

Greedy Tamari Intervals and Constellations

Jean-Bernard is Eighty

IHP

September 24, 2026

Philippe Biane

CNRS, LIGM

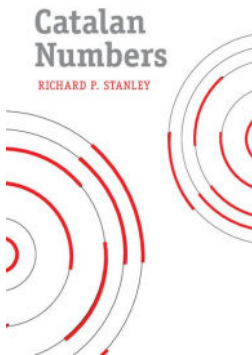
Université Gustave-Eiffel

Joint Work with Wenjie Fang (LIGM)

Catalan numbers

$$c_n = \frac{1}{n+1} \binom{2n}{n}$$

They appear in countless combinatorial problems of



Catalan recurrence

The generating series of Catalan numbers

$$C(t) = 1 + \sum_{n=1}^{\infty} c_n t^n$$

satisfies

$$C(t) = 1 + tC(t)^2$$

or

$$c_{n+1} = \sum_{n_1+n_2=n} c_{n_1} c_{n_2}$$

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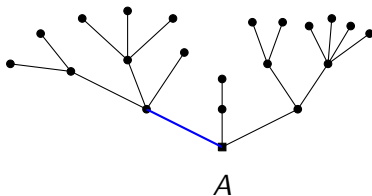
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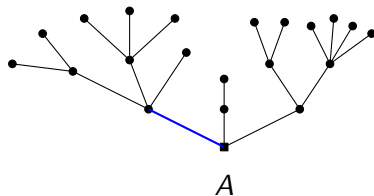
A combinatorial version is a bijective recurrence between sets:

$$\mathcal{C}_{n+1} \longleftrightarrow \bigcup_{n_1+n_2=n} \mathcal{C}_{n_1} \times \mathcal{C}_{n_2}$$

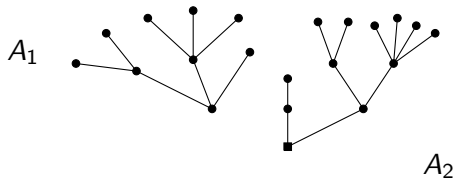
Example: rooted plane trees



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Taking out the root edge



$\mathcal{A}_n$ = rooted plane trees with n edges

$$\mathcal{A}_{n+1} \longleftrightarrow \bigcup_{n_1+n_2=n} \mathcal{A}_{n_1} \times \mathcal{A}_{n_2}$$

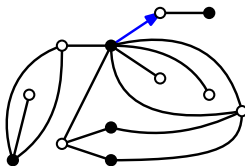
Generalizations I: Fuss-Catalan

$$C^{(m)}(t) = 1 + tC(t)^{m+1}$$

Counts planar rooted trees with mn edges and all ascents multiple of m (or $m+1$ -ary trees).

Generalizations II: Bipartite planar maps

A rooted bipartite planar map



I will explain a well known recurrence for the enumeration of such maps

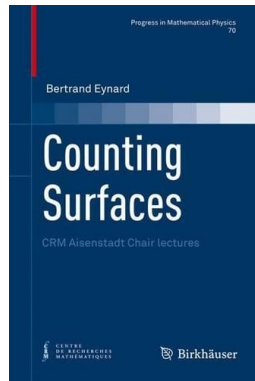
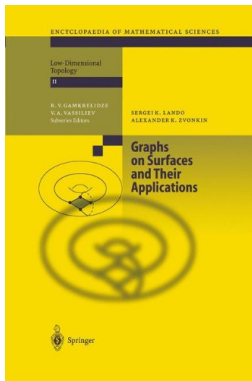
Map enumeration

Map enumeration was first considered by Tutte in the 60's.

