

Dynamics of galactic disks

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01/10/2026

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- 1) Context
- 2) Dynamics of self-gravitating systems
- 3) Linear response theory : the response matrix
- 4) Application
- 5) Future

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Context

- Observations: geometry/kinematics

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- Galaxy : stellar+gas disk, bulge, DM halo

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- Observations: geometry/kinematics
- Galaxy : stellar+gas disk, bulge, DM halo
- Evolution: N-body or matrix method

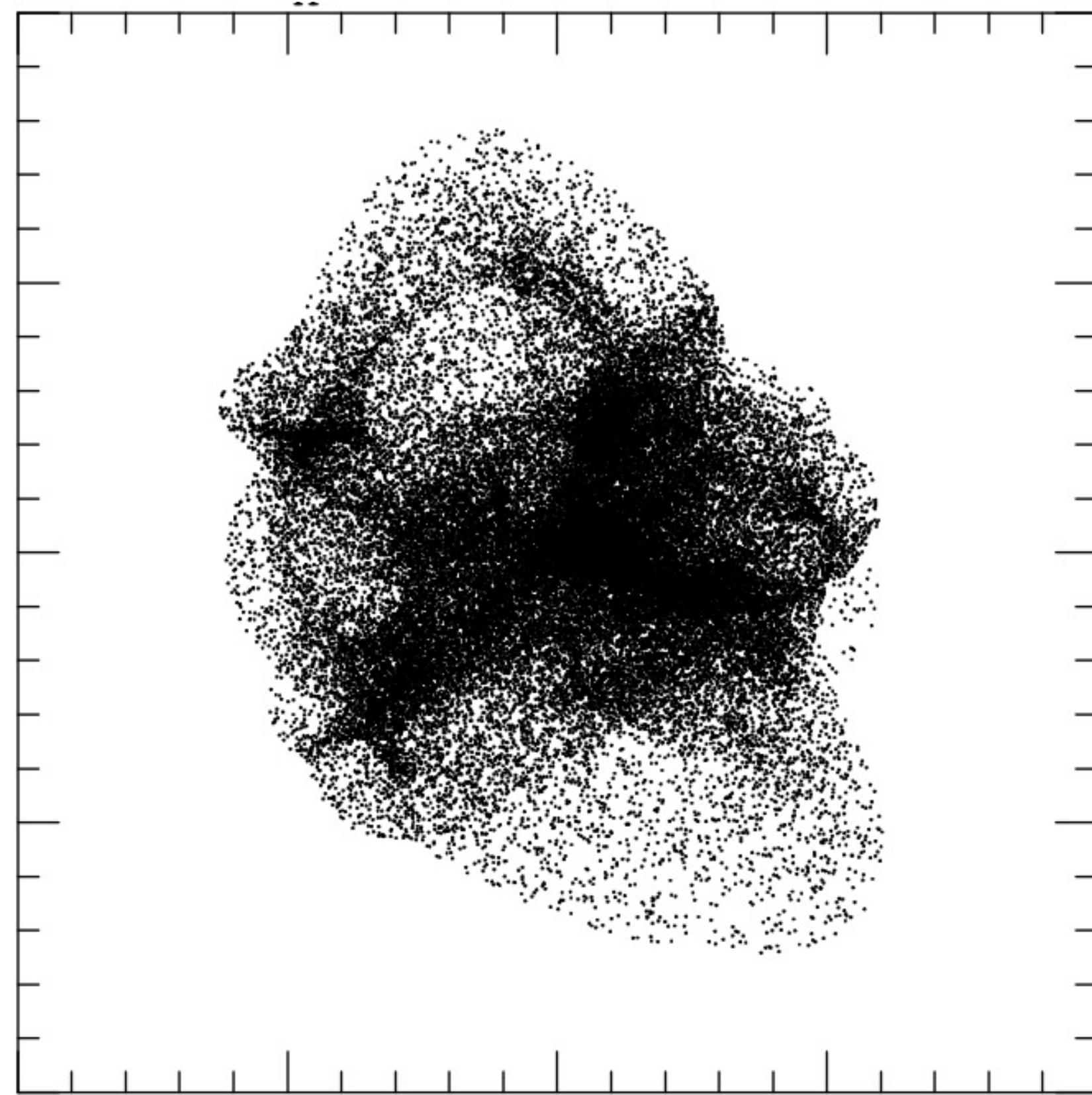
Context

- Observations: geometry/kinematics
- Galaxy : stellar+gas disk, bulge, DM halo
- Evolution: N-body or matrix method
- Historically : modeled by gas only or disk only

Contents

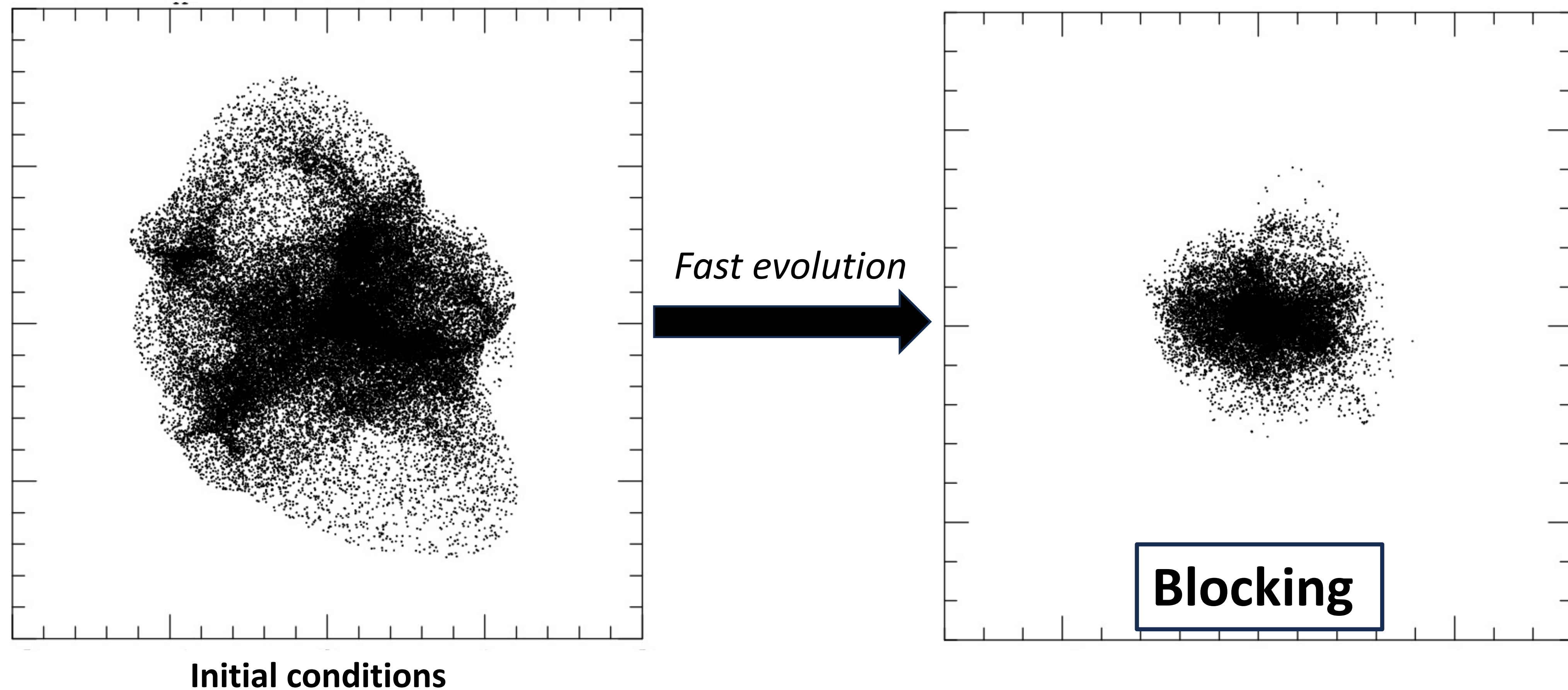
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Violent relaxation

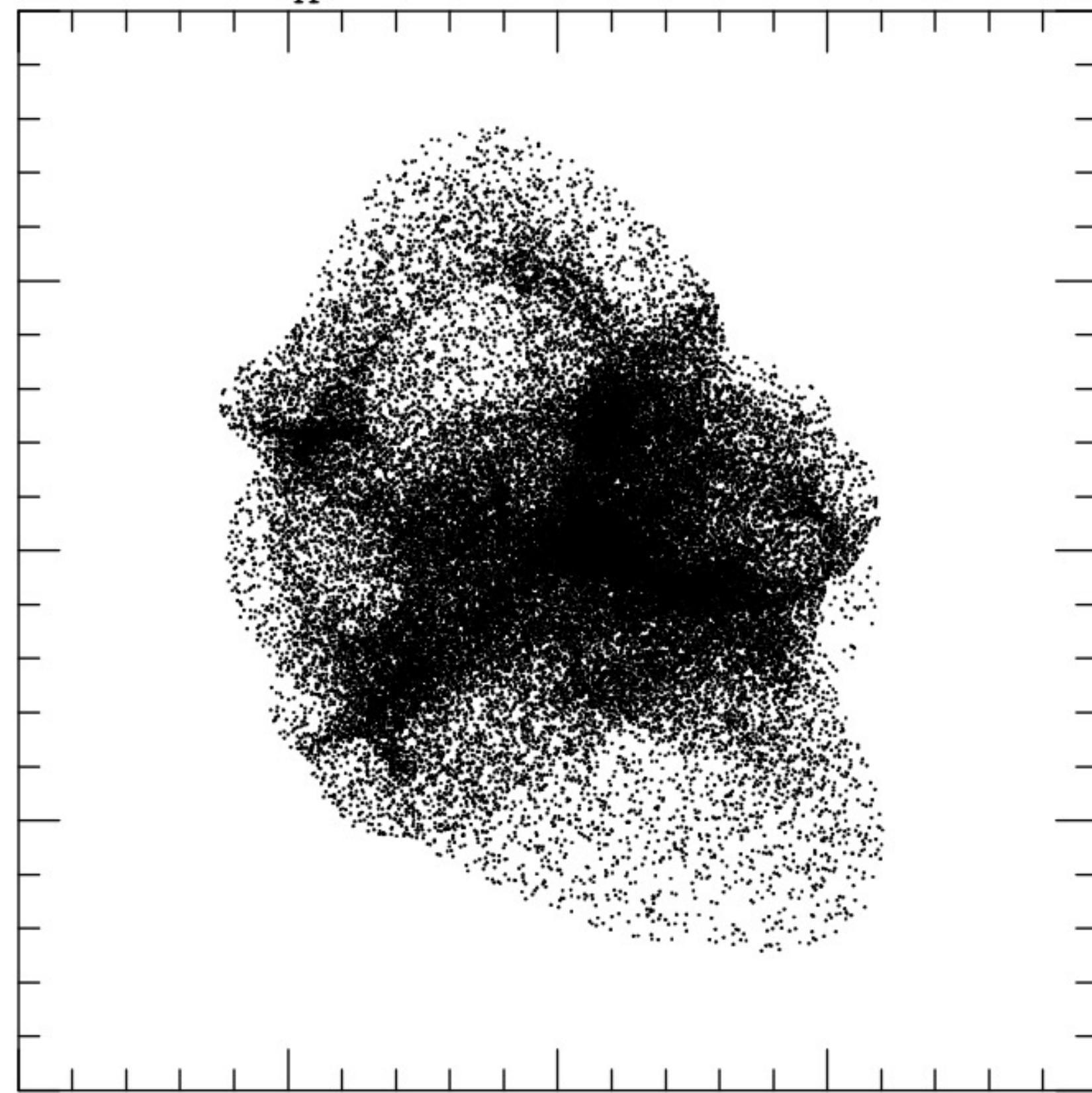


Initial conditions

Violent relaxation

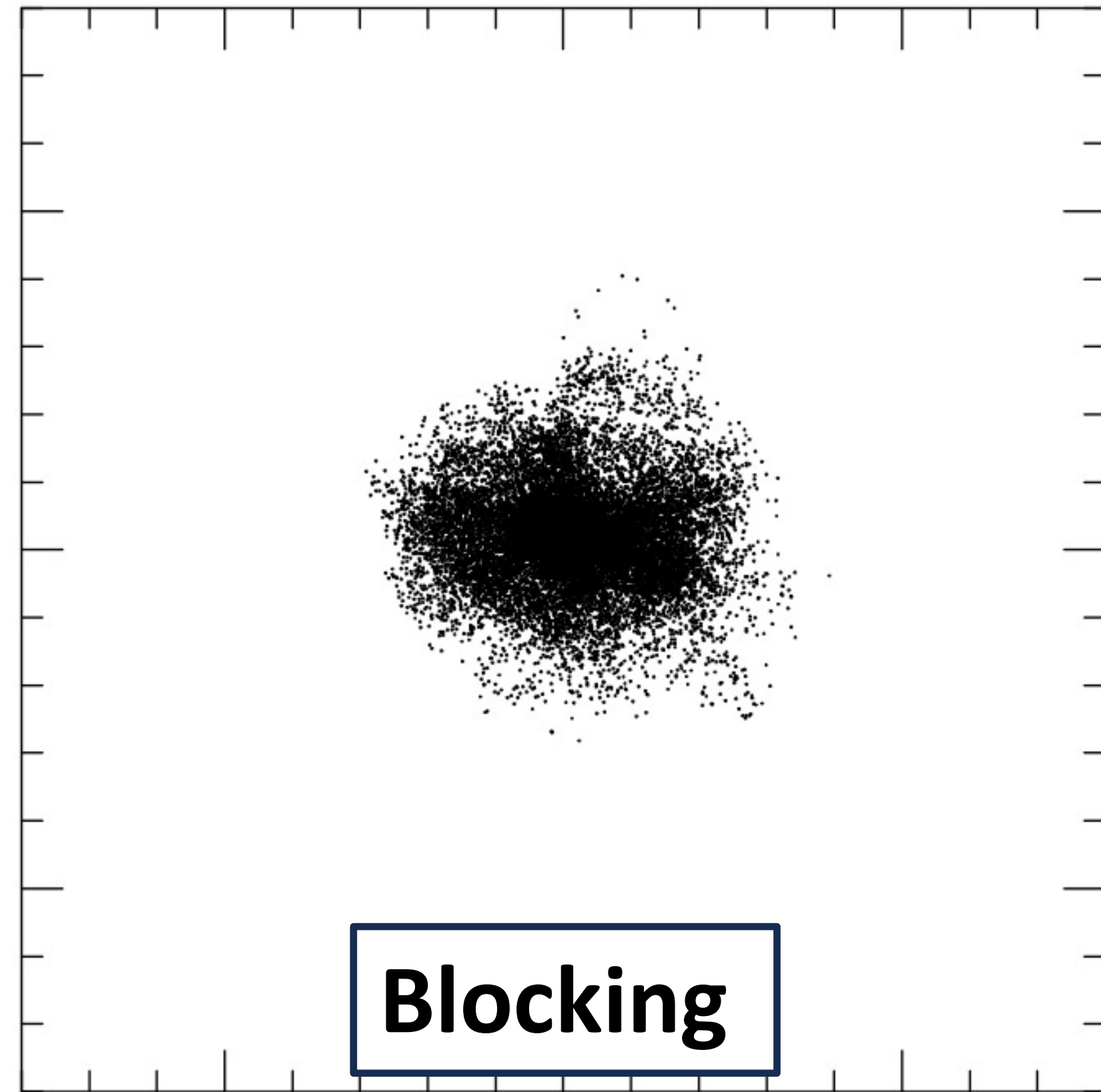


Violent relaxation



Initial conditions

Fast evolution

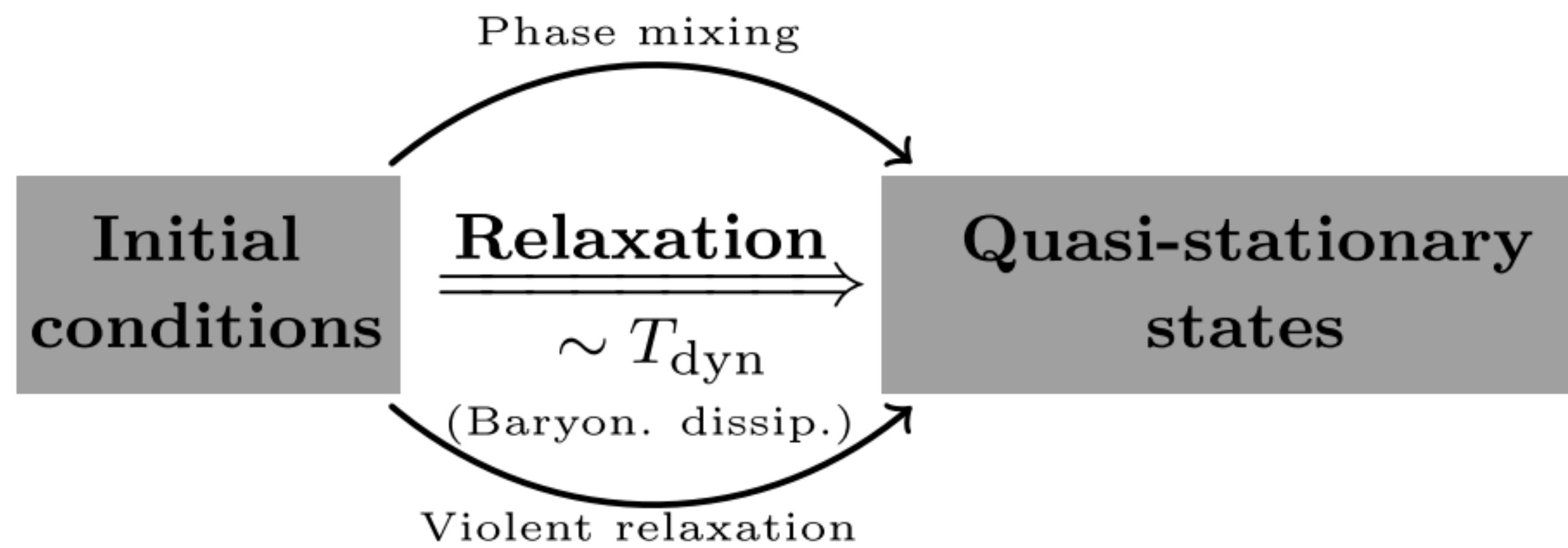


Blocking

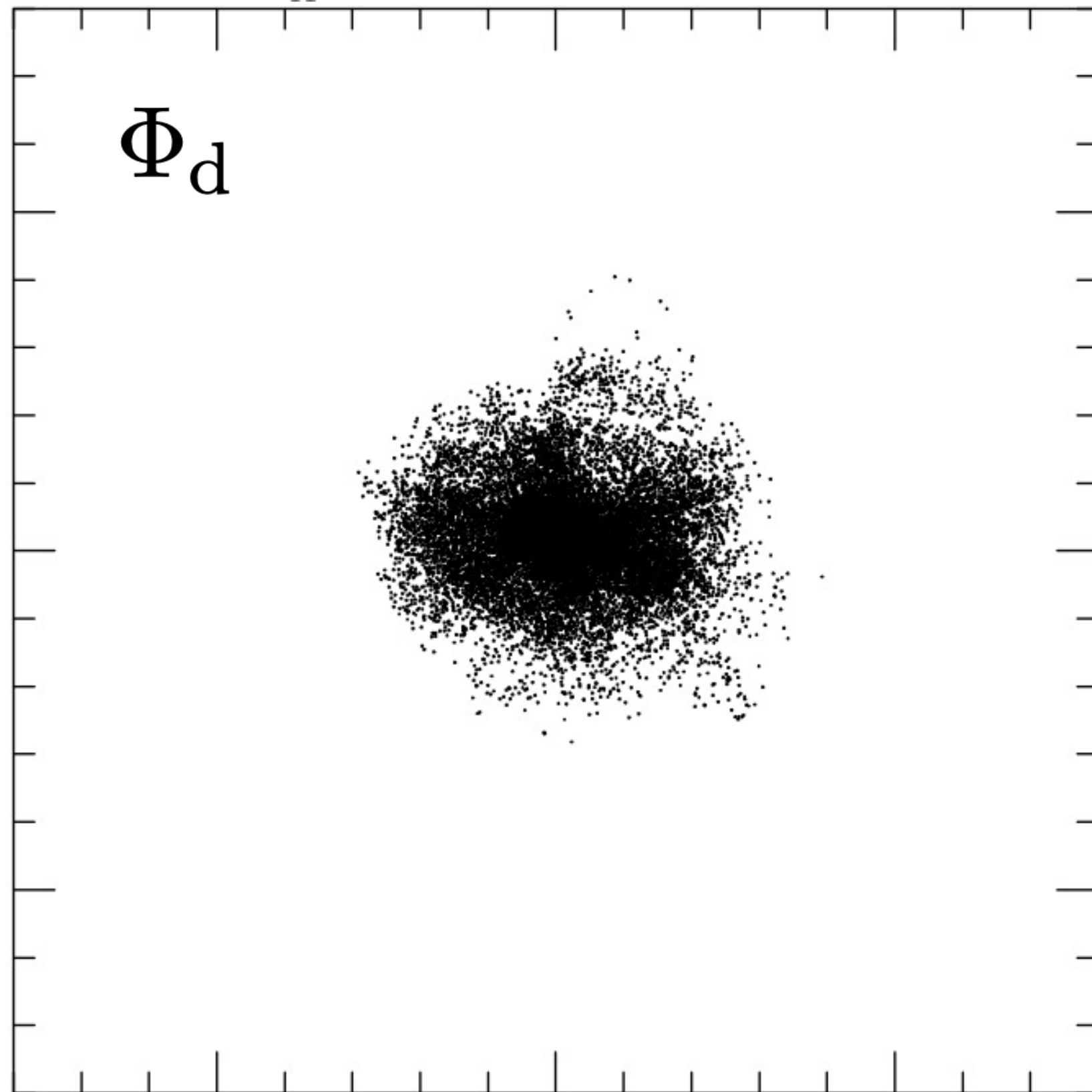
→ Symmetric configuration

→ Quasi-stationary state (QSS)

