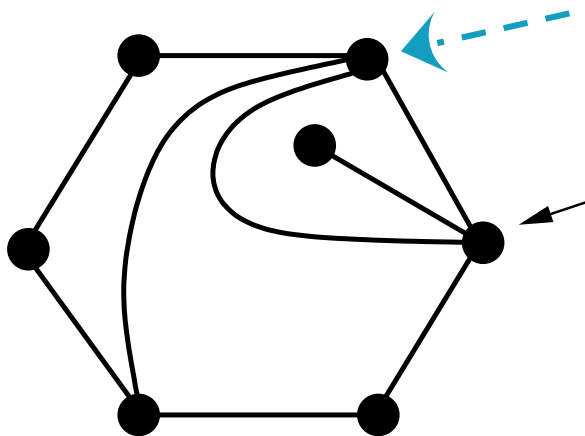
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n faces, perimeter 2ℓ :
order 2ℓ

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Cyclic sieving for maps?

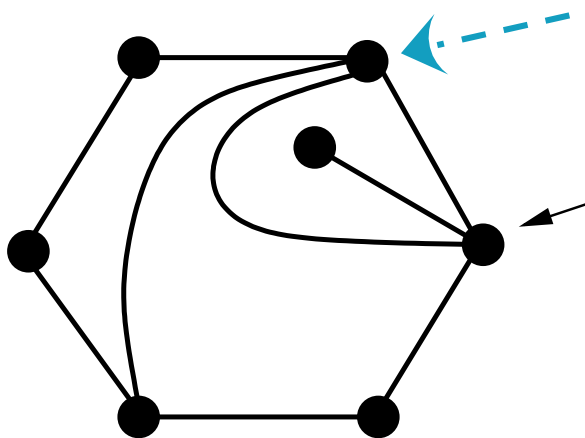
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Quadrangulations
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$$\ell \binom{2\ell}{\ell} \frac{3^n}{(n + \ell + 1)(2n + \ell)} \binom{2n + \ell}{n}$$

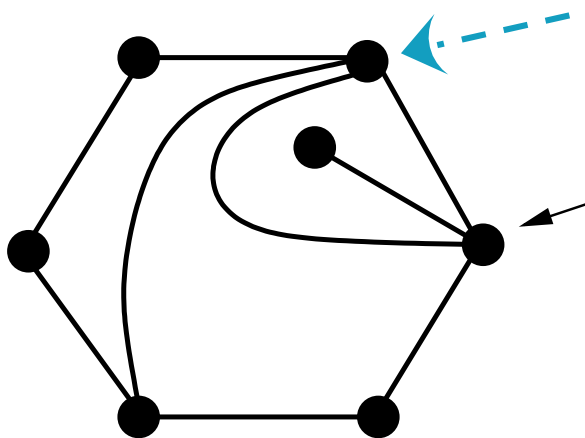
[Tutte 62]

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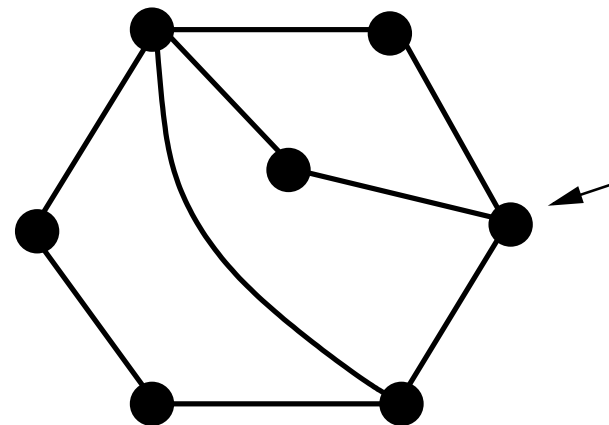


$$\ell \binom{2\ell}{\ell} \frac{3^n}{(n + \ell + 1)(2n + \ell)} \binom{2n + \ell}{n}$$