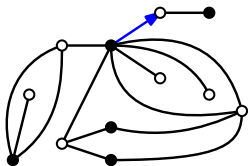
Matrix integral techniques to do such enumeration were introduced by Jean-Bernard and coworkers in several influential papers, cf

- ▶ E. Brézin, C. Itzykson, G. Parisi, J. B. Zuber, Planar diagrams. Comm. Math. Phys. 59 (1978)
- ▶ D. Bessis, C. Itzykson, J.B. Zuber, Quantum field theory techniques in graphical enumeration. Adv. in Appl. Math. 1 (1980)

This is a flourishing subject with several books written e.g.



Generating function of bipartite planar maps



$outdeg(M)$ = half-degree of the outer face of M

$deg(f)$ = the half-degree of a given inner face f .

A generating function for bipartite planar maps in $\mathcal{M}_n$ (=maps with n edges) is defined by

$$F_{\mathcal{M}}(t, x) = \sum_{n \geq 0} t^n \sum_{M \in \mathcal{M}_n} x^{outdeg(M)} \prod_{f \text{ inner face of } M} p_{deg(f)}.$$

Functional equation (Tutte)

Tutte showed that

$$F_{\mathcal{M}}(t, x) = 1 + xtF_{\mathcal{M}}(t, x)^2 + xt\Omega F_{\mathcal{M}}(t, x).$$

where

$$\forall i \geq 0, \quad \Omega x^i = \sum_{k=1}^i x^{i-k} p_k.$$

When $p_i \equiv 0$ this is the equation for Catalan numbers (maps with one face=planar trees).

Bijjective recurrence (Tutte)

$\mathcal{C}_{k,n}$ = maps with n edges and outface of degree k

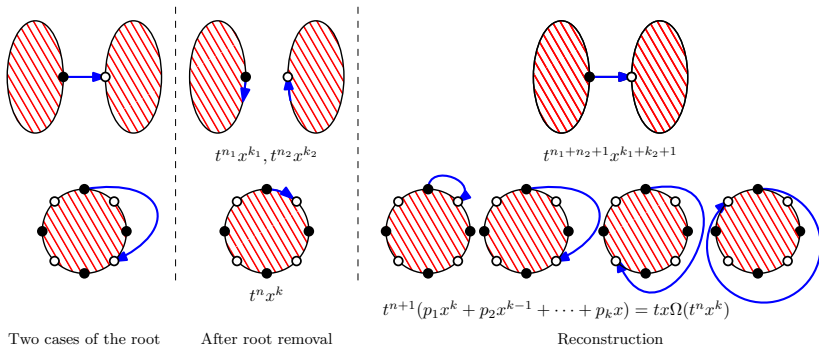
The $\mathcal{C}_{k,n}$ satisfy a bijective recurrence:

$$\mathcal{C}_{k+1,n+1} \longleftrightarrow \left(\bigcup_{\substack{k_1+k_2=k \\ n_1+n_2=n}} \mathcal{C}_{k_1,n_1} \times \mathcal{C}_{k_2,n_2} \right) \cup \left(\bigcup_{l \geq k+1} \mathcal{C}_{l,n+1} \right)$$

which is a bijective version of

$$F_{\mathcal{M}}(t, x) = 1 + xtF_{\mathcal{M}}(t, x)^2 + xt\Omega F_{\mathcal{M}}(t, x).$$

Bijjective recurrence:



Another approach to maps is to find bijections with “simpler” (usually tree-like) objects e.g.:

- ▶ Labelled trees

R. Cori, B. Vauquelin, *Planar maps are well labeled trees*.
Canadian J. Math. 33 (1981)

G. Schaeffer, *Bijjective census and random generation of Eulerian planar maps with prescribed vertex degrees*. Electron. J. Combin. 4 (1997)

- ▶ Mobiles

J. Bouttier, P. Di Francesco, E. Guitter, *Planar maps as labeled mobiles*. Electron. J. Combin. 11 (2004)

and many more variants...