**Initial
conditions**



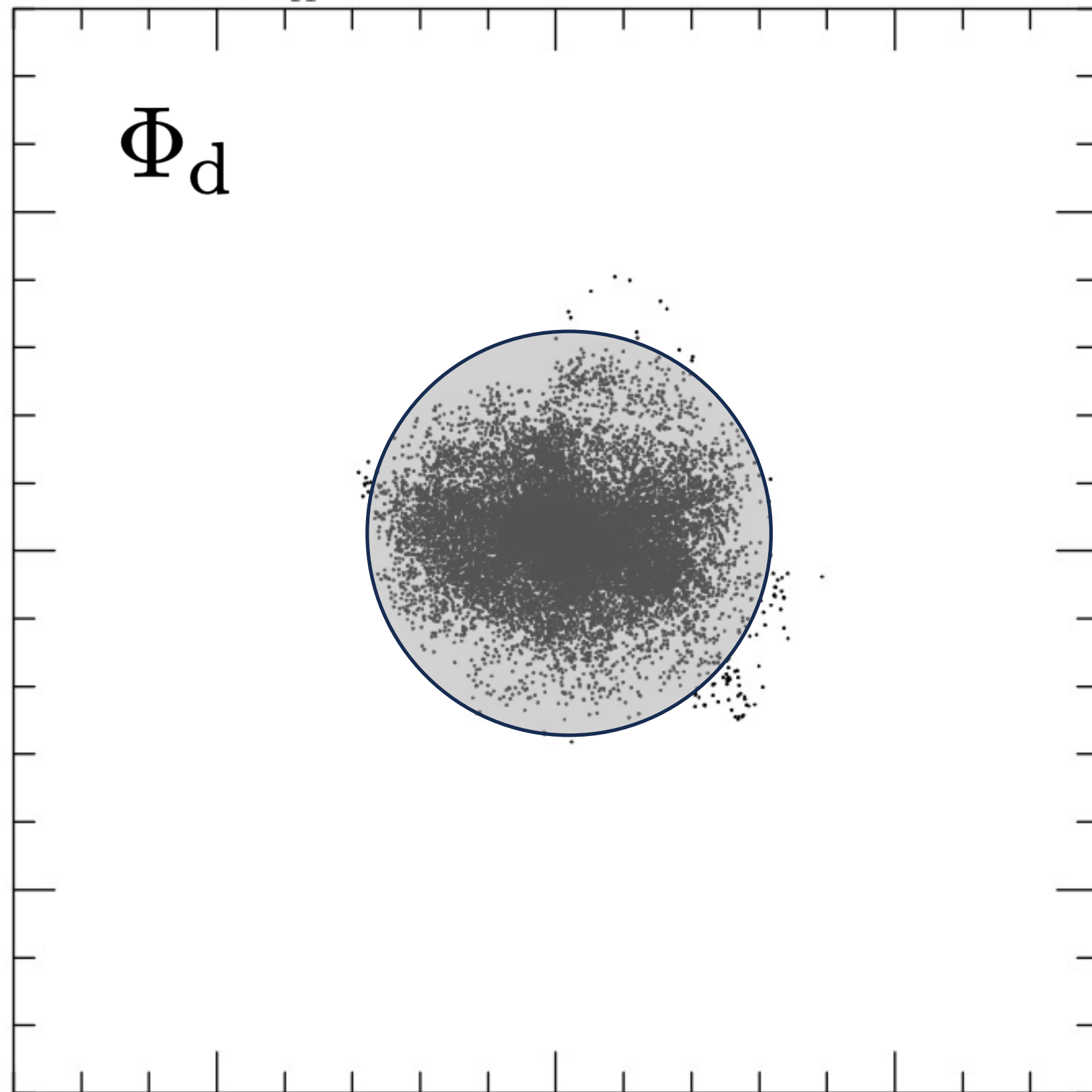
Mean-field limit



Discrete potential

$$\boxed{\Phi_d} = \Phi + \delta\Phi$$

Mean-field limit

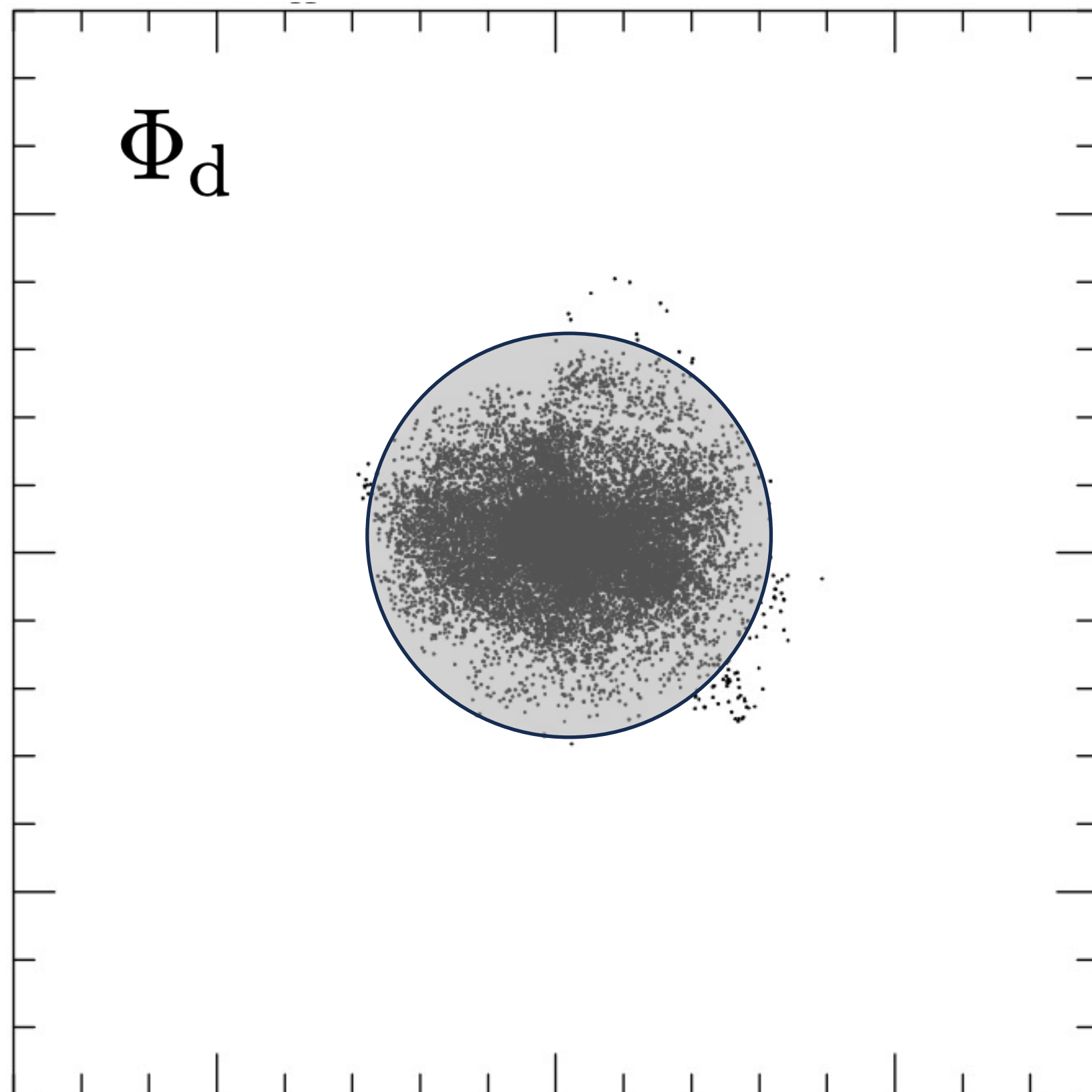


$$\boxed{\Phi_d} = \boxed{\Phi} + \delta\Phi$$

Discrete potential

Mean field

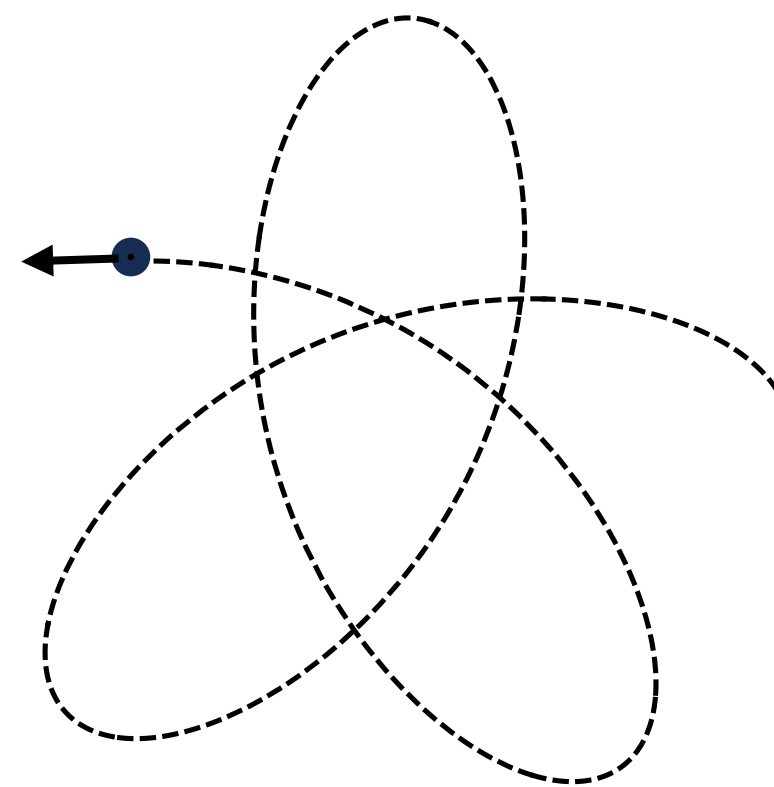
Mean-field limit



$$\boxed{\Phi_d} = \boxed{\Phi} + \delta\Phi$$

Discrete potential

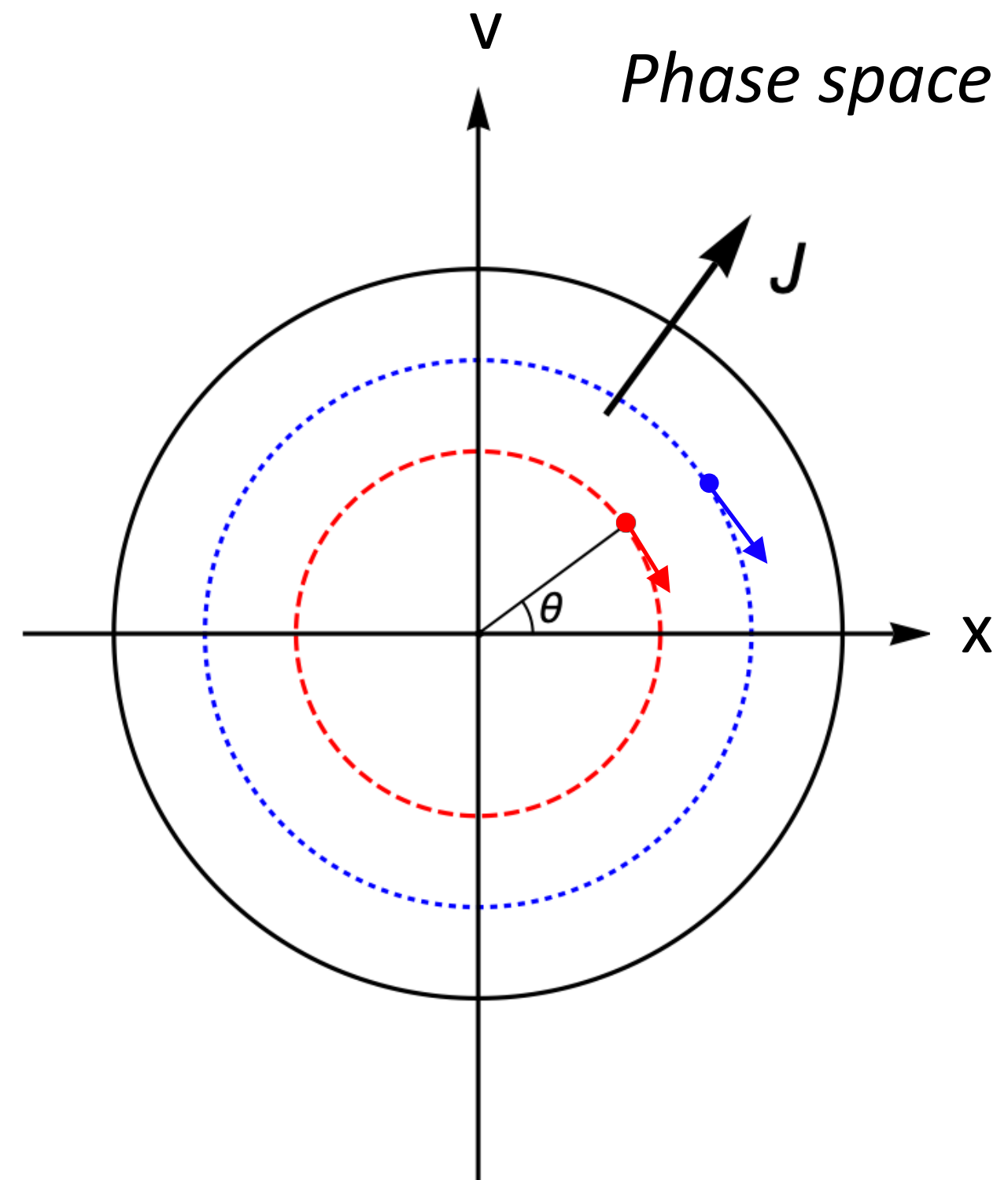
Mean field



- Symmetry of QSS
- Orbit labelling: actions J

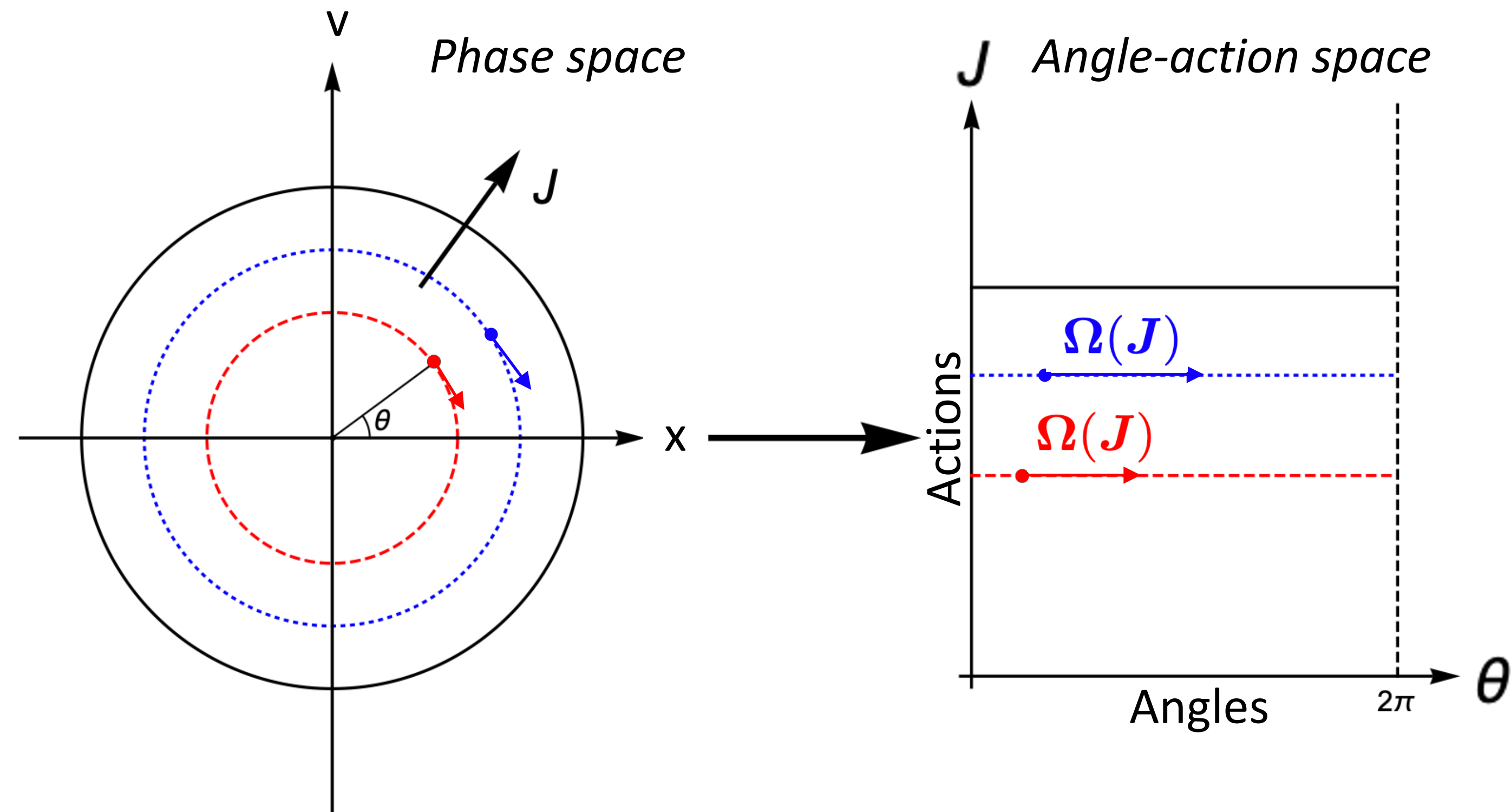
Angle action coordinates

- Action : motion integrals

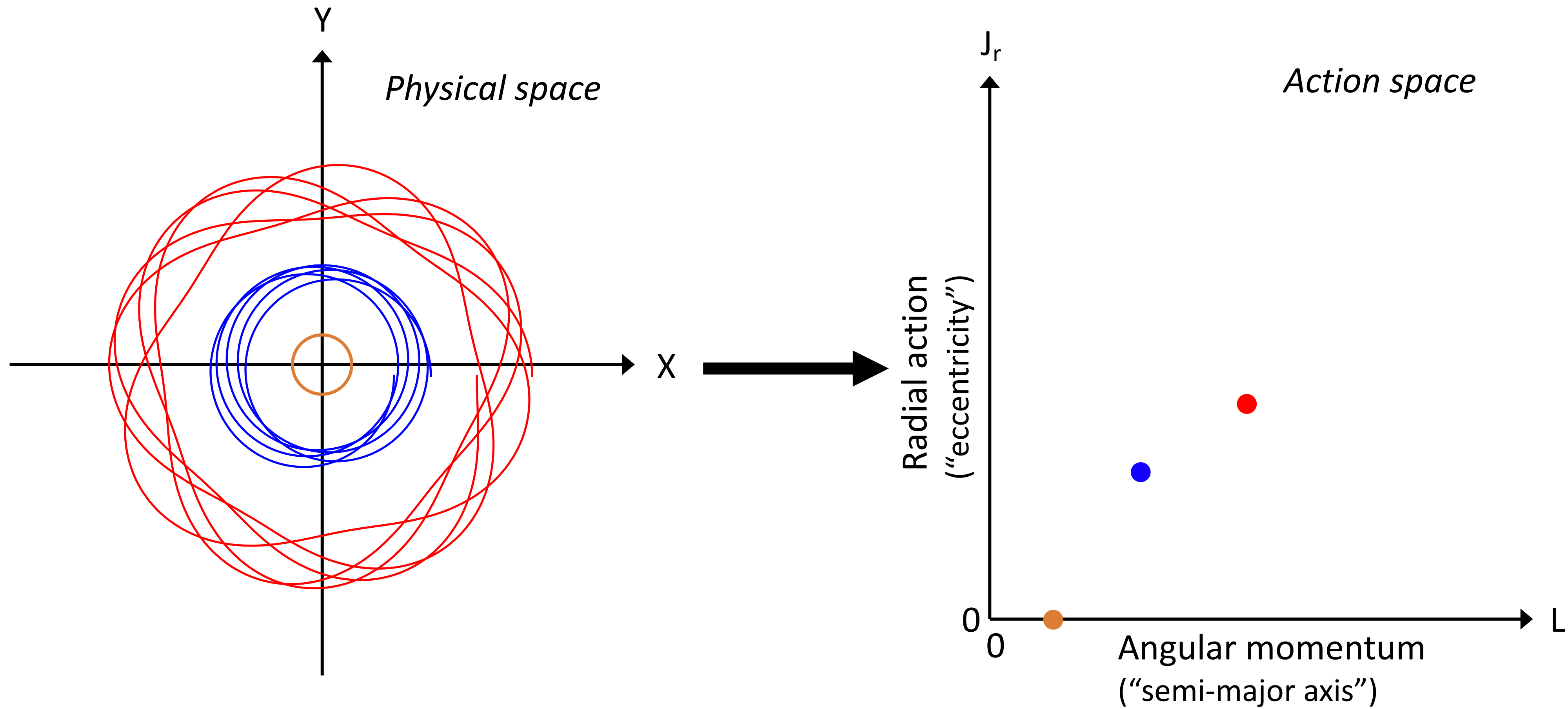


Angle action coordinates

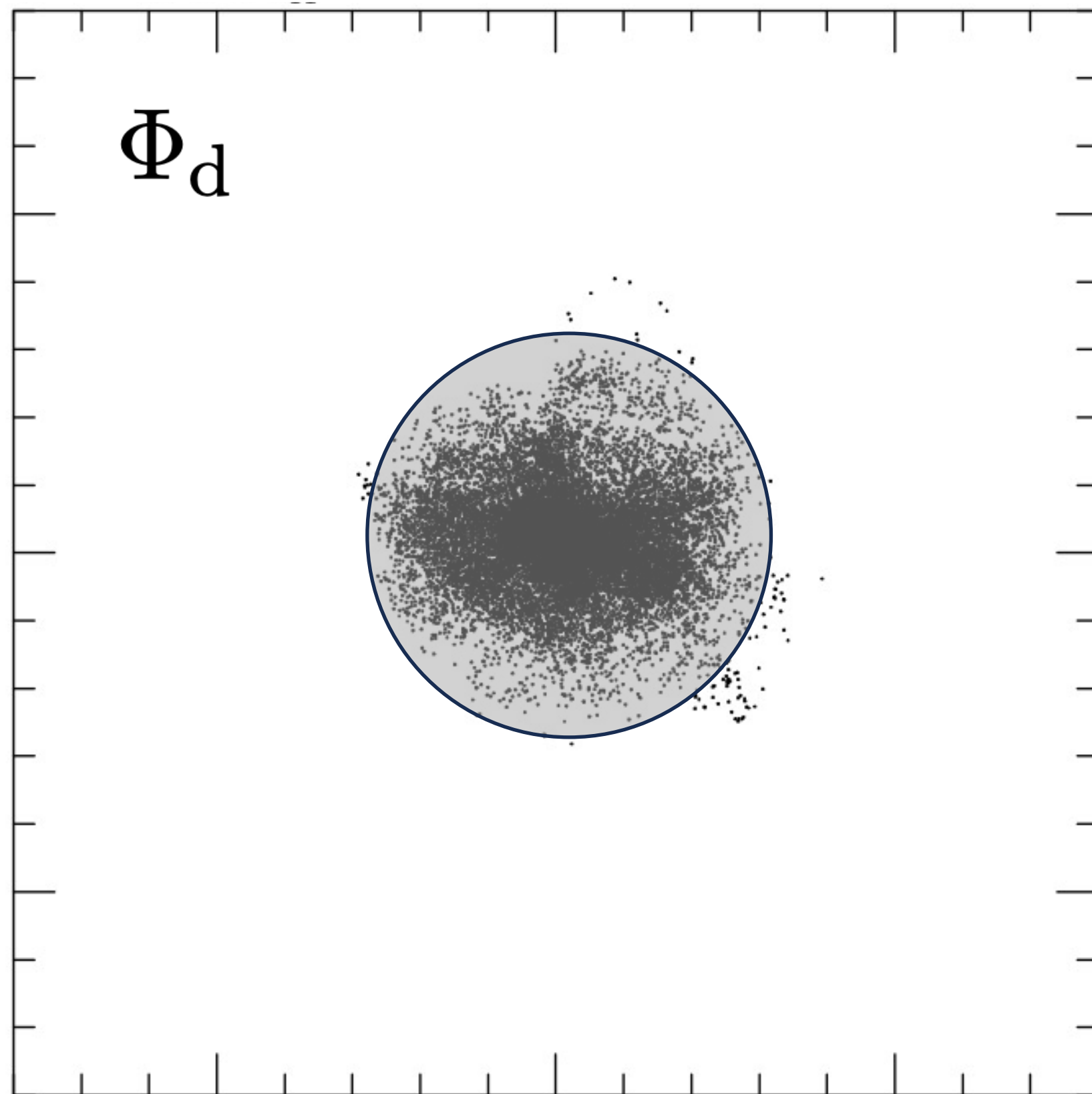
- Action : motion integrals