[Tutte 62]

n faces, perimeter 2ℓ :
order 2ℓ

Simple quadrangulations
(no double edge)



$$\frac{\ell - 1}{n(3n + 1)} \binom{3\ell - 2}{\ell - 1} \binom{3n + 1}{n - \ell + 1}$$

[Brown 65]

Ordinary rotation on trees

Aim. The rotation of the root corner gives an action of the cyclic group of order 2ℓ on (simple) quadrangulations having n faces and perimeter 2ℓ , and a CS phenomenon holds with

$$P_{n,\ell}(q) = \dots$$

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Steps of proof

- | | |
|--|--|
| • Guess a polynomial $P_{n,\ell}(q)$ | |
| • Algebra: $P_{n,\ell}(q)$ is a polynomial | |
| • ... with coefficients in $\mathbb{N}$ | |
| • Enumeration: count quadrangulations fixed by R^e , where e divides 2ℓ | |
| • Evaluation: their number should be $P_{n,\ell}(\omega^e)$, with ω a prim. 2ℓ -th root of 1 | |

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hard with terms c^n

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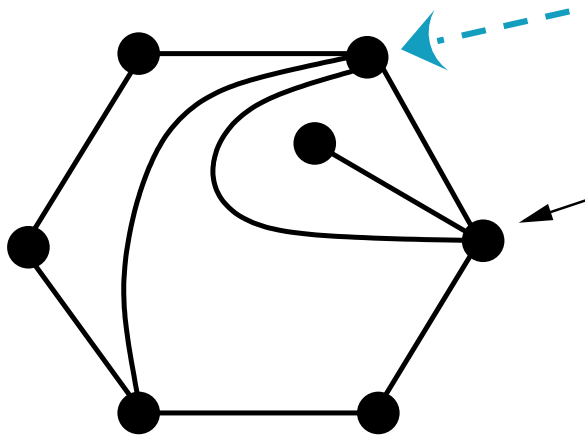
- **Enumeration:** count quadrangulations fixed by R^e , where e divides 2ℓ

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Cyclic sieving for maps?

n faces, perimeter 2ℓ

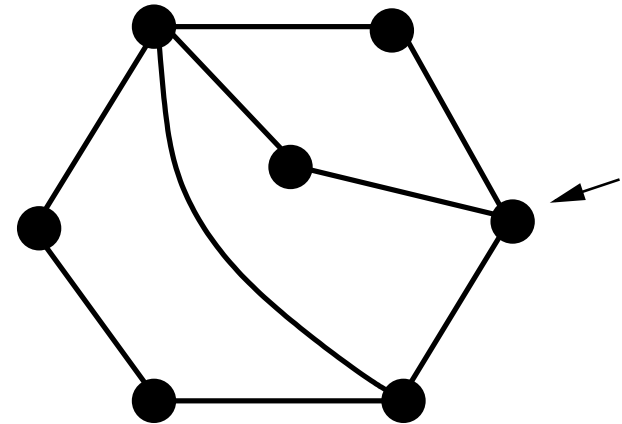
Near-quadrangulations
(all finite faces of degree 4)



$$\ell \binom{2\ell}{\ell} \frac{3^n}{(n + \ell + 1)(2n + \ell)} \binom{2n + \ell}{n}$$

[Tutte 62]

Simple quadrangulations
(no double edge)



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Steps of proof

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hard with terms c^n

- **Algebra:** $P_{n,\ell}(q)$ is a polynomial

routine

- ... with coefficients in $\mathbb{N}$

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Ordinary rotation on trees

Aim. The rotation of the root corner gives an action of the cyclic group of order 2ℓ on (simple) quadrangulations having n faces and perimeter 2ℓ , and a CS phenomenon holds with

$$P_{n,\ell}(q) = \dots$$

Steps of proof

- **Guess a polynomial** $P_{n,\ell}(q)$

hard with terms c^n

- **Algebra:** $P_{n,\ell}(q)$ is a polynomial

routine

- ... with coefficients in $\mathbb{N}$

Ouch!
only for some (n, ℓ)

- **Enumeration:** count quadrangulations fixed by R^e , where e divides 2ℓ

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Thank
you!

Merci !