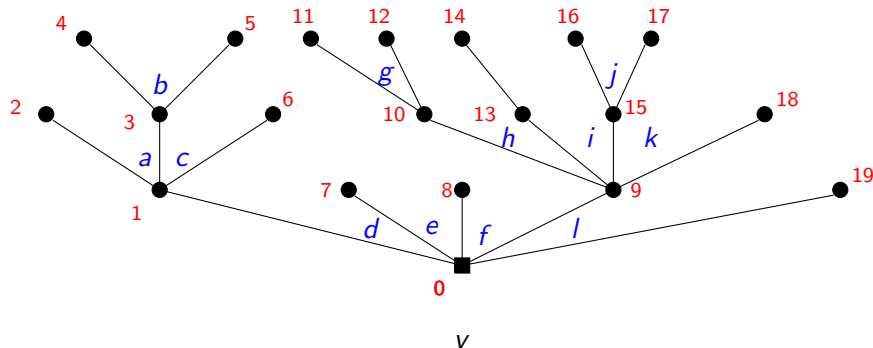
Useful for the asymptotics of random maps cf

J.F. Le Gall, *The topological structure of scaling limits of large planar maps*. Invent. Math. 169 (2007)

Greedy Tamari intervals

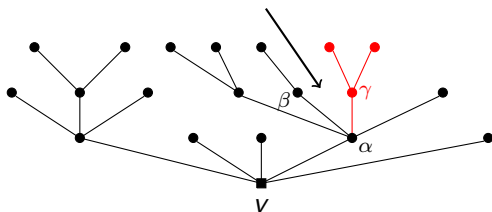
The aim of this talk is to present a bijection with another type of objects, *greedy Tamari intervals*, proving a conjecture of Bousquet-Mélou and Chapoton.

Tamari order on trees

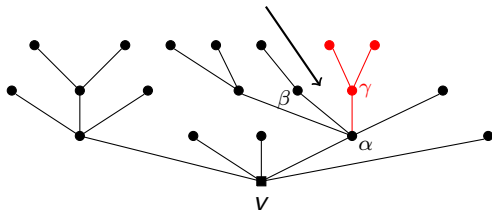


A rooted plane tree v with its vertices enumerated by breadth first search and its valleys (=space between two consecutive children of a node).

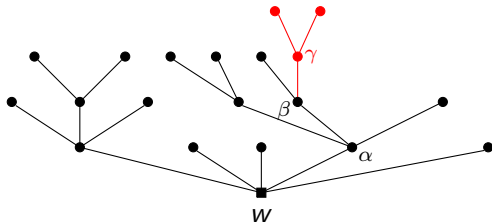
Take a valley, draw in red the tree with root γ :



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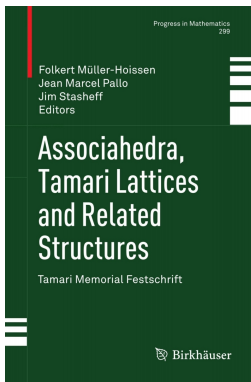


Slide up the red tree along the edge $\alpha - \beta$ to get w :



Relations $v < w$ are cover relations for the *Tamari order*.

There is a huge literature on this structure, related to associahedra, Coxeter groups and many other topics in combinatorics:



Intervals in the Tamari order

Chapoton (2006) found that the number of intervals in the Tamari order is:

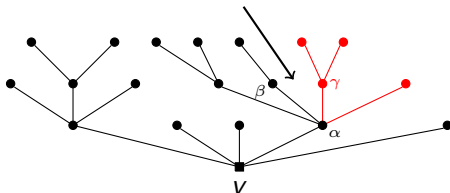
$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}$$

This is also the number of planar triangulations found by Tutte in 1962!

A bijective explanation was found by Bernardi and Bonichon (2008).