Actions



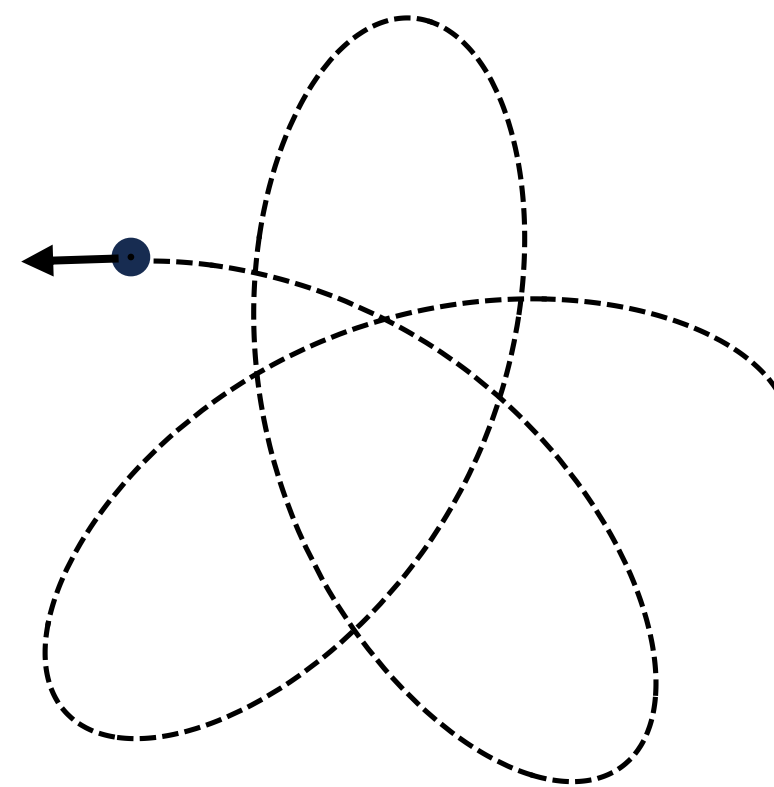
Mean-field limit



$$\boxed{\Phi_d} = \boxed{\Phi} + \delta\Phi$$

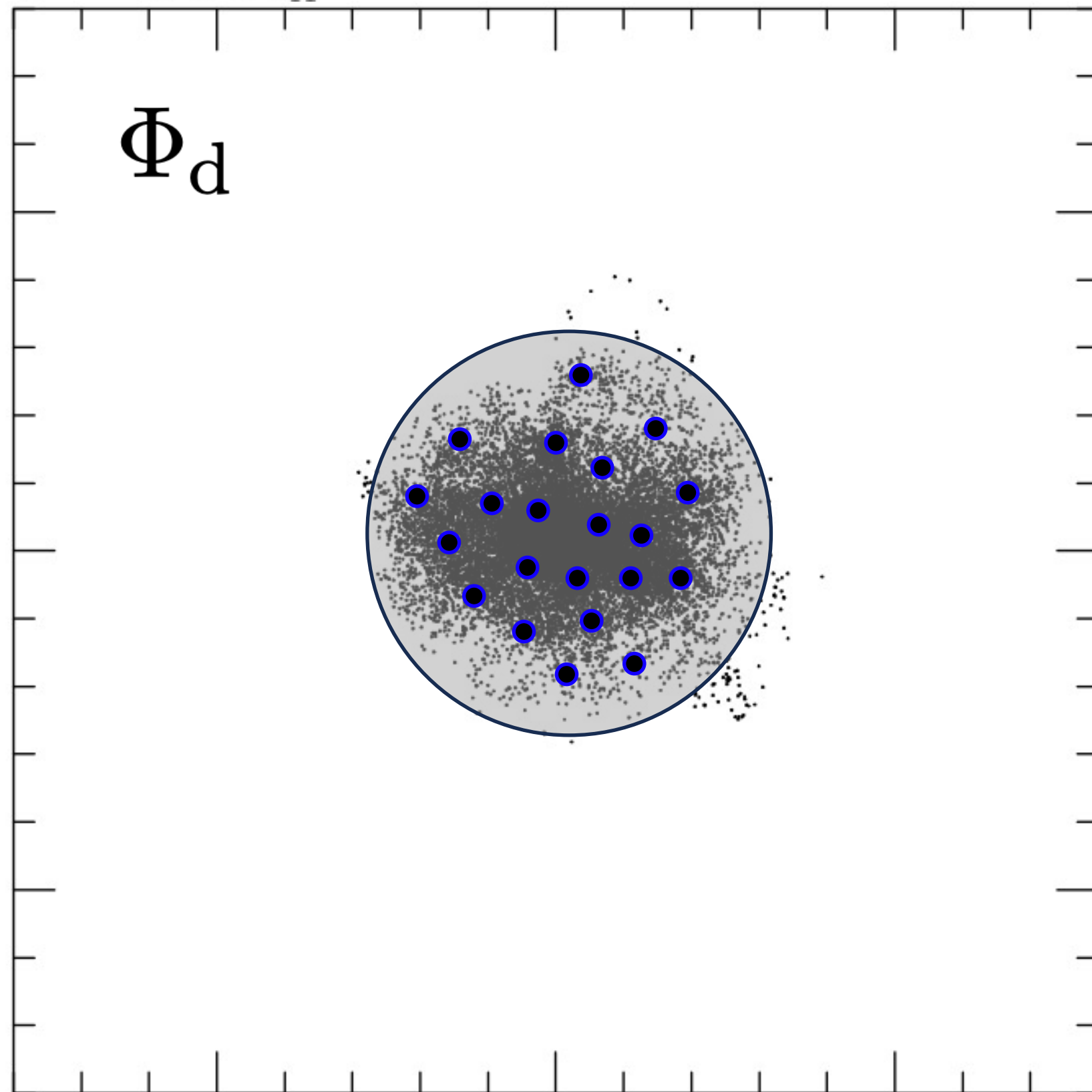
Discrete potential

Mean field



- Symmetry of QSS
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Fluctuations

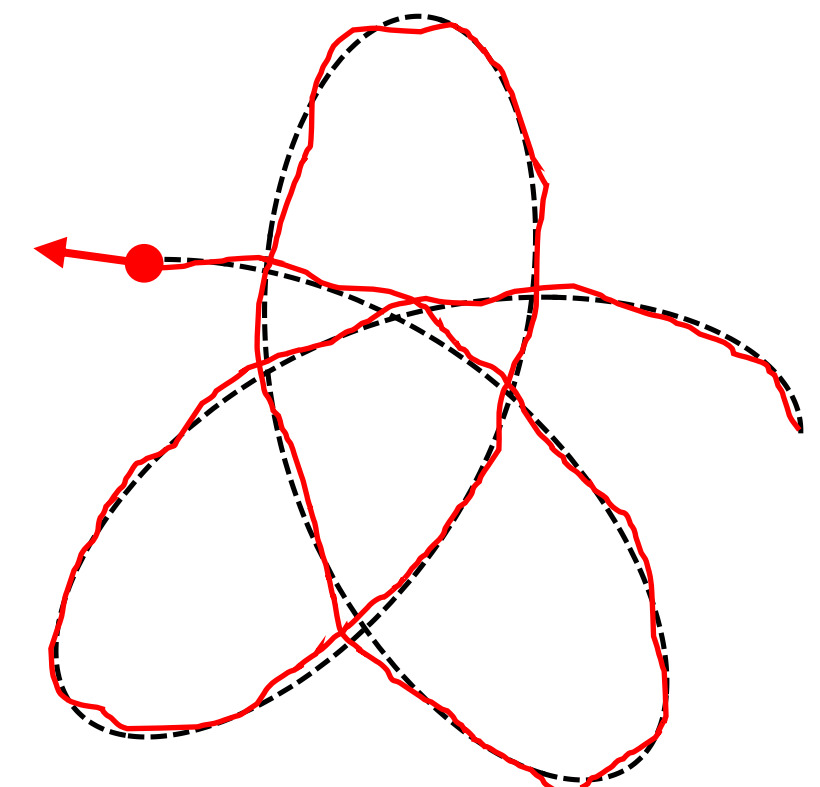
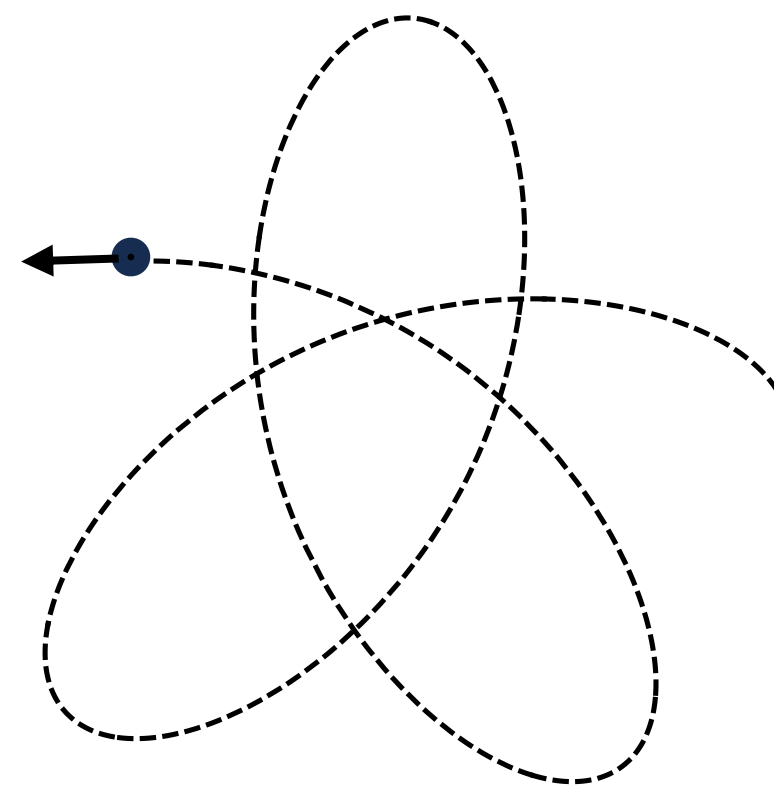


$$\boxed{\Phi_d} = \boxed{\Phi} + \boxed{\delta\Phi}$$

Discrete potential

Mean field

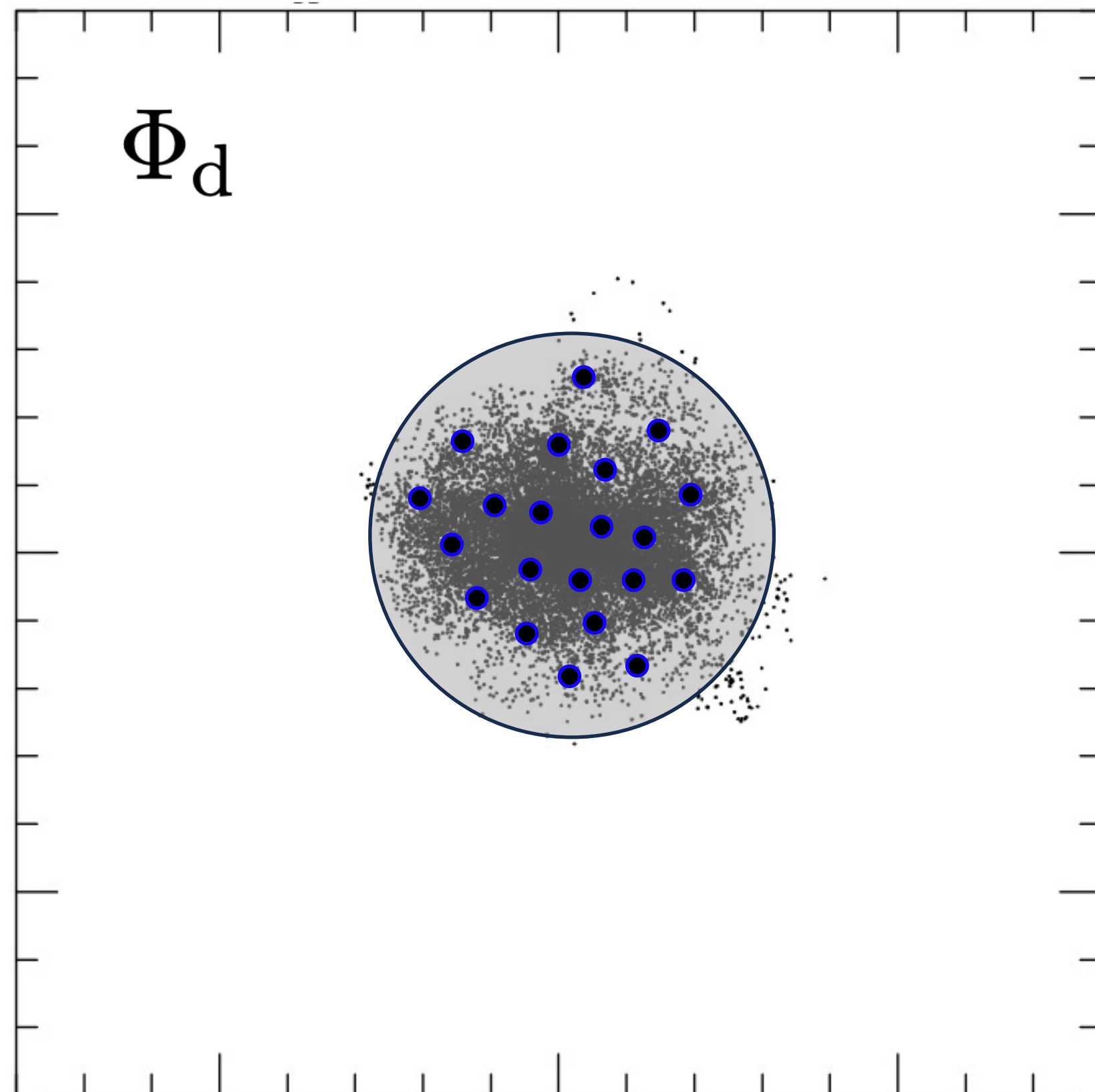
Fluctuations



- Symmetry of QSS
- Orbit labelling: actions J

- Departure from mean-field
- Distortion of the orbits
- Slow evolution of QSS

Fluctuations

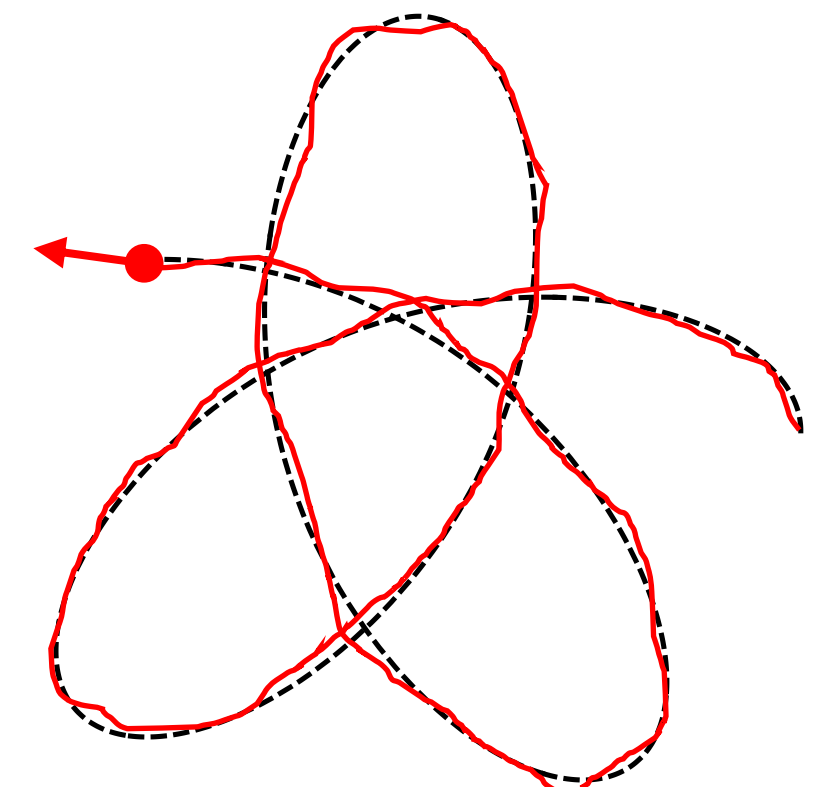
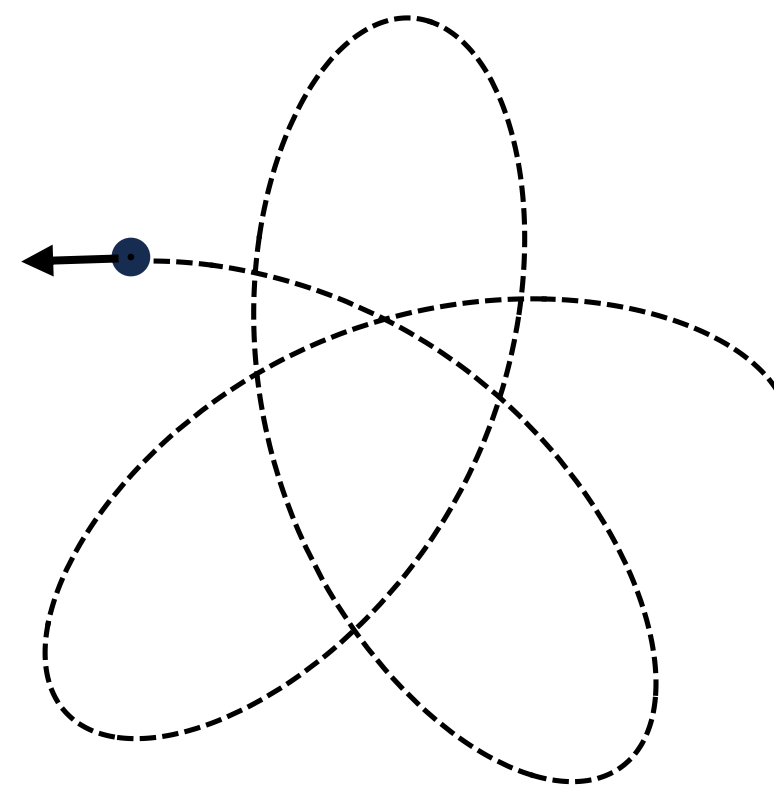