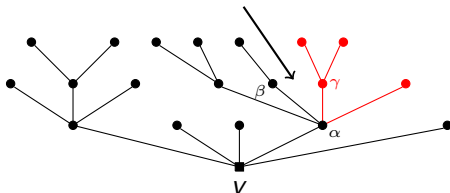
Greedy Tamari order (Dermenjian 2022)

Take a valley between β and γ , draw in red the tree of all descendants of α to the right of γ :

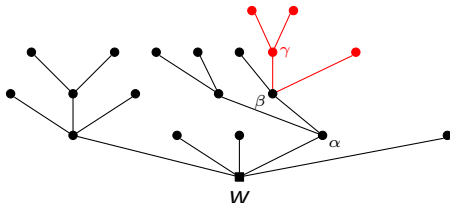


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Take a valley between β and γ , draw in red the tree of all descendants of α to the right of γ :



slide up the red tree along the edge $\alpha - \beta$ to get w :



Relations $v < w$ are the cover relations for the greedy Tamari order.

Intervals in the greedy Tamari poset

In 2024, Bousquet-Mélou and Chapoton enumerated the number of intervals in the greedy Tamari poset by solving a functional equation based on a recursive decomposition and found

$$\frac{3 \cdot 2^{n-1}}{(n+1)(n+2)} \binom{2n}{n}$$

which is also the number of bipartite planar maps with n edges.

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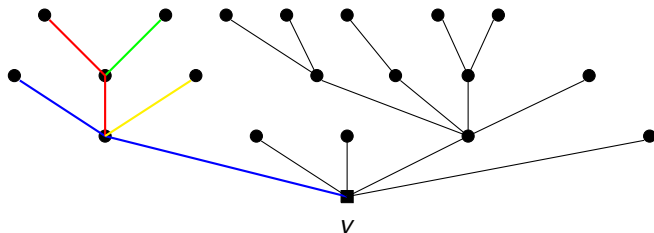
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Ascents of a rooted planar tree

A rooted planar tree has a sequence of *ascents* :



(only the first four ascents are shown here)

Remember the bijective recurrence for bipartite planar maps:

$$\mathcal{C}_{k+1,n+1} \longleftrightarrow \left(\bigcup_{\substack{k_1+k_2=k \\ n_1+n_2=n}} \mathcal{C}_{k_1,n_1} \times \mathcal{C}_{k_2,n_2} \right) \cup \left(\bigcup_{l \geq k+1} \mathcal{C}_{l,n} \right)$$

Let $\mathcal{G}_{k,n}$ = greedy Tamari intervals $[v, w]$ with n edges such that the first ascent of w has k edges.

I will exhibit a recursive bijection:

$$\mathcal{G}_{k+1,n+1} \longleftrightarrow \left(\bigcup_{\substack{k_1+k_2=k \\ n_1+n_2=n}} \mathcal{G}_{k_1,n_1} \times \mathcal{G}_{k_2,n_2} \right) \cup \left(\bigcup_{l \geq k+1} \mathcal{G}_{l,n} \right)$$

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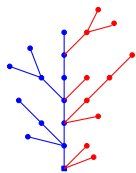
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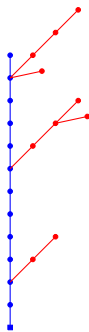
In this bijection the sizes of the ascents of the largest tree in the interval will behave like the sizes of the faces of the maps, as in Bousquet-Mélou and Chapoton conjecture.

Adding a leaf

Start with a greedy interval

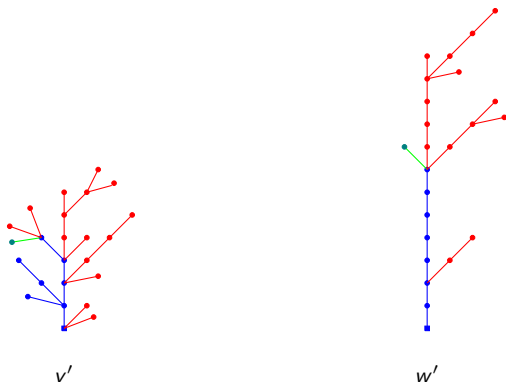


V



W

Add an edge to w cutting its first ascent at height k :



we get a new interval and a map Δ

$$[v, w] \xrightarrow{\Delta} [v', w']$$

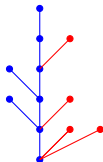
$$\bigcup_{l \geq k+1} \mathcal{G}_{l,n} \rightarrow \mathcal{G}_{k+1,n+1}$$