$$\boxed{\Phi_d} = \boxed{\Phi} + \boxed{\delta\Phi}$$

Discrete potential

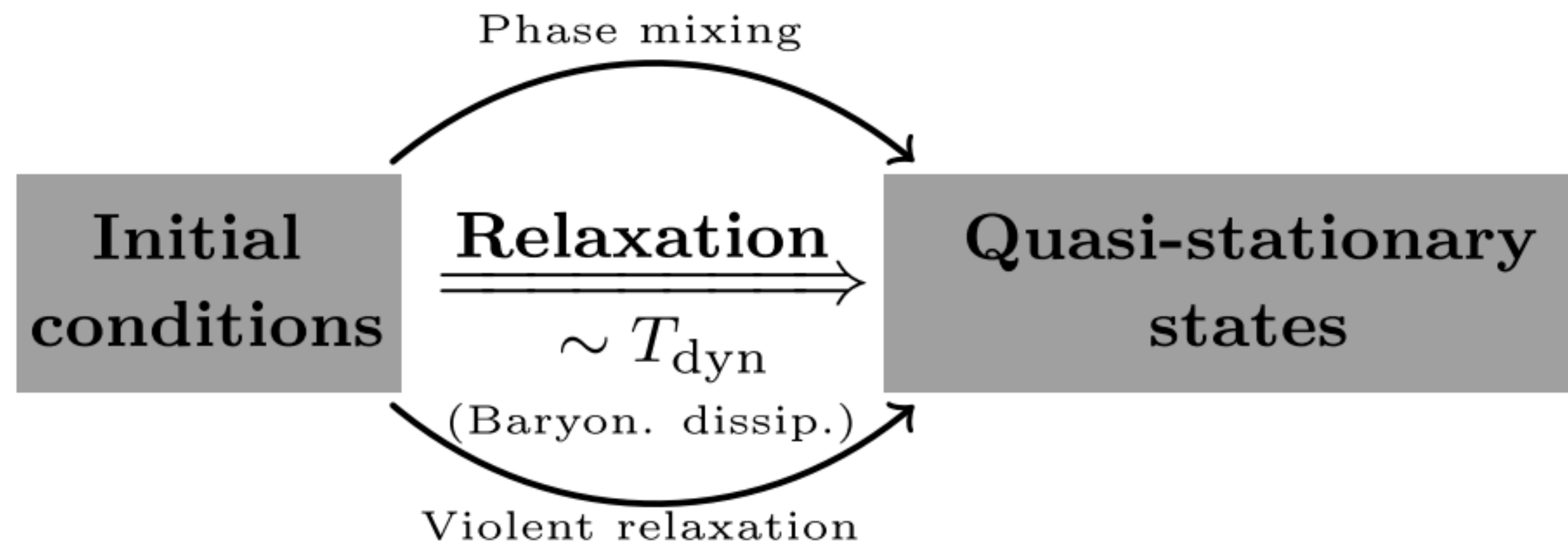
Mean field

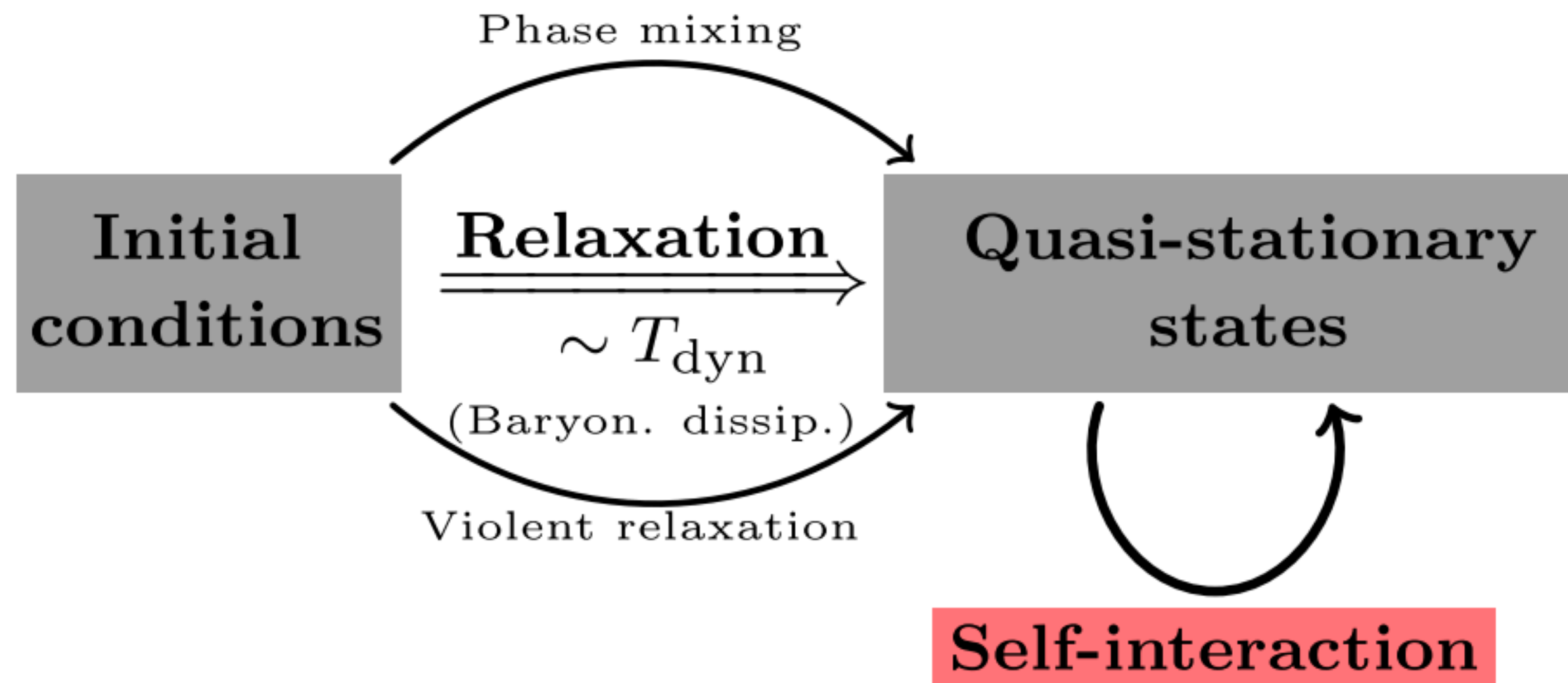
Fluctuations

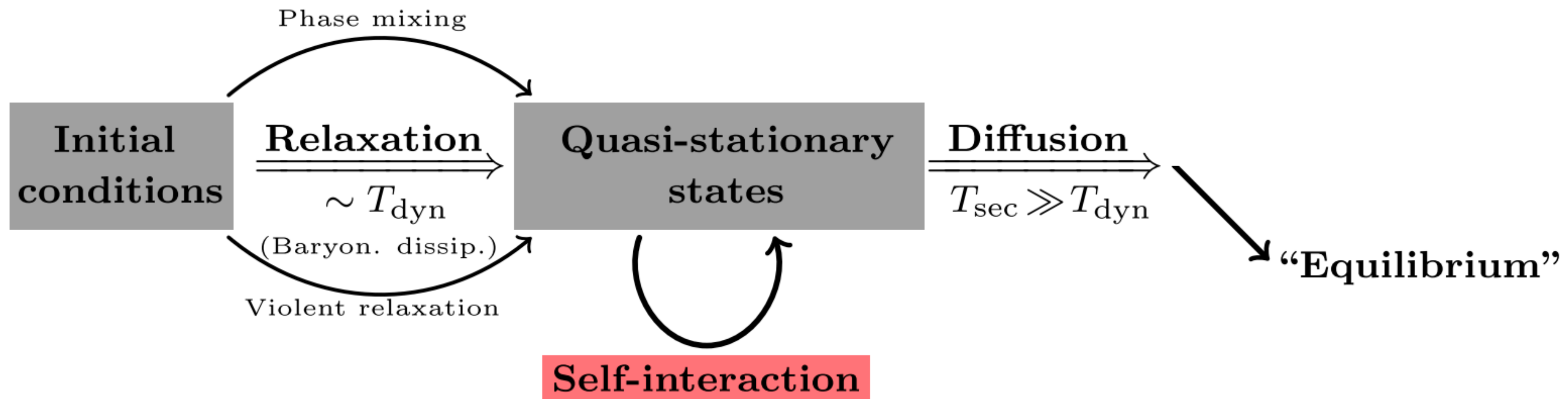


- Symmetry of QSS
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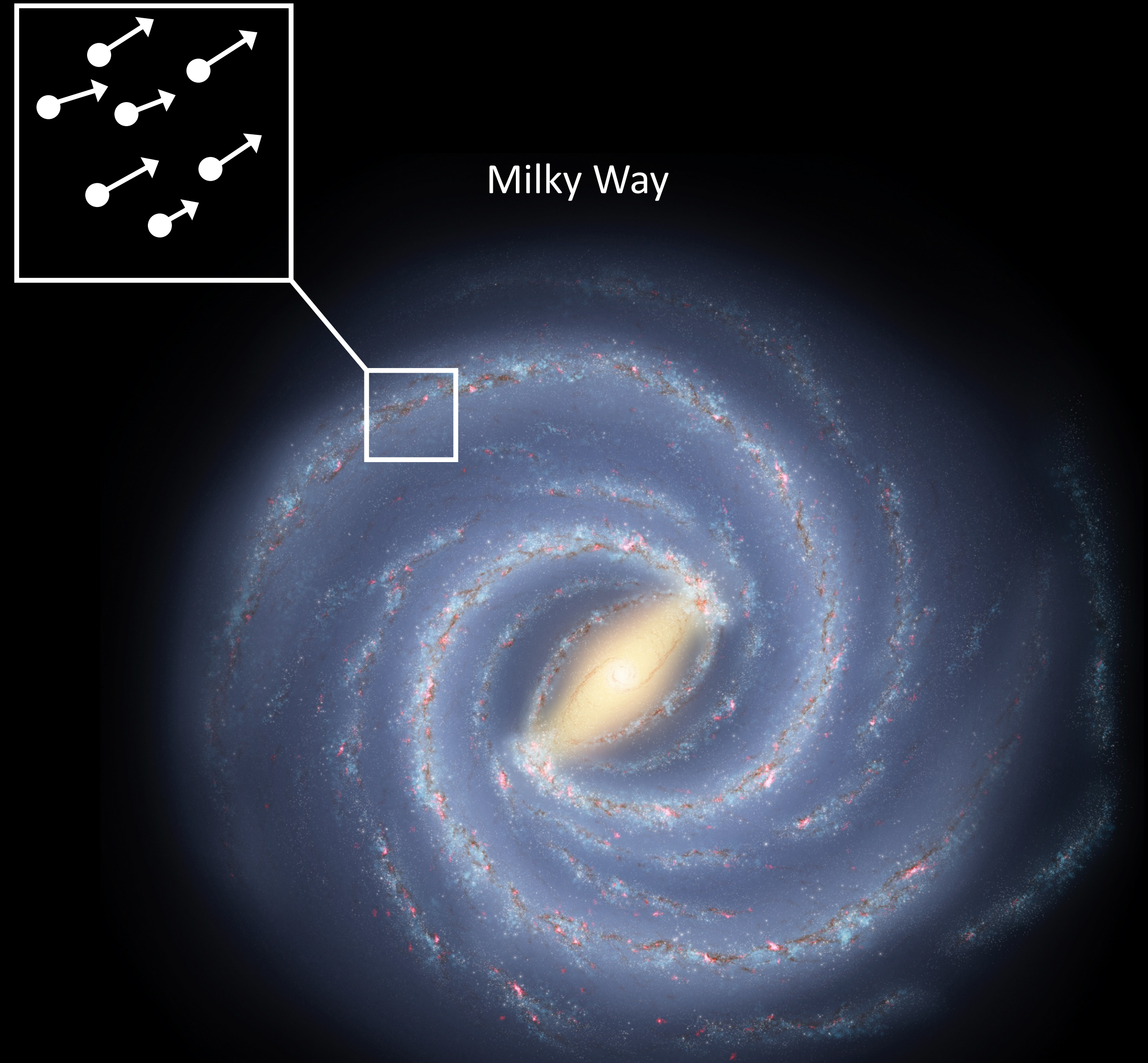
- Departure from mean-field
 - Distortion of the orbits
 - Slow evolution of QSS
- Secular relaxation



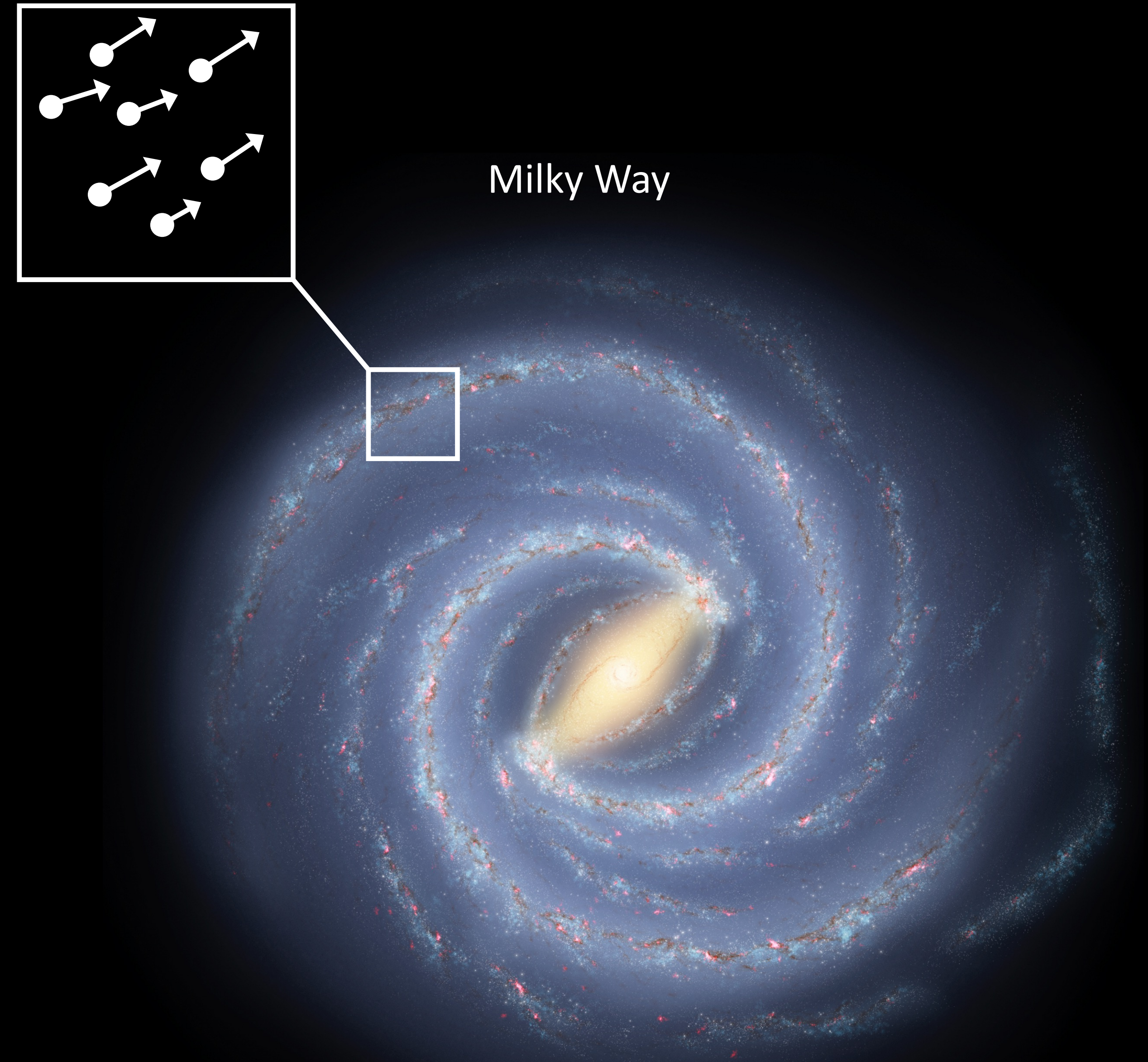
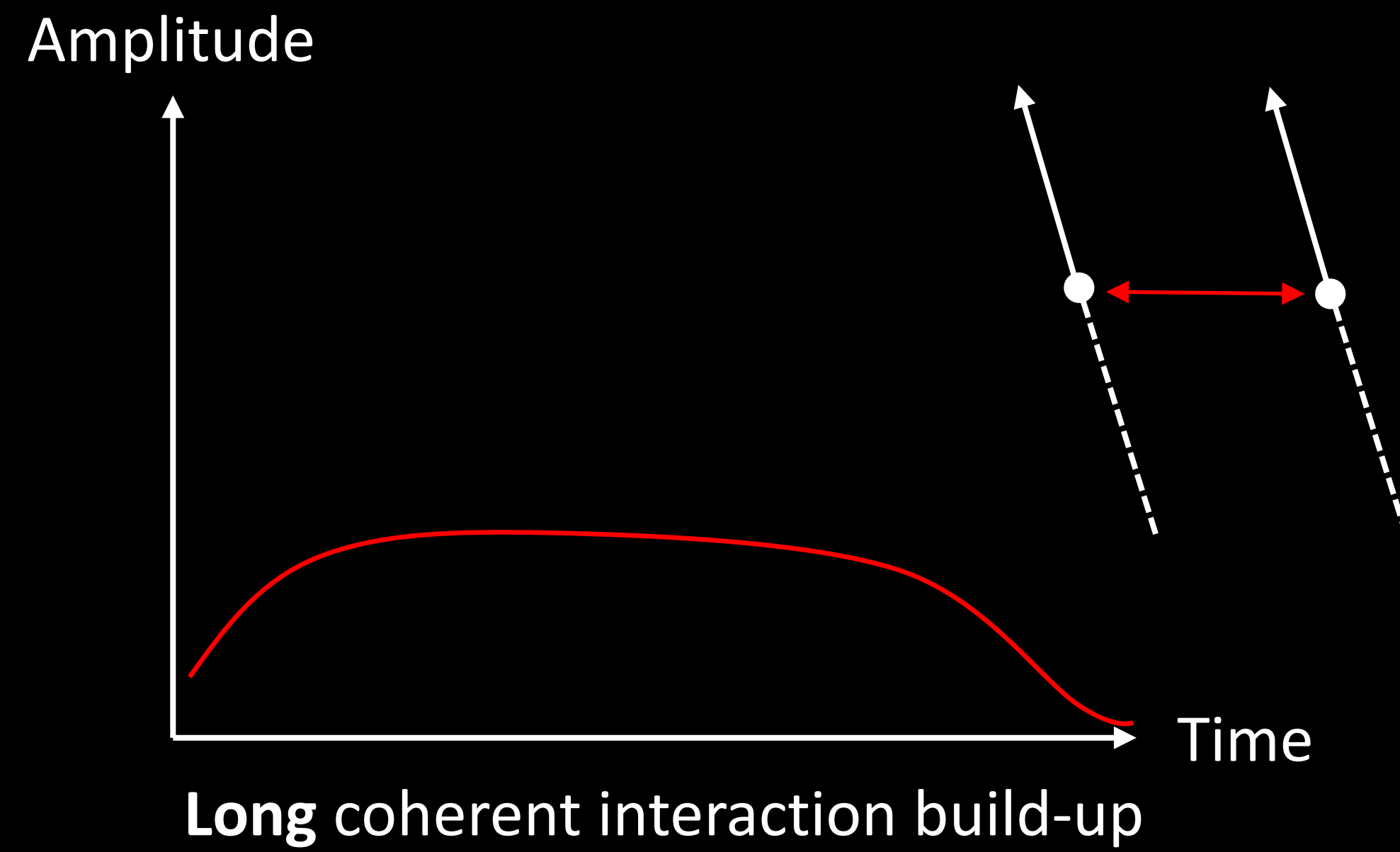




Cold systems

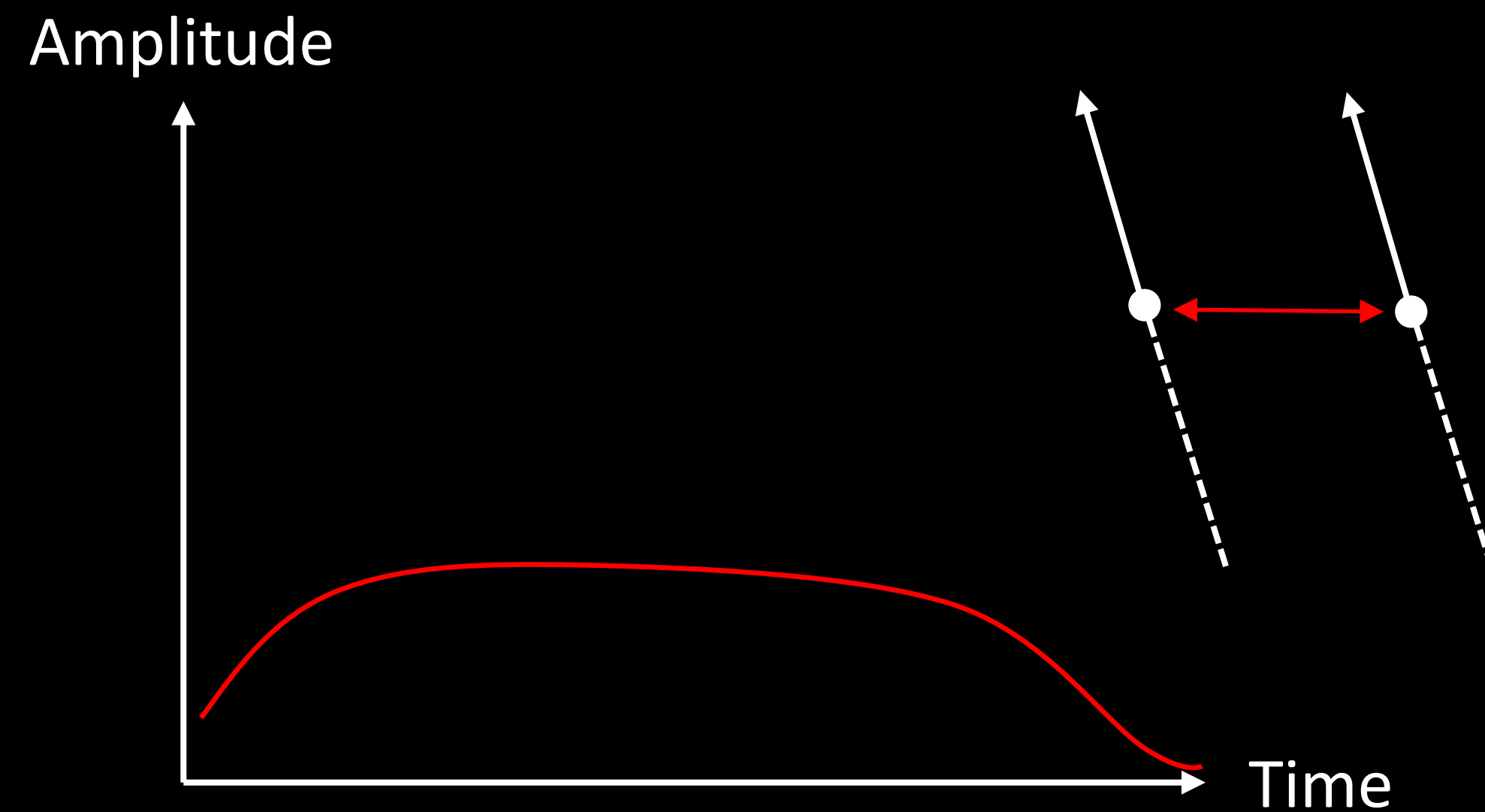


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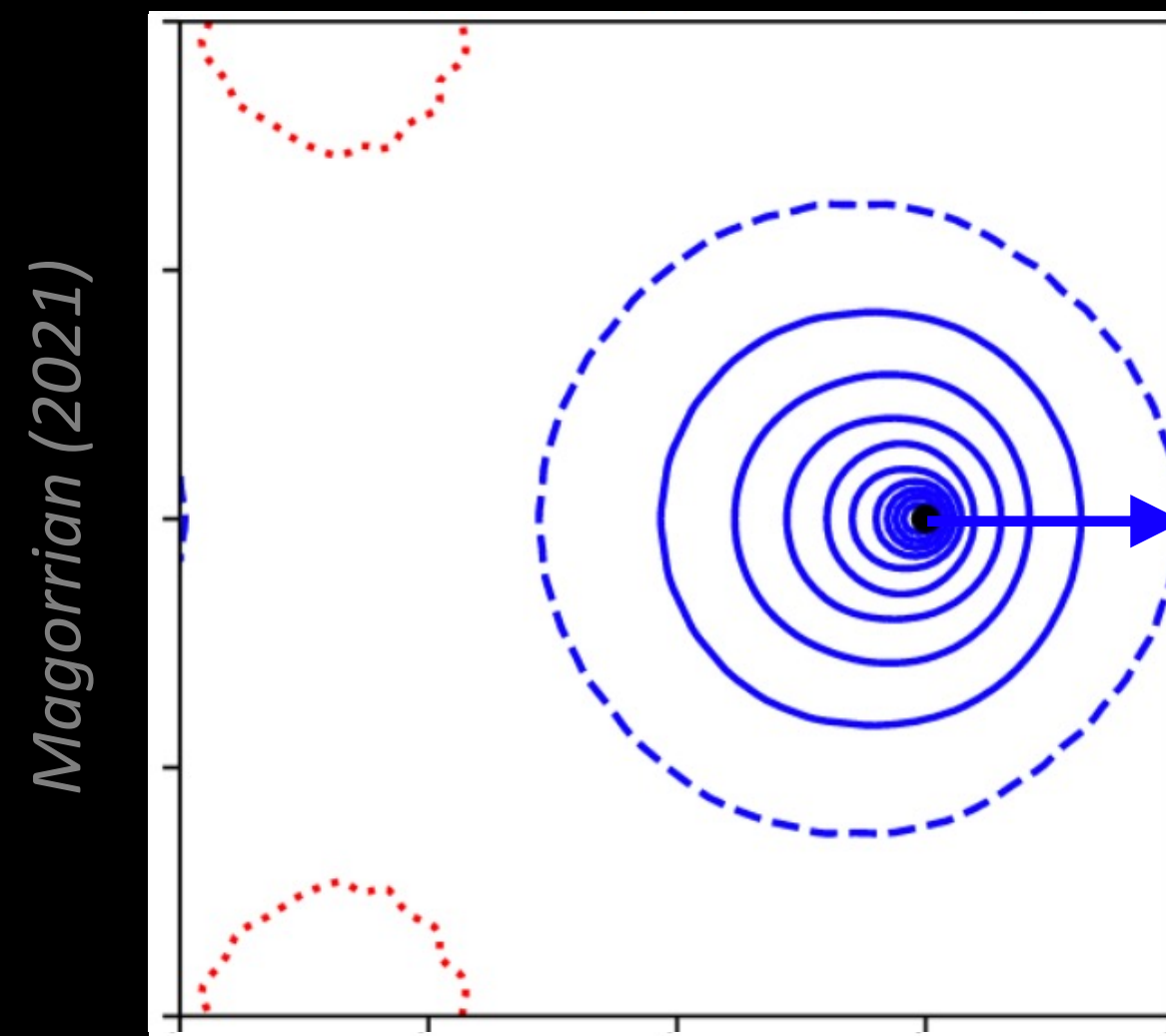


→ Supported by centrifugal force

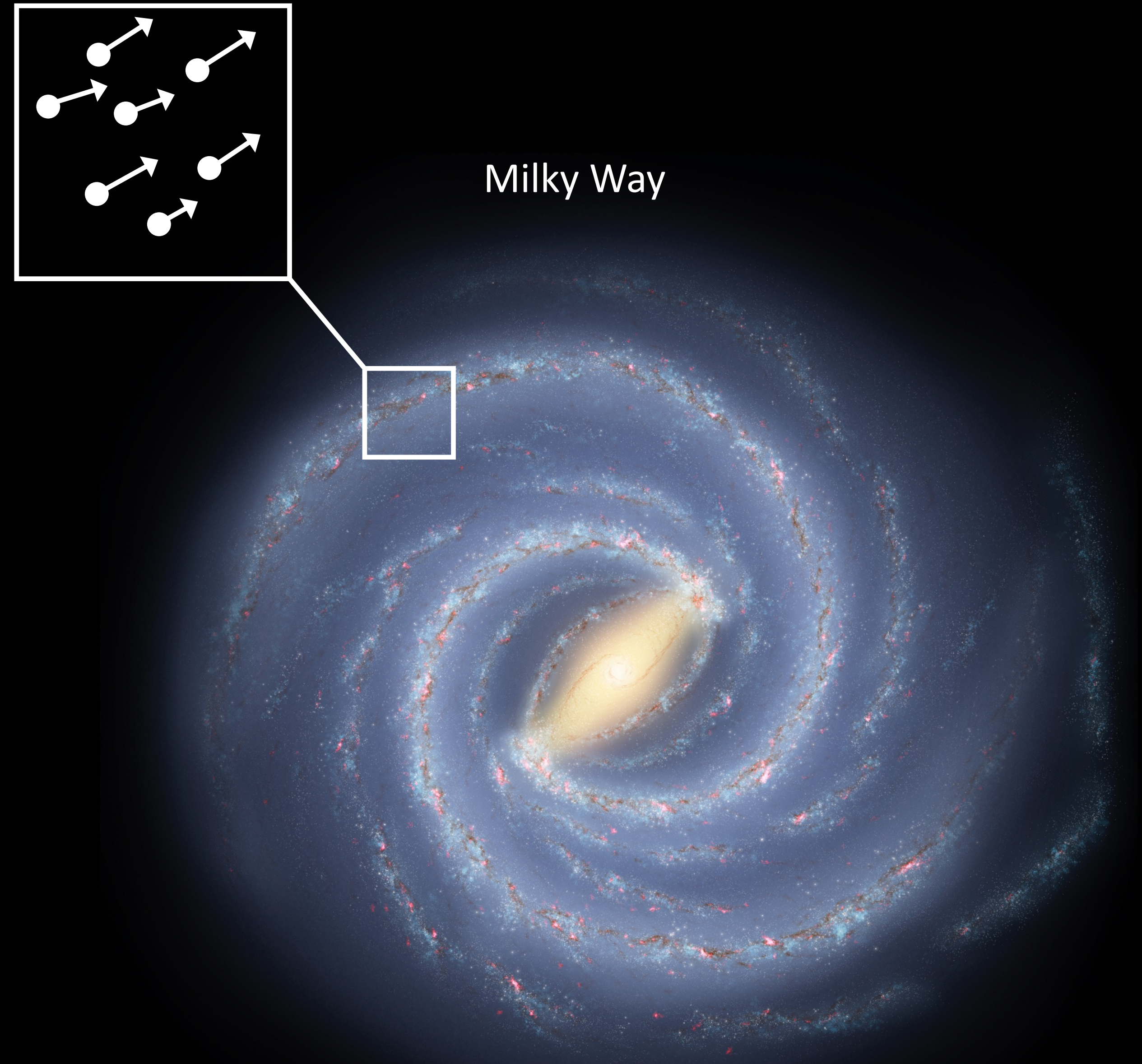
Cold systems



Long coherent interaction build-up



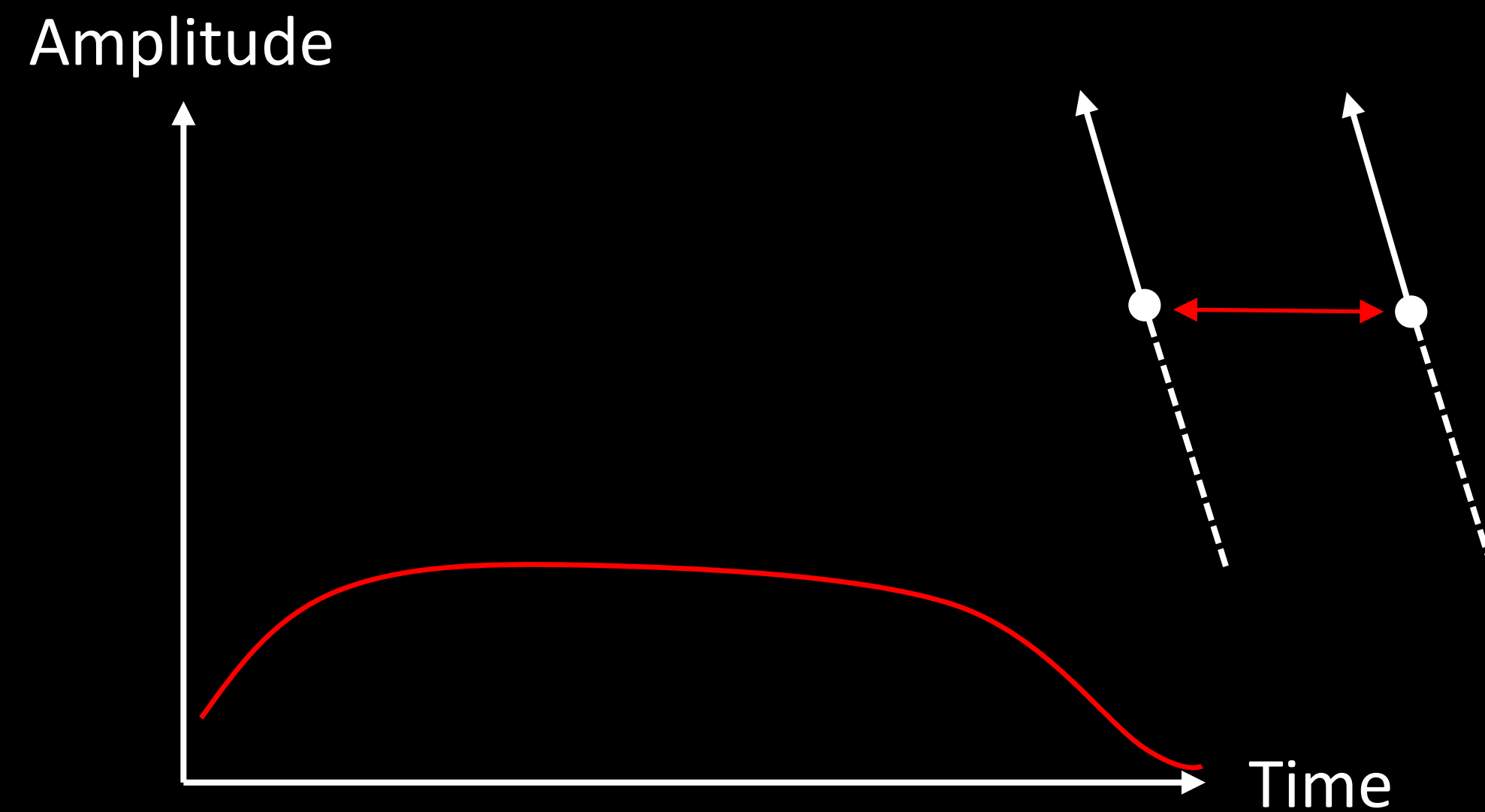
weak gravitational wake



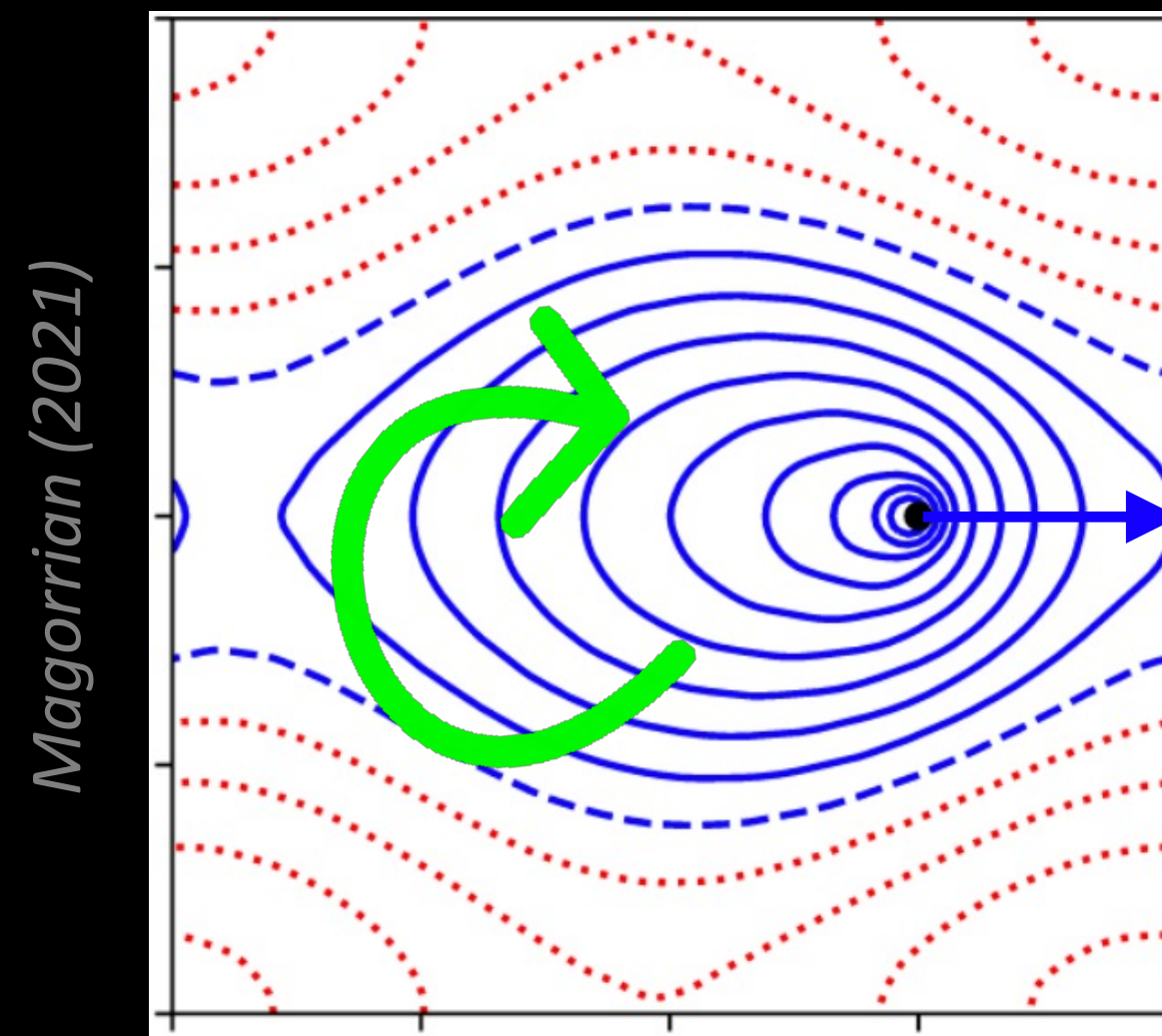
Milky Way

→ Supported by centrifugal force

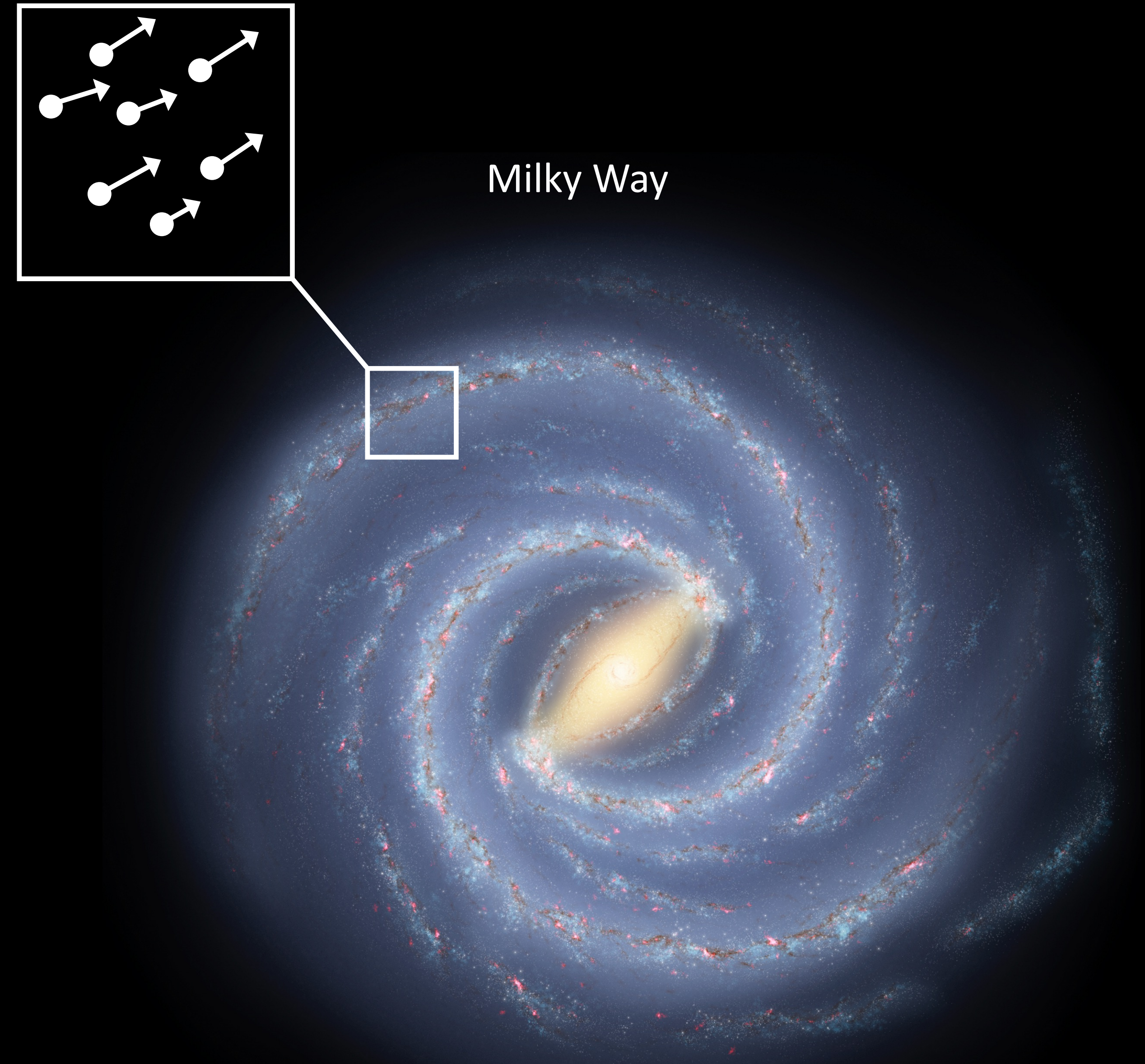
Cold systems



Long coherent interaction build-up

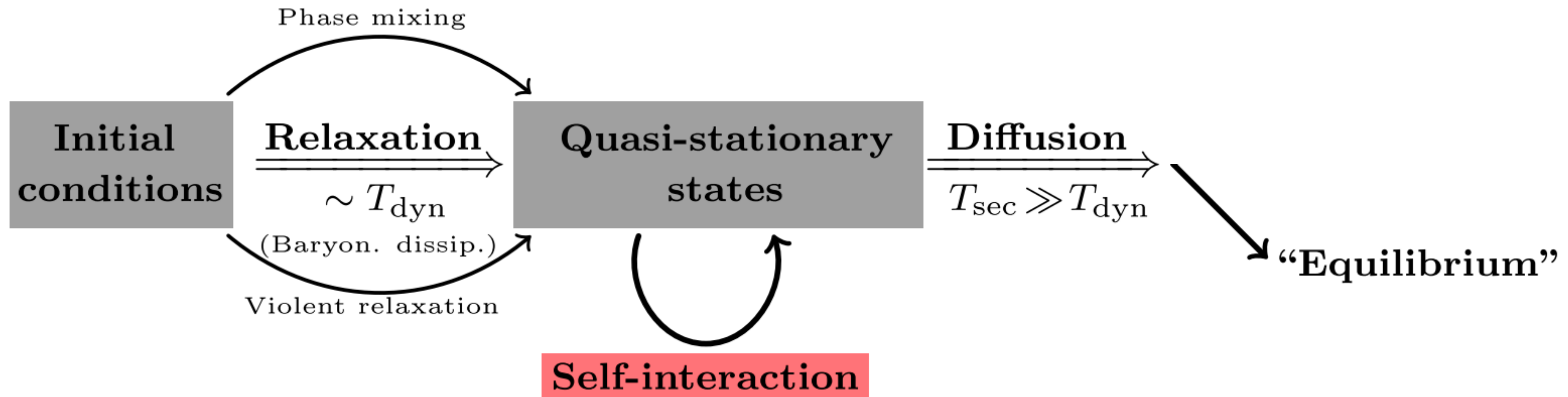


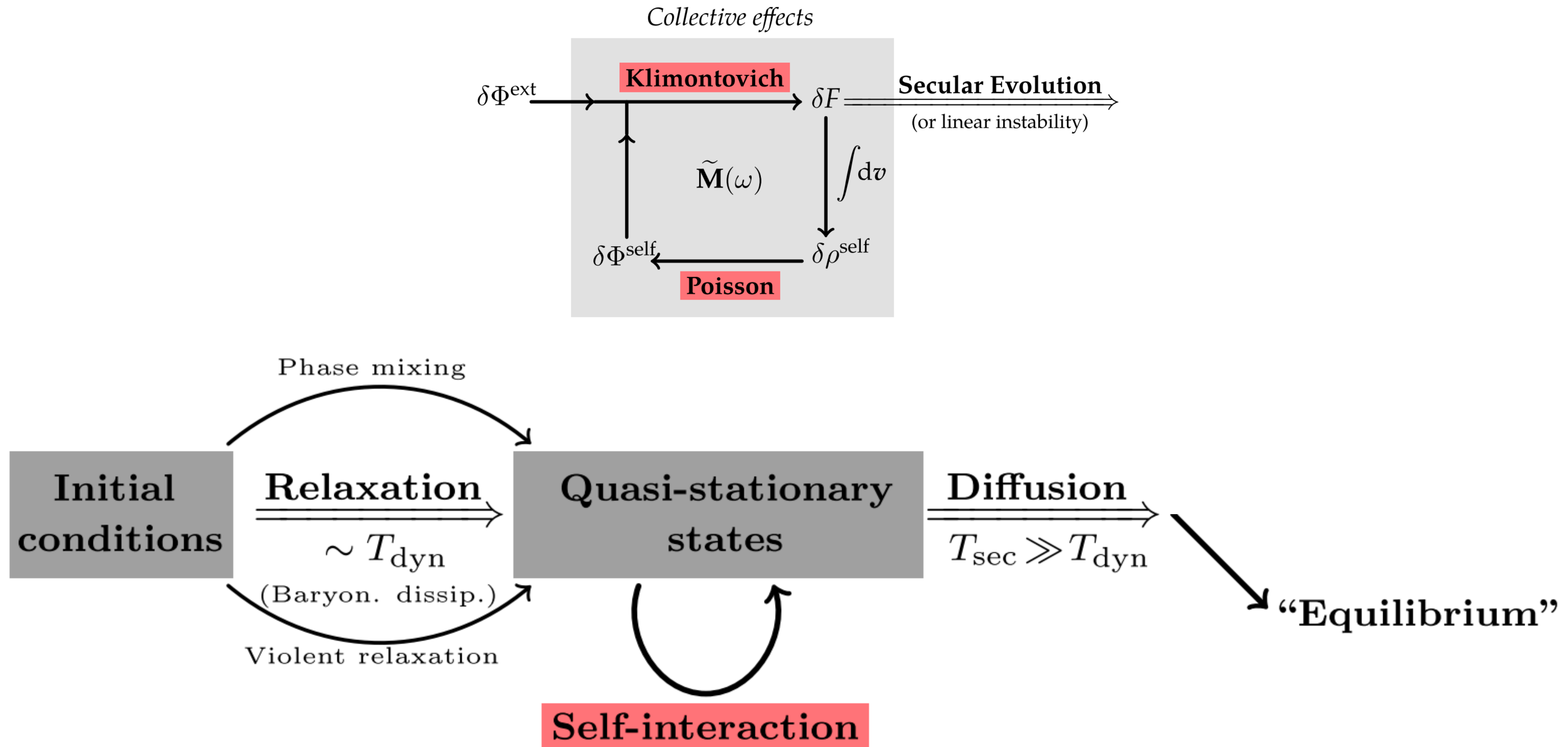
→ Self-amplified gravitational wake

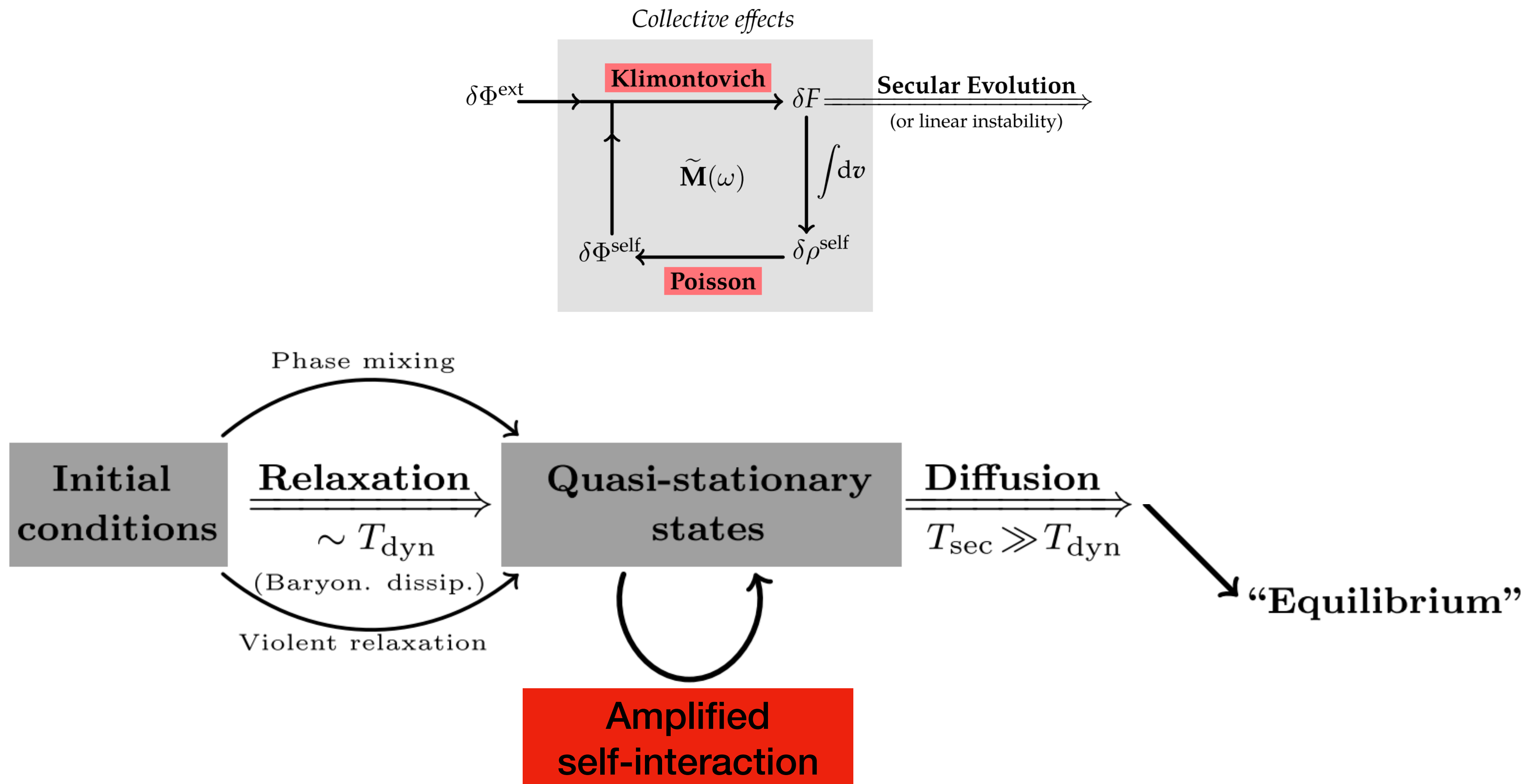


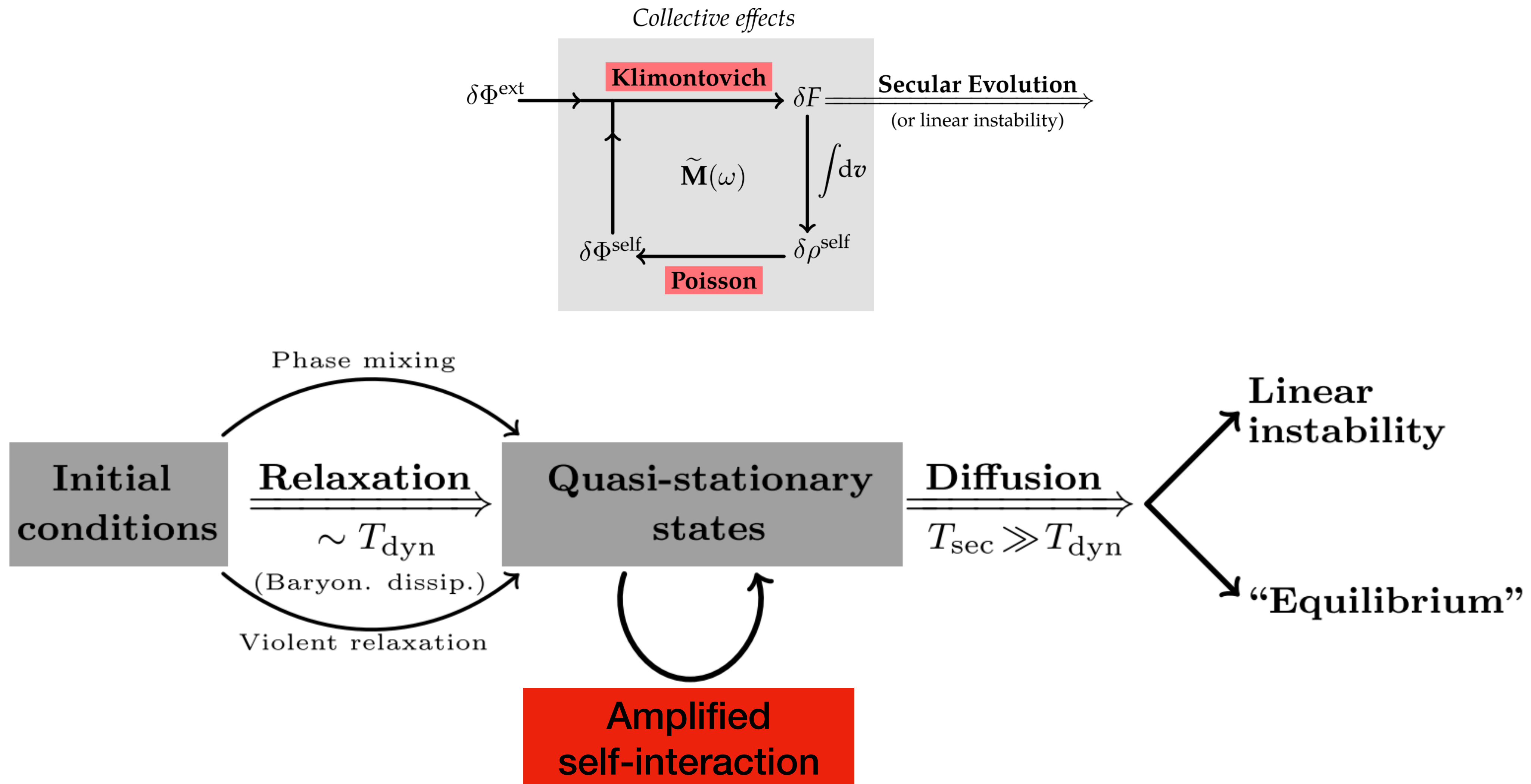
Milky Way

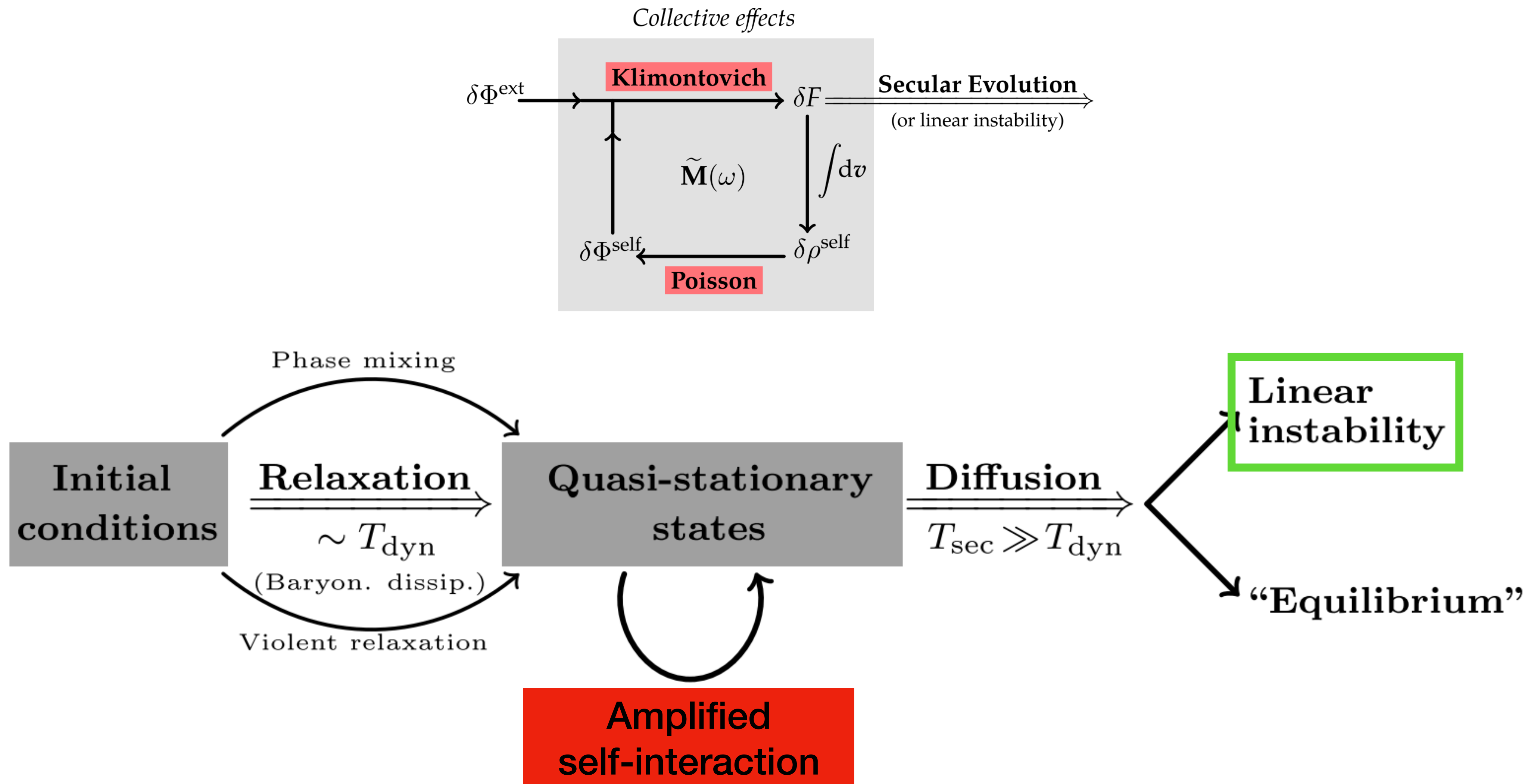
→ Supported by centrifugal force









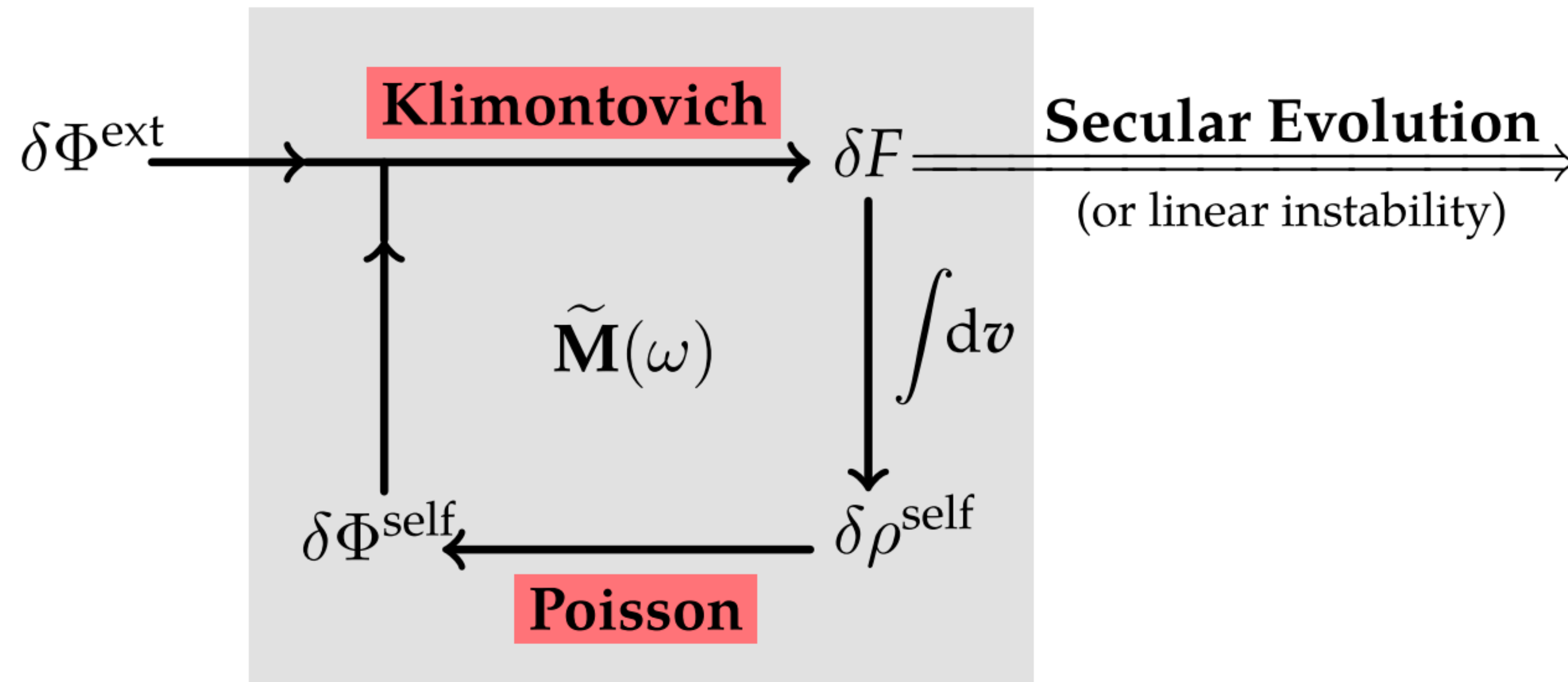


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The response matrix

Collective effects



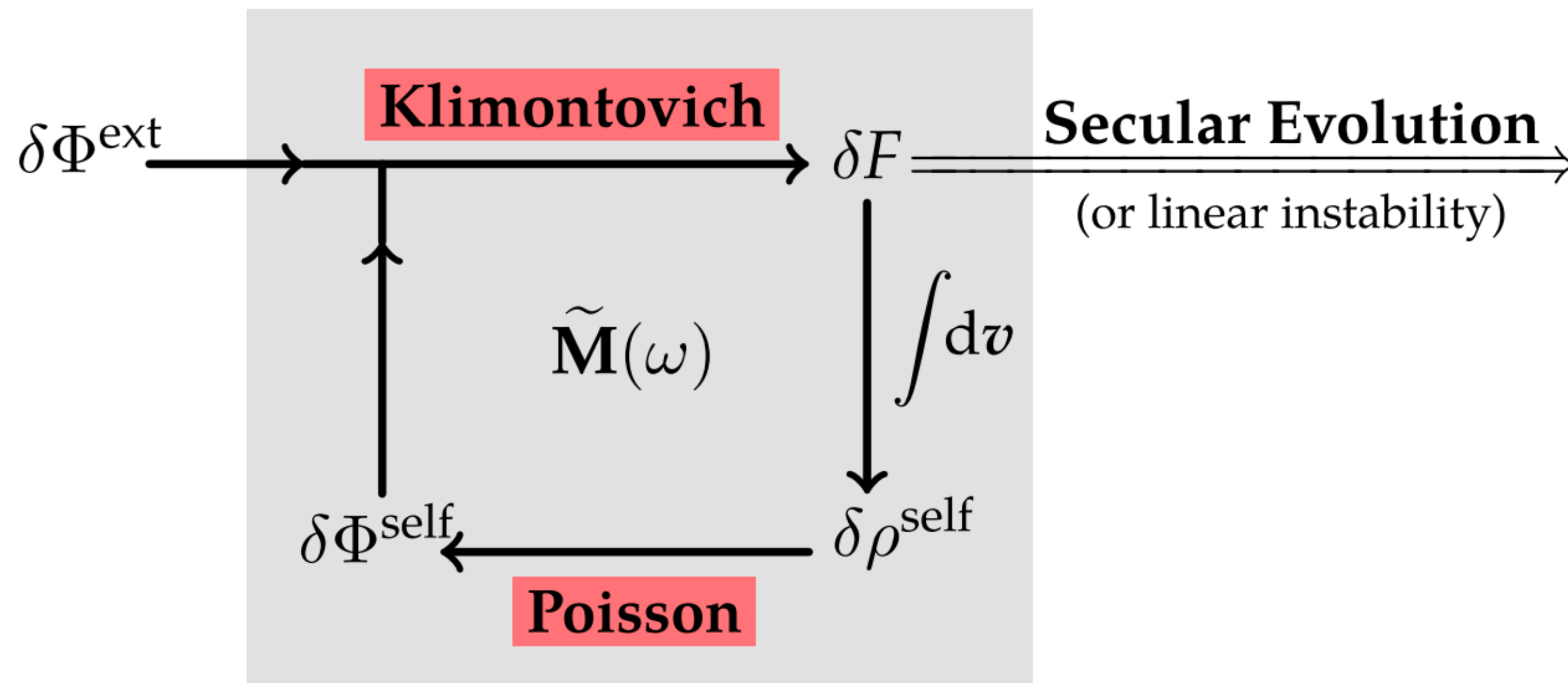
The response matrix

$$F = F(\mathbf{x}, \mathbf{v}) = F(\mathbf{J})$$

$$\rho(\mathbf{x}) = \int d\mathbf{v} F(\mathbf{x}, \mathbf{v})$$

$$\Delta\Phi = 4\pi G\rho$$

Collective effects



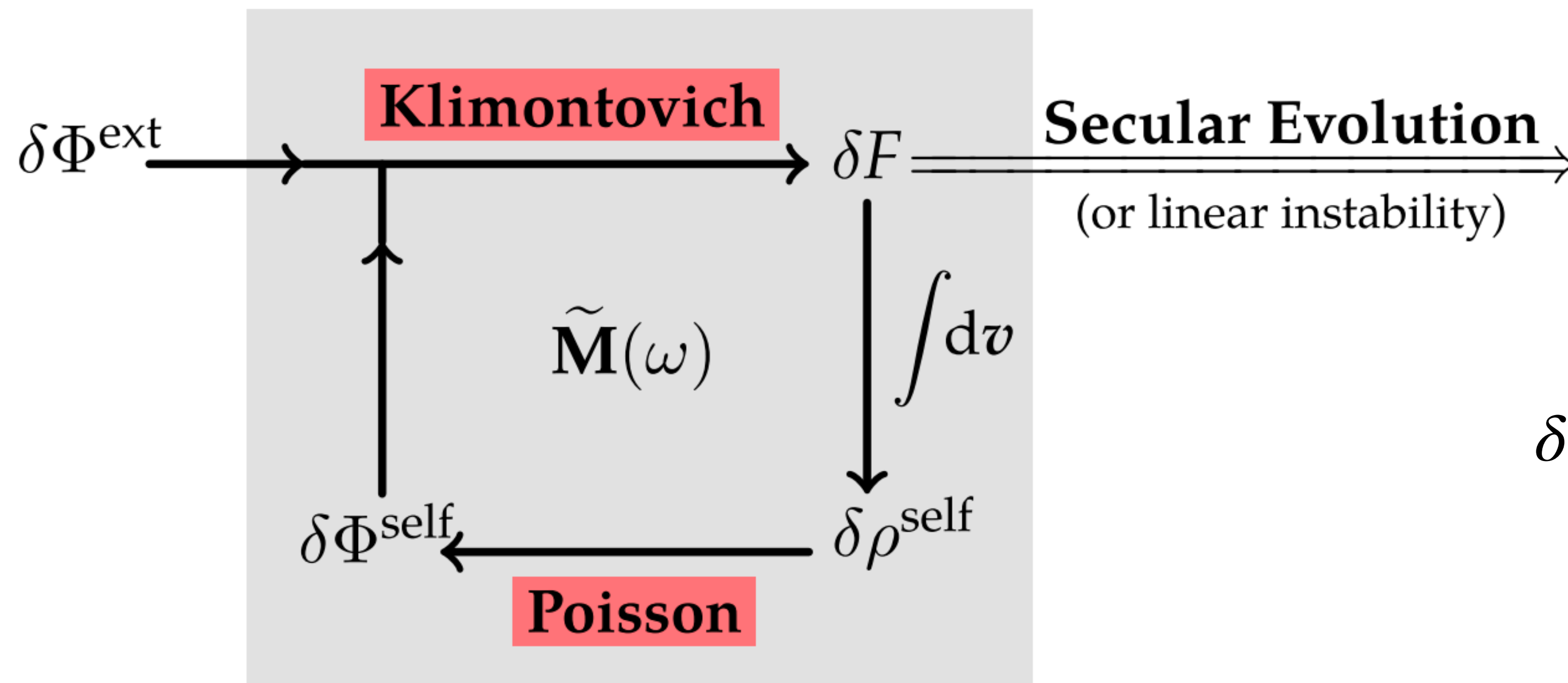