It is possible to use this construction and extend it to a bijection

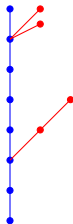
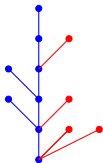
$$\mathcal{G}_{k+1,n+1} \longleftrightarrow \left(\bigcup_{\substack{k_1+k_2=k \\ n_1+n_2=n}} \mathcal{G}_{k_1,n_1} \times \mathcal{G}_{k_2,n_2} \right) \cup \left(\bigcup_{l \geq k+1} \mathcal{G}_{l,n} \right)$$

However we could not make this work for m -constellations (to be defined soon), so instead we modify slightly the map.

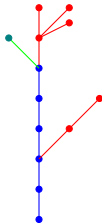
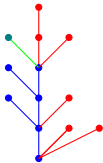
Start with an interval $[v, w]$

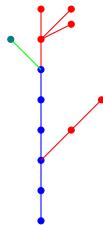
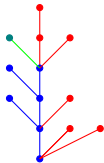


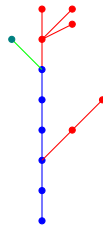
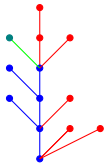
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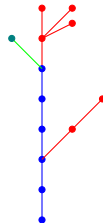
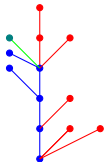
add a leaf at height k







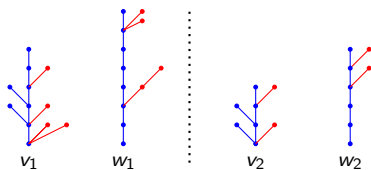
lift left blue trees by 1



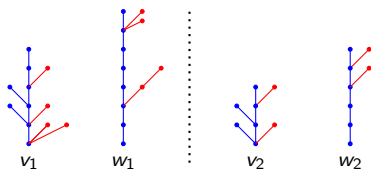
we get a new interval and a map

$$\cup_{l \geq k+1} \mathcal{G}_{l,n} \rightarrow \mathcal{G}_{k+1,n+1}$$

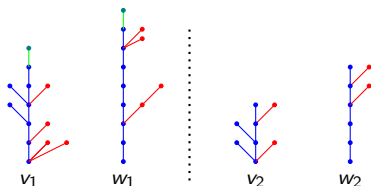
We now define the product of two intervals.



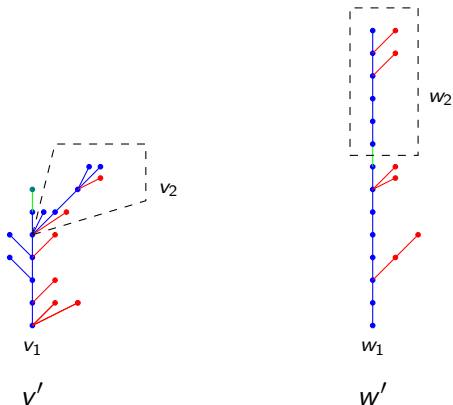
We now define the product of two intervals.



add a leaf at the top of v_1 and w_1 and lift the left blue trees:



Then put w_2 above w_1 and insert v_2 into v_1 just above the highest red tree:



We get maps

$$[v_1, w_1] \times [v_2, w_2] \rightarrow [v', w']$$

$$\mathcal{G}_{k_1, n_1} \times \mathcal{G}_{k_2, n_2} \rightarrow \mathcal{G}_{k_1+k_2+1, n_1+n_2+1}$$

Theorem:

There is a bijection

$$\left(\bigcup_{\substack{k_1+k_2=k \\ n_1+n_2=n}} \mathcal{G}_{k_1,n_1} \times \mathcal{G}_{k_2,n_2} \right) \cup \left(\bigcup_{l \geq k+1} \mathcal{G}_{l,n} \right) \longleftrightarrow \mathcal{G}_{k+1,n+1}$$