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Collective effects



$$\delta\Phi^{\text{self}} = \tilde{\mathbf{M}}(\omega) \cdot (\delta\Phi^{\text{ext}})$$

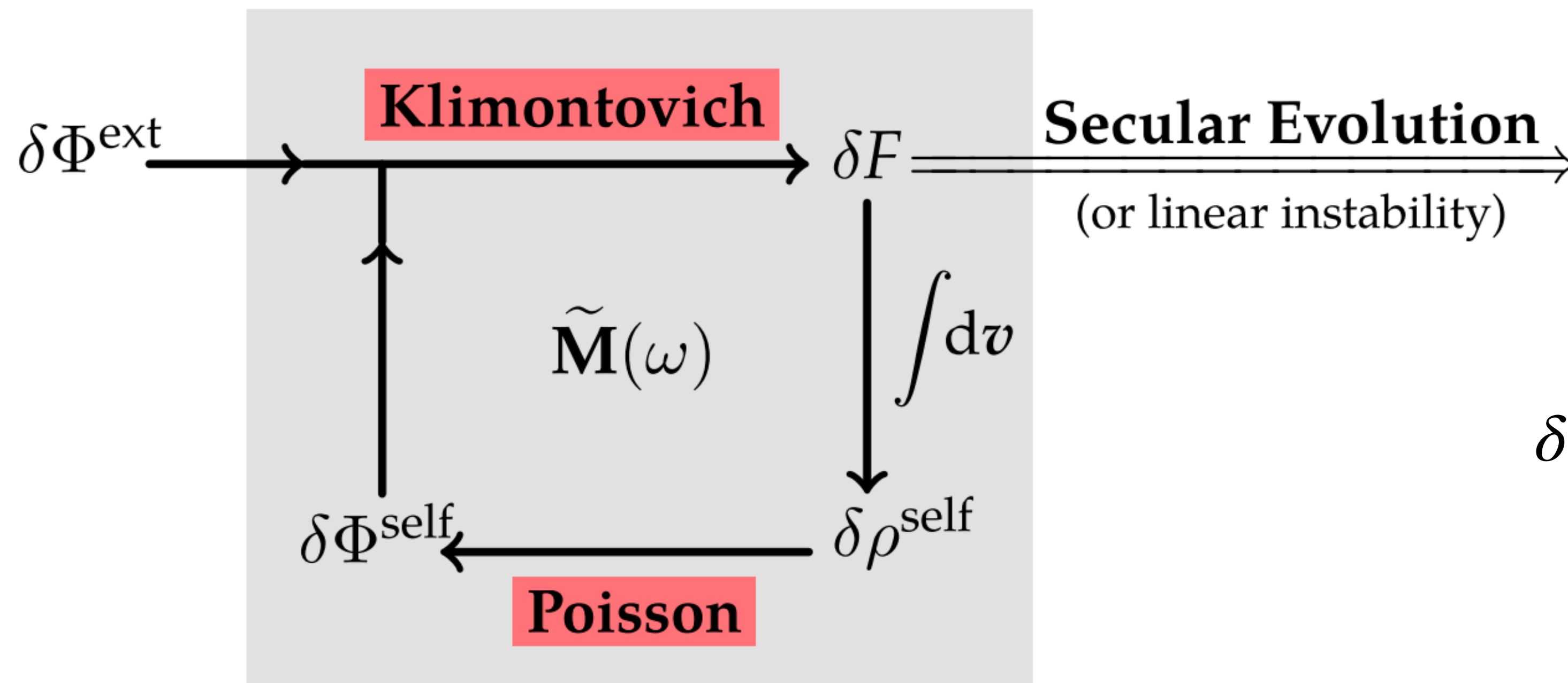
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Collective effects



$$\delta\Phi^{\text{self}} = \tilde{\mathbf{M}}(\omega) \cdot (\delta\Phi^{\text{self}} + \delta\Phi^{\text{ext}})$$

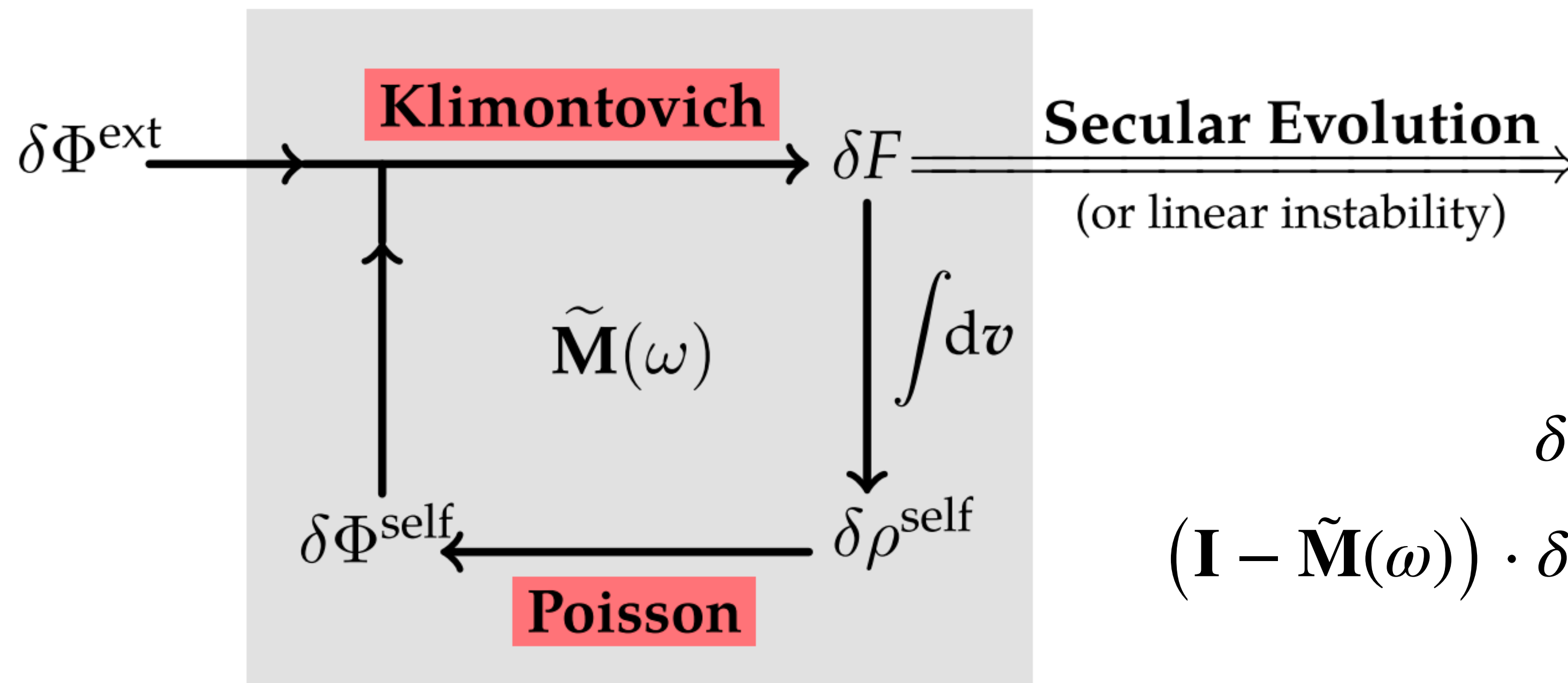
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Collective effects



$$\delta\Phi^{\text{self}} = \tilde{\mathbf{M}}(\omega) \cdot (\delta\Phi^{\text{self}} + \delta\Phi^{\text{ext}})$$

$$(\mathbf{I} - \tilde{\mathbf{M}}(\omega)) \cdot \delta\Phi^{\text{self}} = \tilde{\mathbf{M}}(\omega) \cdot \delta\Phi^{\text{ext}}$$

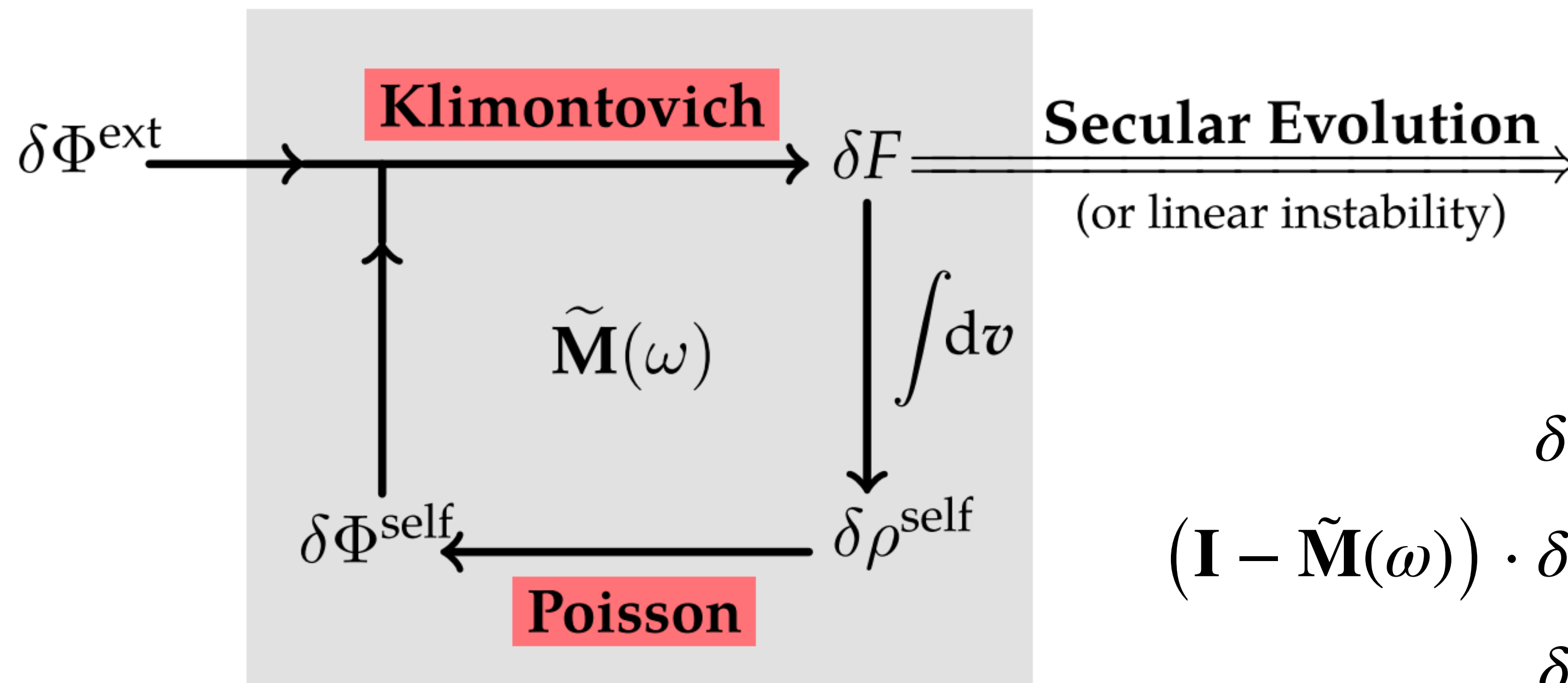
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Collective effects



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$$\delta\Phi^{\text{self}} = (\mathbf{I} - \tilde{\mathbf{M}}(\omega))^{-1} \cdot \tilde{\mathbf{M}}(\omega) \cdot \delta\Phi^{\text{ext}}$$

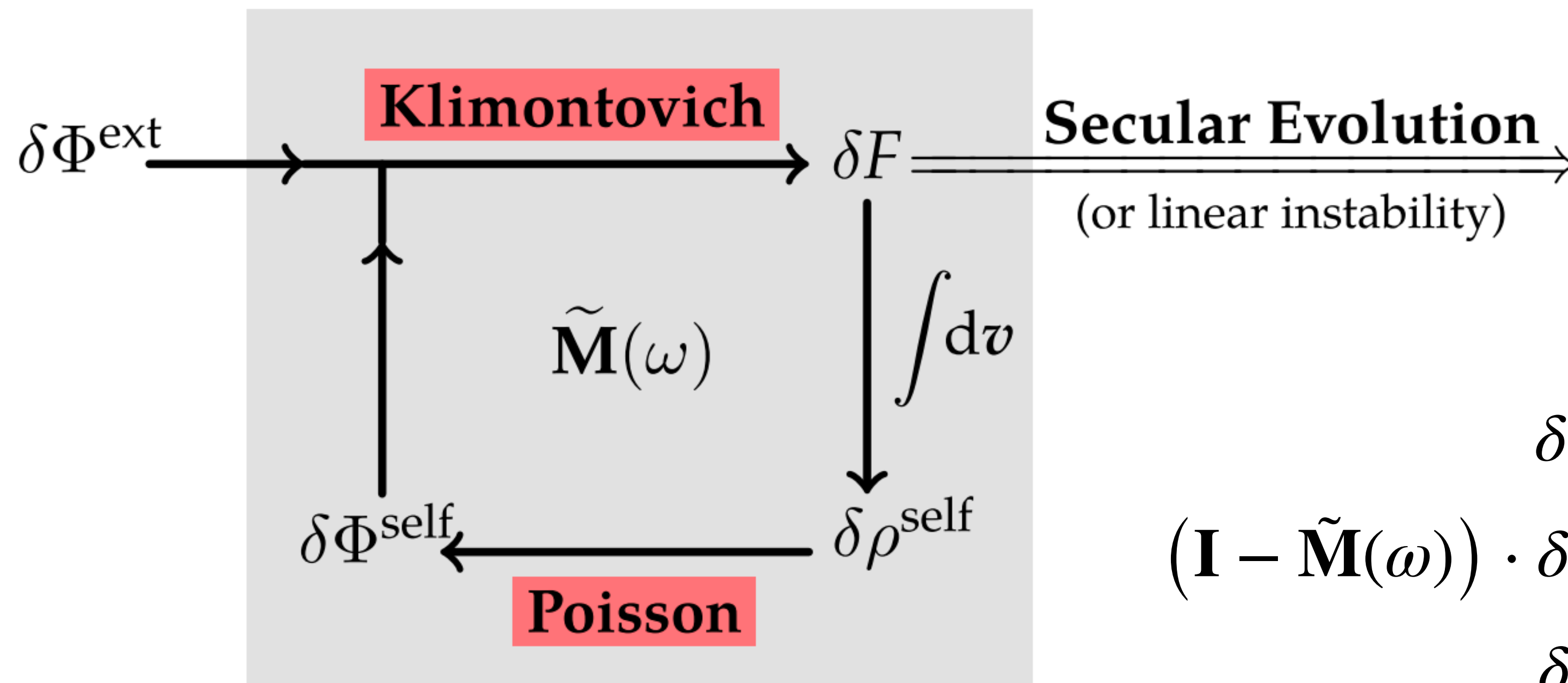
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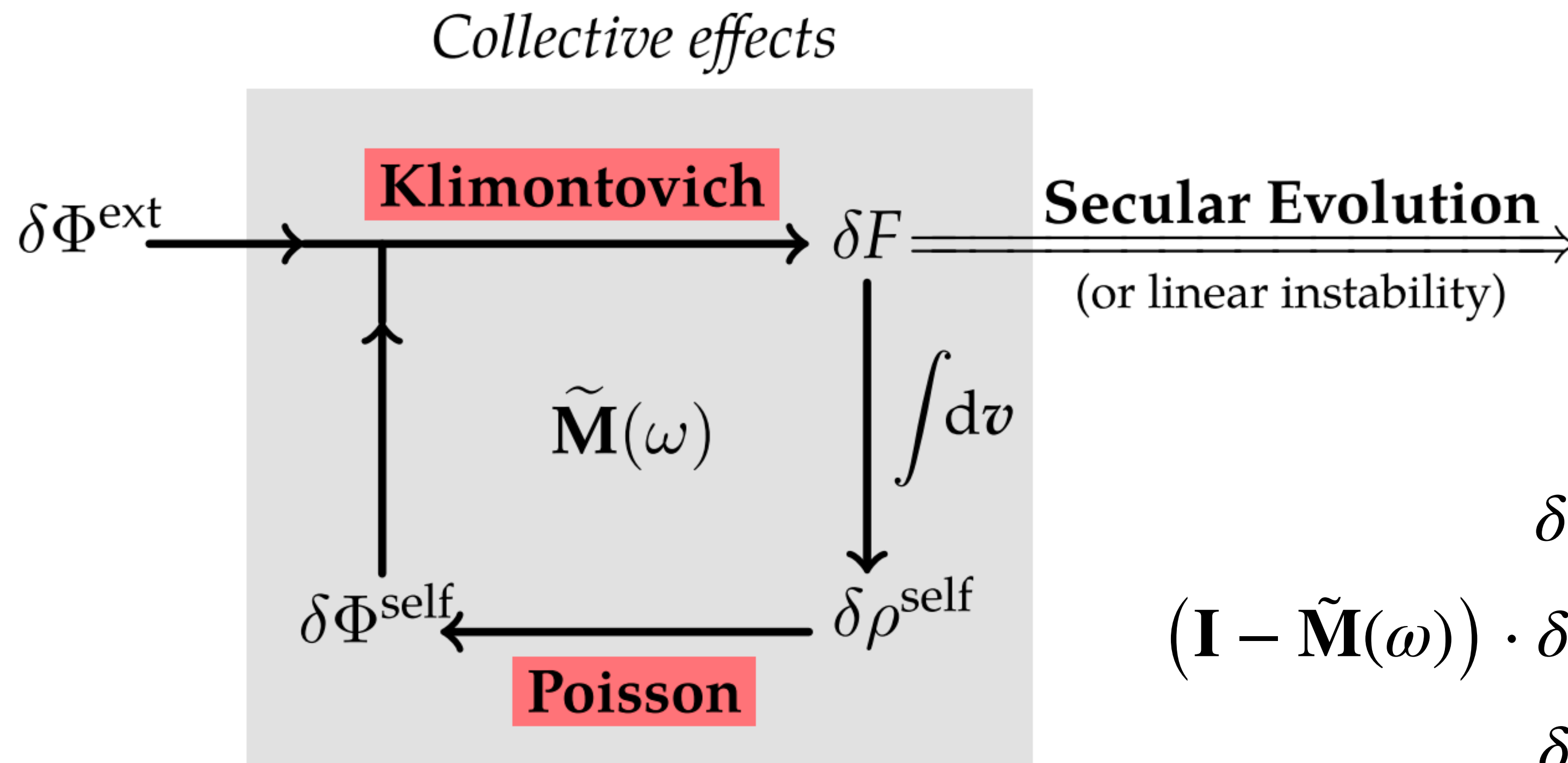


$$\delta\Phi^{\text{self}} = \tilde{\mathbf{M}}(\omega) \cdot (\delta\Phi^{\text{self}} + \delta\Phi^{\text{ext}})$$

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$$\delta\Phi^{\text{self}} = (\mathbf{I} - \tilde{\mathbf{M}}(\omega))^{-1} \cdot \underbrace{\tilde{\mathbf{M}}(\omega) \cdot \delta\Phi^{\text{ext}}}_{\text{bare response}}$$

The response matrix



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$$\rho(\mathbf{x}) = \int d\mathbf{v} F(\mathbf{x}, \mathbf{v})$$

$$\Delta\Phi = 4\pi G\rho$$

$$\delta\Phi^{\text{self}} = \tilde{\mathbf{M}}(\omega) \cdot (\delta\Phi^{\text{self}} + \delta\Phi^{\text{ext}})$$

$$(\mathbf{I} - \tilde{\mathbf{M}}(\omega)) \cdot \delta\Phi^{\text{self}} = \tilde{\mathbf{M}}(\omega) \cdot \delta\Phi^{\text{ext}}$$

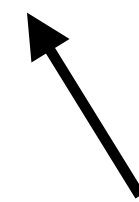
$$\delta\Phi^{\text{self}} = \underbrace{(\mathbf{I} - \tilde{\mathbf{M}}(\omega))^{-1}}_{\text{self-amplification}} \cdot \underbrace{\tilde{\mathbf{M}}(\omega)}_{\text{bare response}} \cdot \delta\Phi^{\text{ext}}$$

The response matrix

$$M_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

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$$M_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$



Mode frequency

The response matrix

$$M_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

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Orbital distribution

Mode frequency

The response matrix

Scan over actions

Orbital distribution

$$M_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Mode frequency

The response matrix

$$M_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Diagram illustrating the components of the response matrix equation:

- Scan over actions**: Points to the integration over $\mathbf{J}$ ($\int d\mathbf{J}$).
- Orbital distribution**: Points to the function F in the numerator $\mathbf{k} \cdot \partial F / \partial \mathbf{J}$.
- Sum over resonances**: Points to the summation over $\mathbf{k}$ ($\sum_{\mathbf{k}}$).
- Mode frequency**: Points to the denominator term $\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})$, which is highlighted with a red box.

The response matrix

Scan over actions

Orbital distribution

FT Basis functions

$$M_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Sum over resonances

Mode frequency

$$U(\mathbf{r}, \mathbf{r}') = - \sum_p \psi^{(p)}(\mathbf{r}) \psi^{(p)}(\mathbf{r}')^*$$

$$M_{pq}(\omega) \sim \langle \psi^{(p)} | \mathbf{M}(\omega) | \psi^{(q)} \rangle$$

The response matrix

$$\det(\mathbf{I} - \mathbf{M}[\omega]) = 0$$

$$M_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Scan over actions $\rightarrow$ $\int d\mathbf{J}$
 Orbital distribution $\rightarrow$ $\psi_{\mathbf{k}}^{(p)*}(\mathbf{J})$
 FT Basis functions $\rightarrow$ $\psi_{\mathbf{k}}^{(q)}(\mathbf{J})$
 Sum over resonances $\rightarrow$ $\sum_{\mathbf{k}}$
 Mode frequency $\rightarrow$ ω

$$U(\mathbf{r}, \mathbf{r}') = - \sum_p \psi^{(p)}(\mathbf{r}) \psi^{(p)}(\mathbf{r}')^*$$

$$M_{pq}(\omega) \sim \langle \psi^{(p)} | \mathbf{M}(\omega) | \psi^{(q)} \rangle$$

The response matrix

$$\det(\mathbf{I} - \mathbf{M}[\omega]) = 0$$

↑
Infinite matrix

Scan over actions

Orbital distribution

FT Basis functions

$$M_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Sum over resonances

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$$M_{pq}(\omega) \sim \langle \psi^{(p)} | \mathbf{M}(\omega) | \psi^{(q)} \rangle$$

The response matrix

$$\det(\mathbf{I} - \mathbf{M}[\omega]) = 0$$

$$M_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Diagram annotations:

- Scan over actions: points to $\int d\mathbf{J}$
- Orbital distribution: points to $\psi_{\mathbf{k}}^{(p)*}(\mathbf{J})$
- FT Basis functions: points to $\psi_{\mathbf{k}}^{(q)}(\mathbf{J})$
- Sum over resonances: points to $\sum_{\mathbf{k}}$
- Mode frequency: points to $\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})$

$$\text{Mode } \omega = \omega_0 + i\gamma \Leftrightarrow \psi \sim e^{-i\omega t} = e^{\gamma t} e^{-i\omega_0 t}$$

The response matrix: flat disc 2D

- Angle actions:
- Basis functions:
- DF:

$$M_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

The response matrix: flat disc 2D

- Angle actions: J_R, L_z
- Basis functions:
- DF:

$$M_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

The response matrix: flat disc 2D

- Angle actions: J_R, L_z
- Basis functions: Kalnajs 76, Clutton-Brock 72, Aoki 78 ...
- DF:

$$M_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

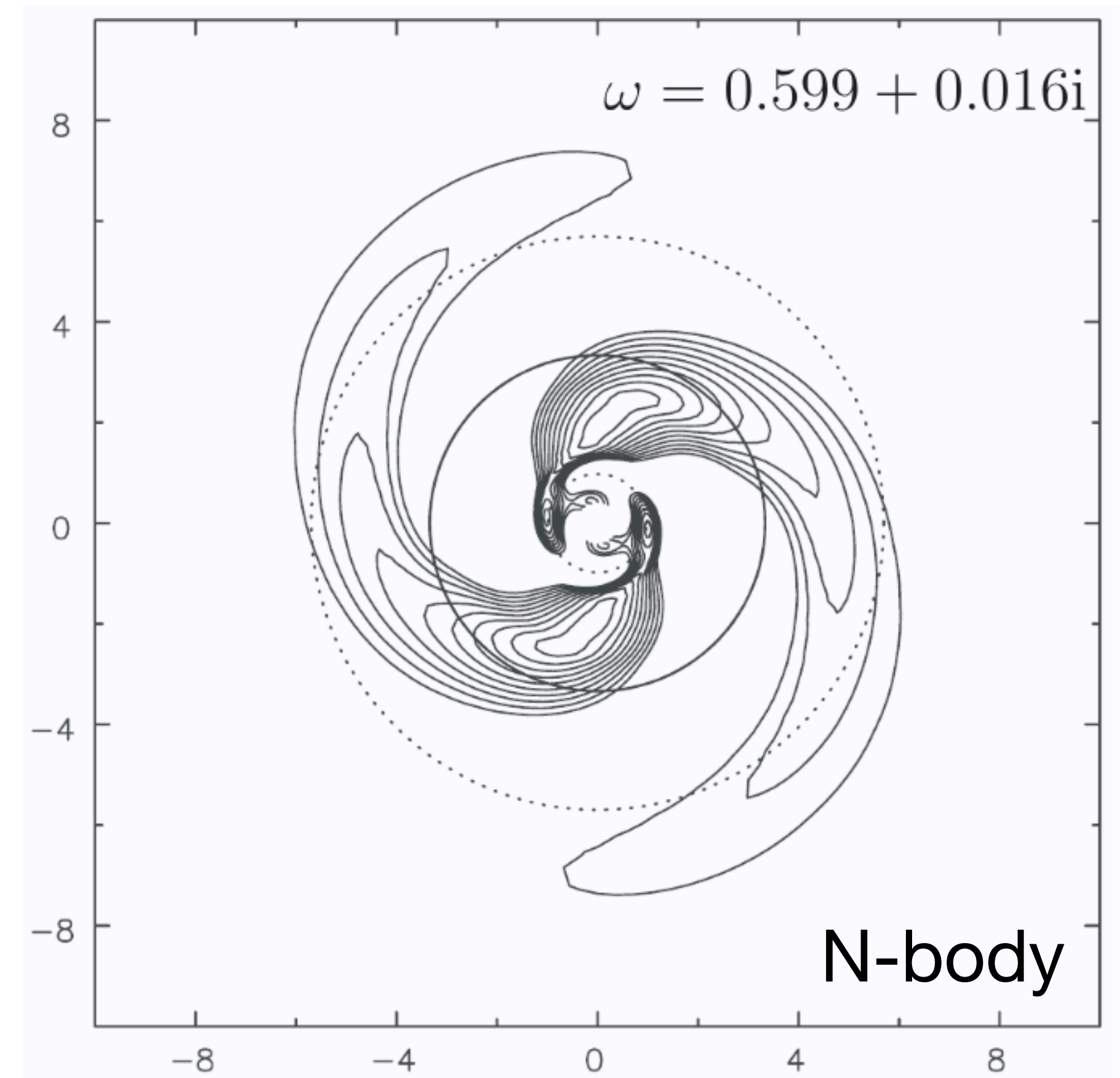
The response matrix: flat disc 2D

- Angle actions: J_R, L_z
- Basis functions: Kalnajs 76, Clutton-Brock 72, Aoki 78 ...
- DF: Schwarzschild, Isochrone, Mestel, Cored exponential, Miyamoto, ...

$$M_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{k}} \int d\mathbf{J} \frac{\mathbf{k} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

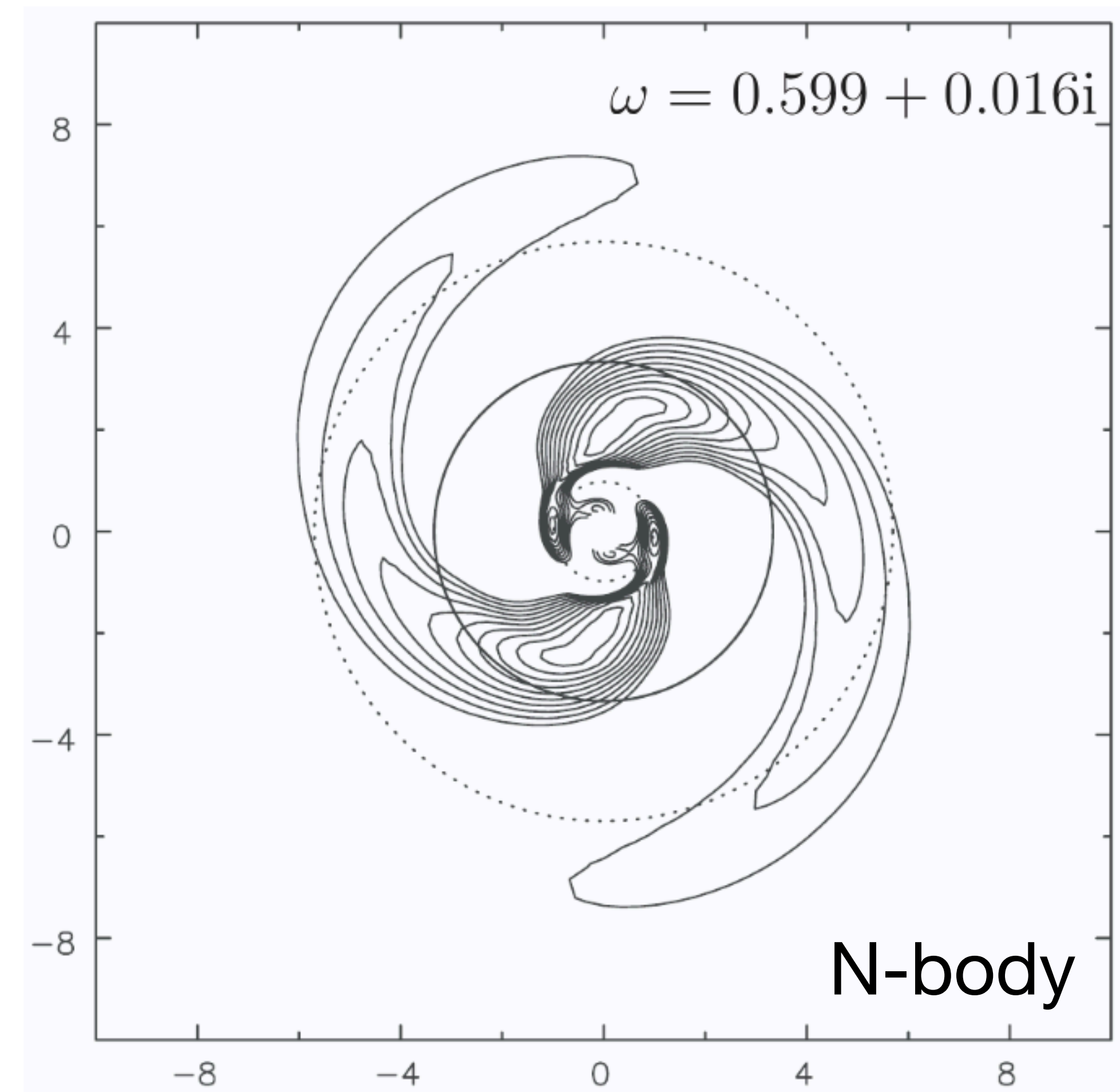
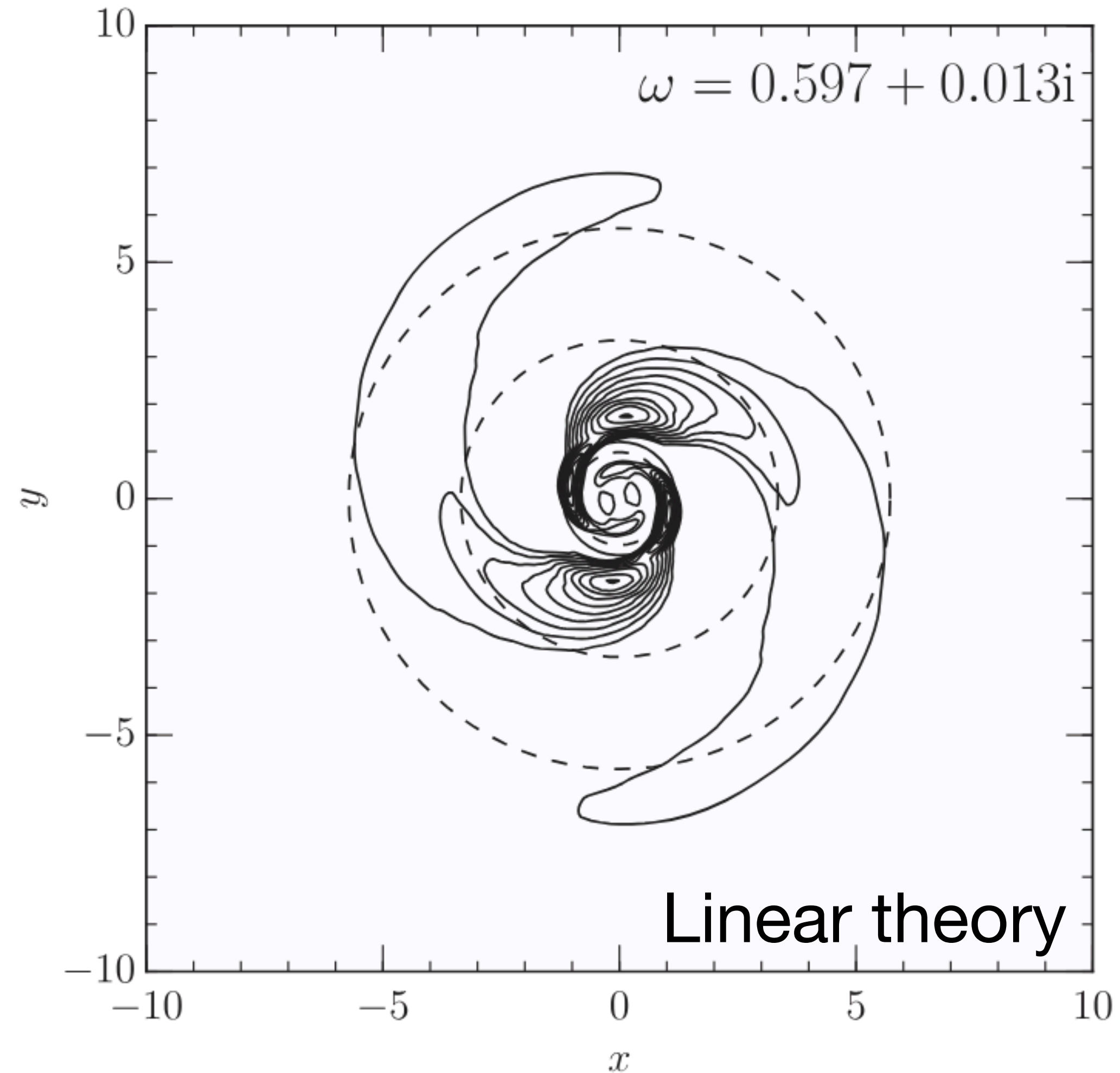
The response matrix: flat disc 2D

Mestel disk



The response matrix: flat disc 2D

Mestel disk



Contents

- 1) Context
- 2) Dynamics of self-gravitating systems
- 3) Linear response theory : the response matrix
- 4) Application
- 5) Future

The system

- Stellar component : disk, ...