Corollary: By induction one gets a bijection

$$\Phi : \mathcal{C}_{k,n} \longleftrightarrow \mathcal{G}_{k,n}$$

If $\Phi(C) = [v, w]$ then the lengths of ascents of w correspond to half-degrees of faces of C .

m -constellations

A *planar m -constellation* is a planar map where faces are colored black and white, with the outer face colored white and each edge bordering a black and a white face. Black faces are all of degree $m + 1$. White faces are all of degree divisible by $m + 1$.

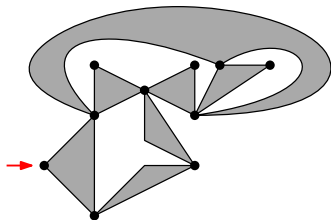
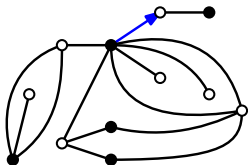


Figure: Example of an m -constellation



Bipartite planar maps encode equations in the symmetric group

$$\sigma_{\bullet}\sigma_{\circ} = \sigma_{\diamond}$$

$\sigma_{\bullet}$ =permutation of edges with cycles indexed by $\bullet$ vertices

$\sigma_{\circ}$ =permutation of edges with cycles indexed by $\circ$ vertices

$\sigma_{\diamond}$ =permutation of edges with cycles indexed by (half)faces

Equations of the form:

$$\sigma_1\sigma_2\ldots\sigma_{m+1} = \sigma_{\diamond}$$

are encoded by m -constellations:

Recurrence for constellations

A generating function $F_{\mathcal{C}^{(m)}}$ of m -constellations is given by

$$F_{\mathcal{C}^{(m)}}(t, x) = \sum_{n \geq 0} t^n \sum_{C \in \mathcal{C}_n^{(m)}} x^{\text{outdeg}(C)/(m+1)} \prod_{f \text{ inner face of } C} p_{\text{deg}(f)/(m+1)}.$$

W. Fang has given a recursive decomposition of planar m -constellations leading to the following functional equation:

$$F_{\mathcal{C}^{(m)}}(t, x) = 1 + tx(F_{\mathcal{C}^{(m)}} + \Omega)^m F_{\mathcal{C}^{(m)}}$$

(m, r) branch trees

Given $m \geq 2$ and $r \geq 0$, an (m, r) -*branch tree* is a rooted plane tree T such that the length of the ascent of every leaf is divisible by m , except for the leftmost leaf, where it is of the form $ma + r$ for some integer $a \geq 0$.

The bijective recurrence above restricts to (m, r) branch trees and gives

Theorem

Given $m \geq 1$ and $1 \leq r \leq m - 1$, the generating function $F_{\mathcal{I}(m,r)}(t, x)$ satisfies the quadratic recurrences

$$F_{\mathcal{I}(m,r+1)}(t, x) = (F_{\mathcal{I}(m,0)}(t, x) + \Omega)F_{\mathcal{I}(m,r)}, \quad 0 \leq r \leq m - 2,$$

$$F_{\mathcal{I}(m,0)}(t, x) = 1 + xt(F_{\mathcal{I}(m,0)}(t, x) + \Omega)F_{\mathcal{I}(m,m-1)}$$

This implies

$$F_{\mathcal{I}(m,0)}(t, x) = (F_{\mathcal{I}(m,0)}(t, x) + \Omega)^m F_{\mathcal{I}(m,0)}$$

which is the functional equation of $(m + 1)$ -constellations. We get a recursive bijection between m constellations and intervals in the m -greedy Tamari lattice.

A last question

Chains of length 1 in the greedy Tamari poset correspond to trees.

Chains of length 2 correspond to planar maps.

Is there a geometric model for chains of higher length?

HAPPY BIRTHDAY
JEAN-BERNARD!