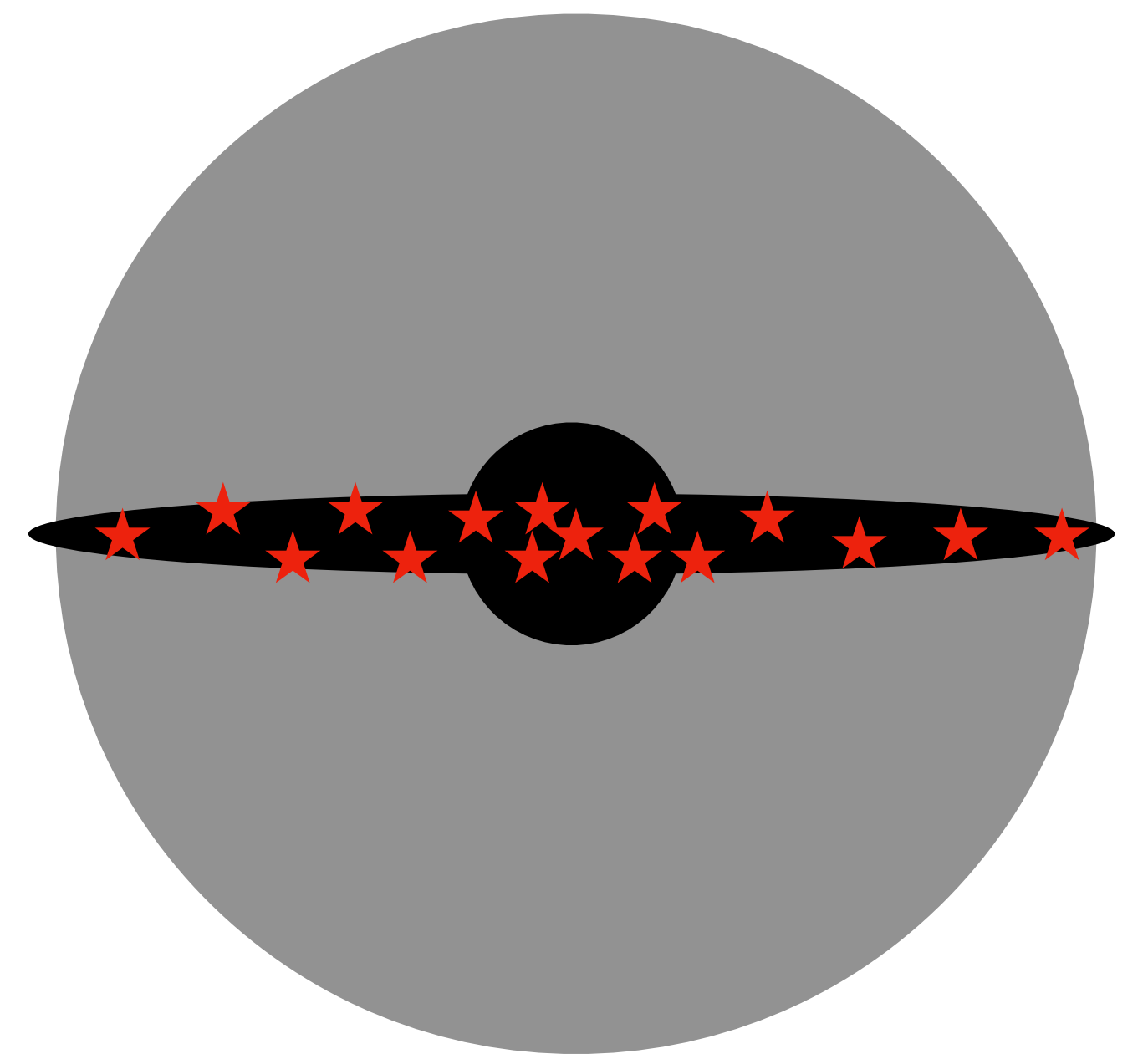
The system

- Stellar component : disk, ...
- Gas component : disk, bulge



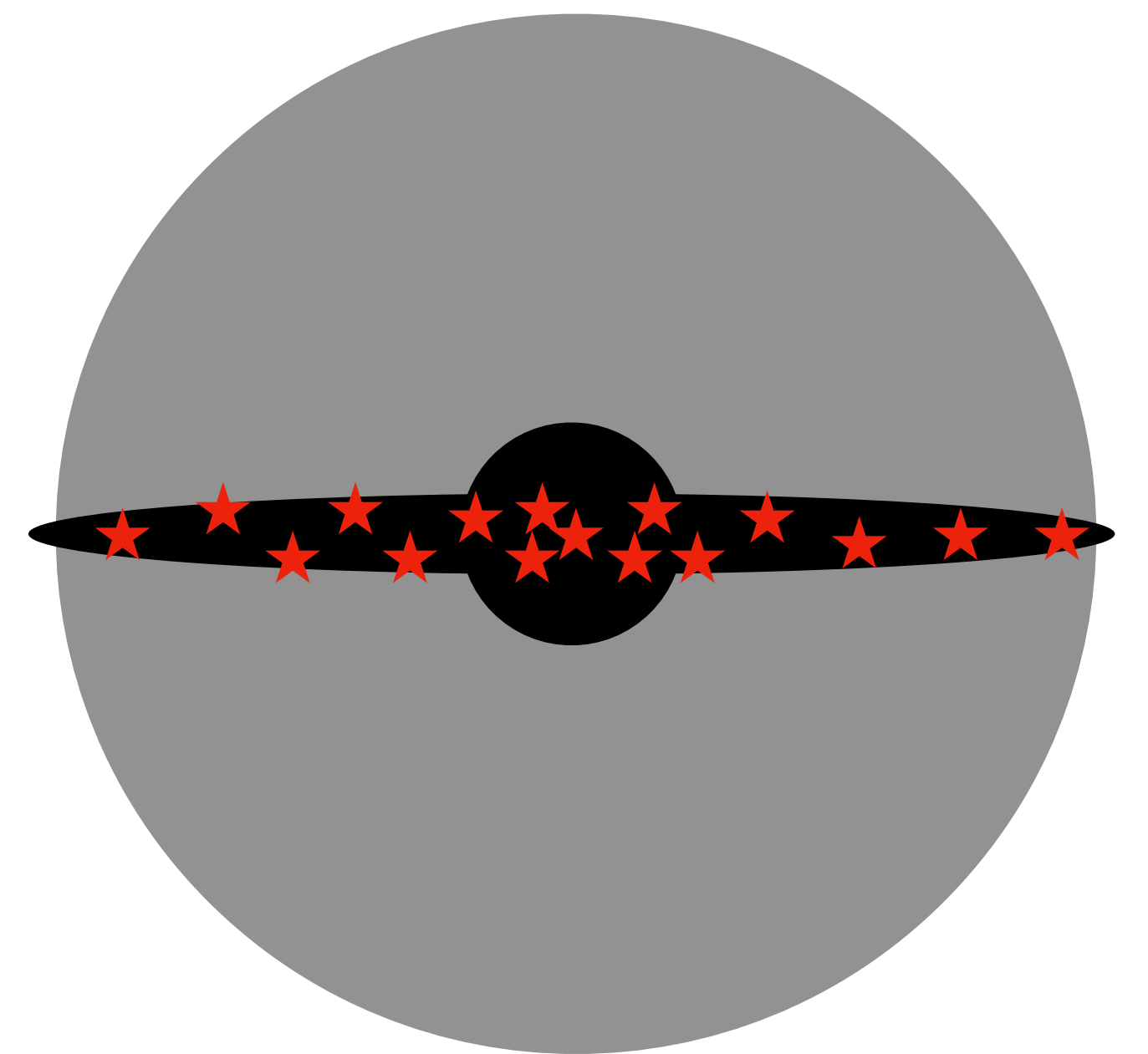
The system

- Stellar component : disk, ...
- Gas component : disk, bulge
- DM halo : spherical



The system

- Stellar component : disk, ...
- Gas component : disk, bulge
- DM halo : spherical

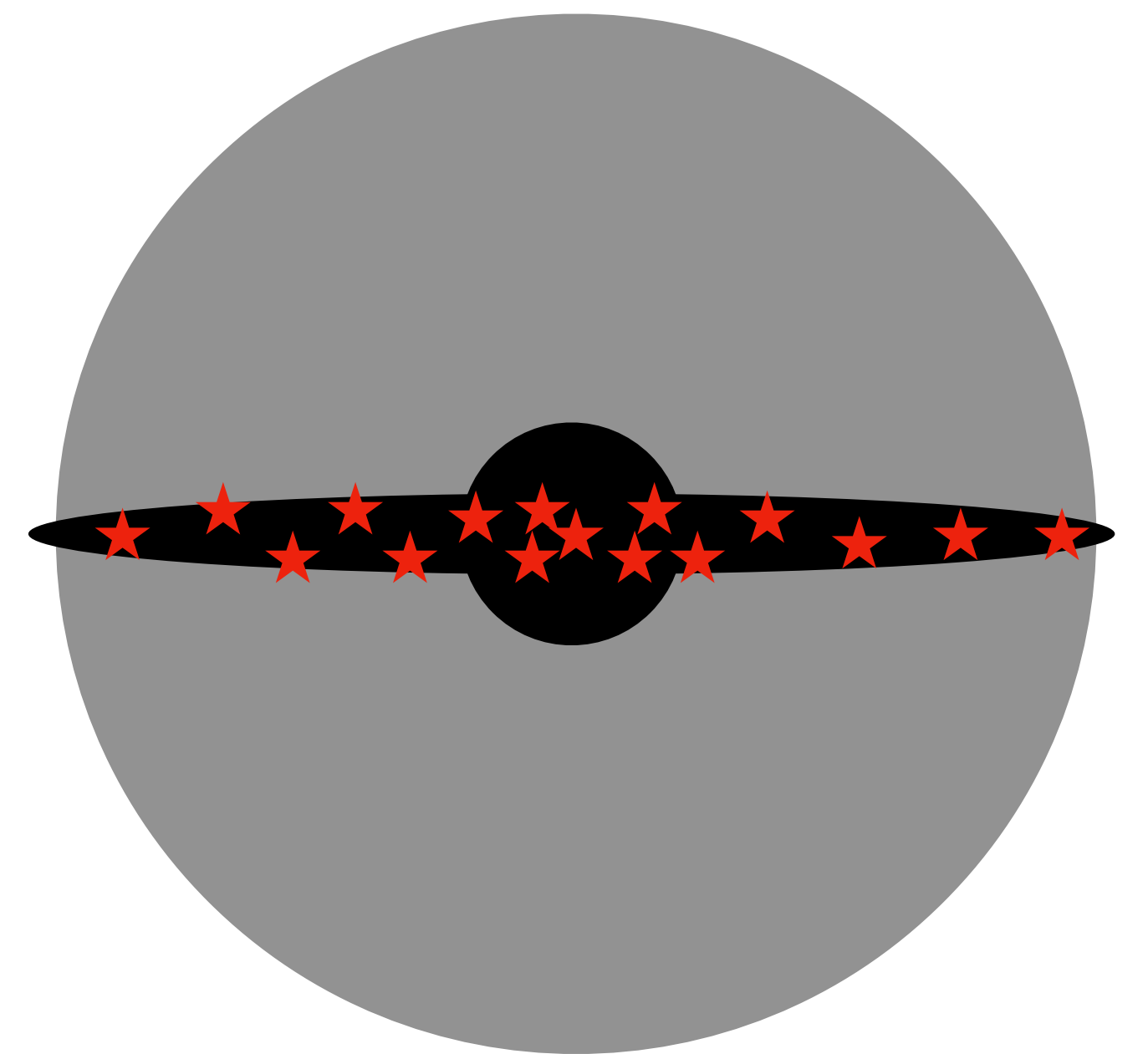


The system

- Stellar component : disk, ...
- Gas component : disk, bulge
- DM halo : spherical

Spiral arms and bar : $m=2$ instability

=> How to explain ?



Pure stellar

- Pure stellar disk



Pure stellar

- Pure stellar disk
- Boltzmann equation



$$\frac{\partial F}{\partial t} + [F, H] = 0.$$

Pure stellar

- Pure stellar disk
- Boltzmann equation + Linearization



$$\frac{\partial F}{\partial t} + [F, H] = 0. \Rightarrow \frac{\partial \delta F}{\partial t} + \Omega \cdot \frac{\partial \delta F}{\partial \theta} - \frac{\partial F_0}{\partial J} \cdot \frac{\partial \delta \psi}{\partial \theta} = 0,$$

Pure stellar

- Pure stellar disk
- Boltzmann equation + Linearization + FT



$$\frac{\partial F}{\partial t} + [F, H] = 0. \Rightarrow \frac{\partial \delta F}{\partial t} + \Omega \cdot \frac{\partial \delta F}{\partial \theta} - \frac{\partial F_0}{\partial J} \cdot \frac{\partial \delta \psi}{\partial \theta} = 0, \Rightarrow \delta \widehat{F}_k = -\frac{\mathbf{k} \cdot \partial F_0 / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \Omega} \delta \widehat{\psi}_k$$

Pure stellar

- Pure stellar disk
- Boltzmann equation + Linearization + FT



$$\delta \widehat{\Sigma}_m(R, \omega) = \sum_n \widehat{a}_n(\omega) \mathcal{D}_n^m(R),$$

$$\delta \widehat{\psi}_m(R, \omega) = \sum_n \widehat{a}_n(\omega) \mathcal{U}_n^m(R).$$

$$\frac{\partial F}{\partial t} + [F, H] = 0. \Rightarrow \frac{\partial \delta F}{\partial t} + \Omega \cdot \frac{\partial \delta F}{\partial \theta} - \frac{\partial F_0}{\partial J} \cdot \frac{\partial \delta \psi}{\partial \theta} = 0, \Rightarrow \delta \widehat{F}_k = -\frac{\mathbf{k} \cdot \partial F_0 / \partial \mathbf{J}}{\omega - \mathbf{k} \cdot \Omega} \delta \widehat{\psi}_k.$$

Pure stellar

- Pur stellar disk
- Boltzmann equation + Linearization + FT



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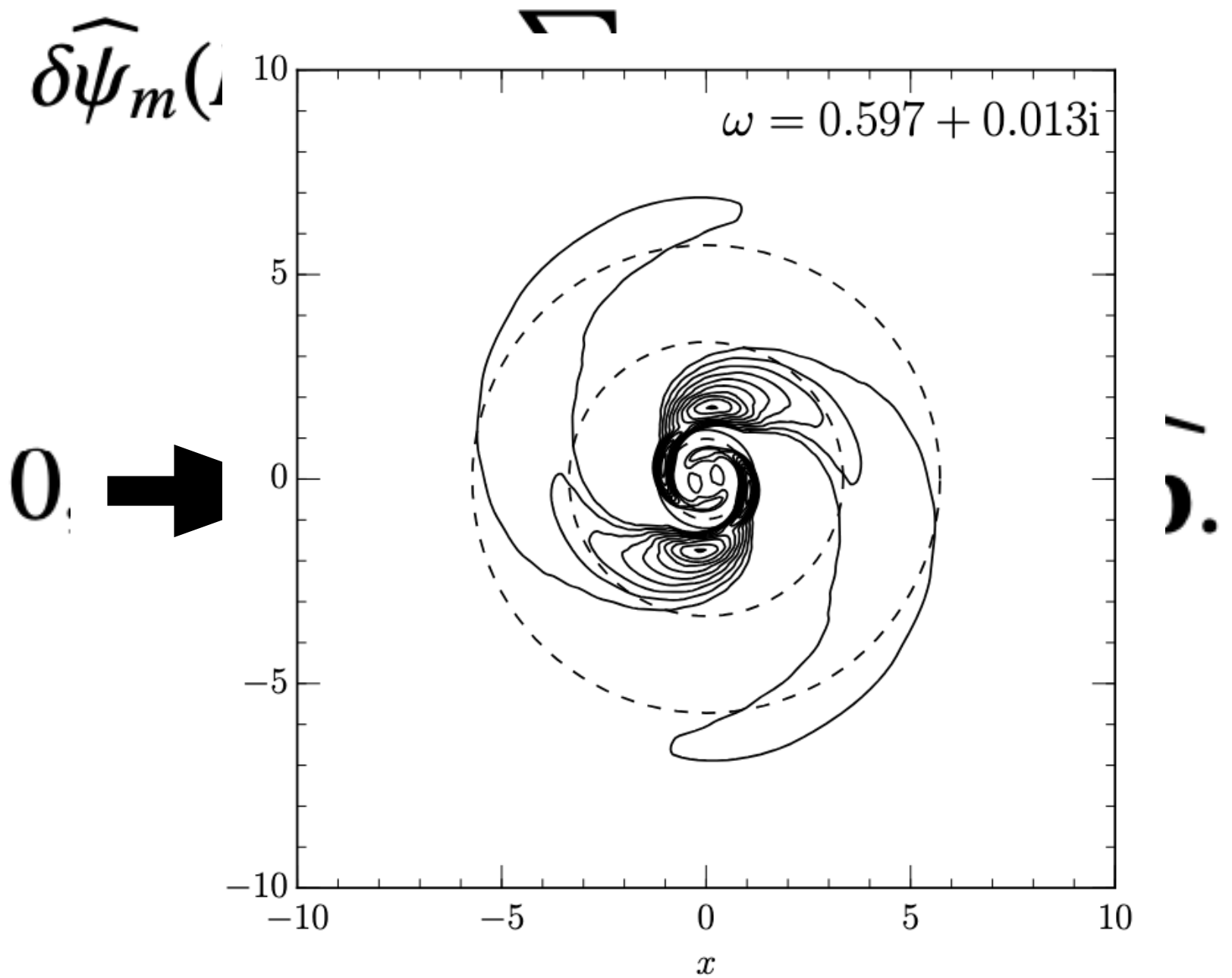
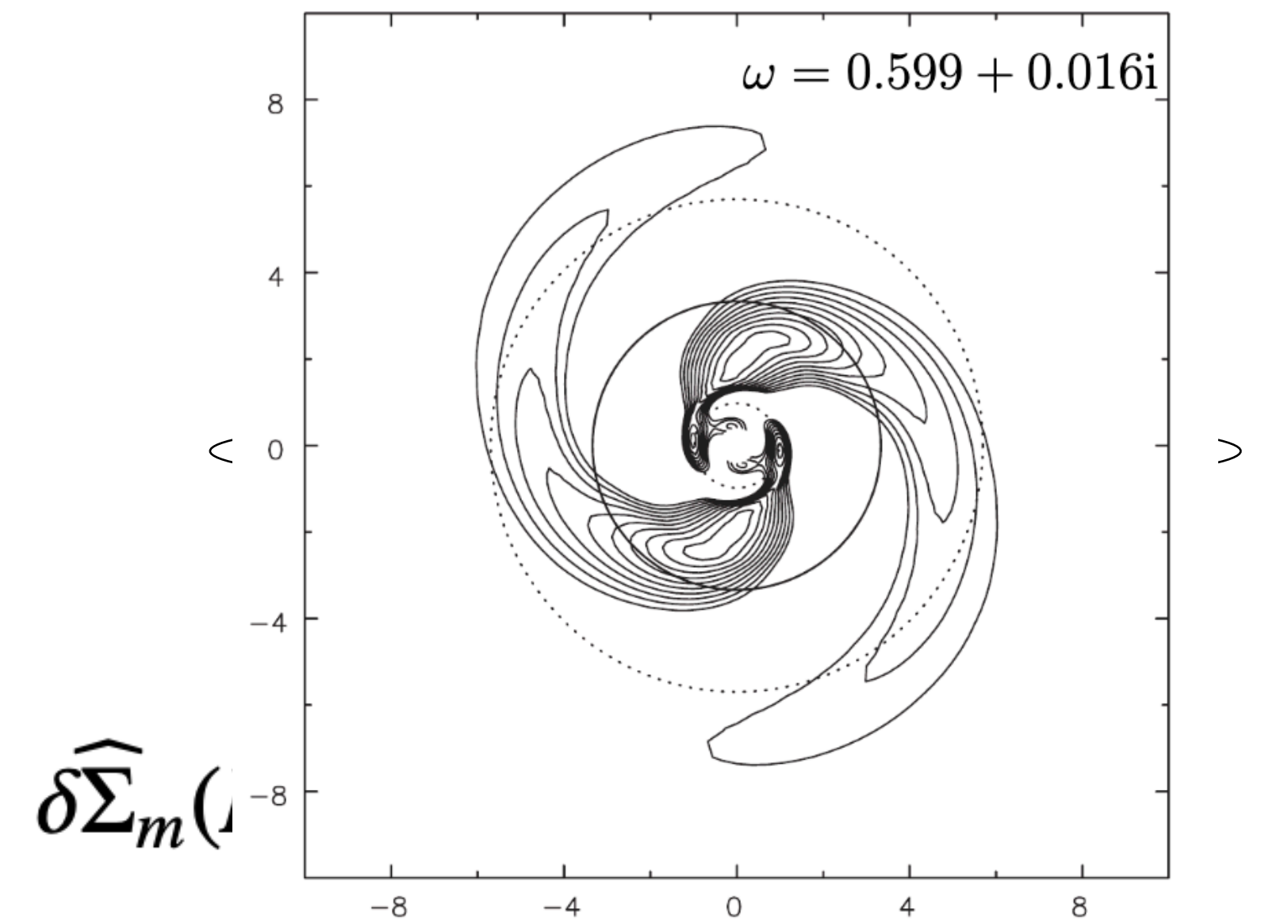
$$\delta \widehat{\psi}_m(R, \omega) = \sum_n \widehat{a}_n(\omega) \mathcal{U}_n^m(R).$$

$$\frac{\partial F}{\partial t} + [F, H] = 0. \Rightarrow \frac{\partial \delta F}{\partial t} + \Omega \cdot \frac{\partial \delta F}{\partial \theta} - \frac{\partial F_0}{\partial J} \cdot \frac{\partial \delta \psi}{\partial \theta} = 0, \Rightarrow \widehat{\mathbf{b}} = \mathbf{H} \cdot \widehat{\mathbf{b}}.$$

Pure stellar

- Pur stellar disk
- Boltzmann equation + Linearization + FT

$$\frac{\partial F}{\partial t} + [F, H] = 0. \Rightarrow \frac{\partial \delta F}{\partial t} + \Omega \cdot \frac{\partial \delta F}{\partial \theta} - \frac{\partial F_0}{\partial J} \cdot \frac{\partial \delta \psi}{\partial \theta} = 0 \Rightarrow$$



De Rijcke & al. (2019)

Pure gas

- Toomre disk of pure gas



Pure gas

- Toomre disk of pure gas
- Euler-Poisson equation

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} [R \Sigma v_R] + \frac{1}{R} \frac{\partial}{\partial \phi} [\Sigma v_\phi] = 0,$$

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{1}{\Sigma} \frac{\partial P}{\partial R} - \frac{\partial \psi}{\partial R},$$

$$\frac{\partial v_\phi}{\partial t} + v_R \frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_\phi}{\partial \phi} + \frac{v_R v_\phi}{R} = -\frac{1}{R \Sigma} \frac{\partial P}{\partial \phi} - \frac{1}{R} \frac{\partial \psi}{\partial \phi}.$$

Aoki & al. (1979)

Pure gas

- Toomre disk of pure gas
- Euler-Poisson equation + linearization

$$\frac{\partial \delta \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} [R \Sigma_g \delta v_R] + \frac{\Sigma_g}{R} \frac{\partial}{\partial \phi} [\delta v_\phi] + \Omega_g \frac{\partial}{\partial \phi} [\delta \Sigma] = 0,$$

$$\frac{\partial \delta v_R}{\partial t} + \Omega_g \frac{\partial \delta v_R}{\partial \phi} - 2\Omega_g \delta v_\phi = -\frac{\partial W}{\partial R},$$

$$\frac{\partial \delta v_\phi}{\partial t} + \frac{\kappa_g^2}{2\Omega_g} \delta v_R + \Omega_g \frac{\partial \delta v_\phi}{\partial \phi} = -\frac{1}{R} \frac{\partial W}{\partial \phi},$$

Aoki & al. (1979)

Pure gas

- Toomre disk of pure gas
- Euler-Poisson equation + linearization + FT

$$(-i\omega + im\Omega_g) \delta\widehat{\Sigma}_m + \frac{1}{R} \frac{d}{dR} [R\Sigma_g \delta\widehat{v}_{R,m}] + \frac{\Sigma_g(im)}{R} \delta\widehat{v}_{\phi,m} = 0,$$

$$(-i\omega + im\Omega_g) \delta\widehat{v}_{R,m} - 2\Omega_g \delta\widehat{v}_{\phi,m} = -\frac{d\widehat{W}_m}{dR},$$

$$(-i\omega + im\Omega_g) \delta\widehat{v}_{\phi,m} + \frac{\kappa_g^2}{2\Omega_g} \delta\widehat{v}_{R,m} = -\frac{(im)}{R} \widehat{W}_m.$$

Aoki & al. (1979)

Pure gas

- Toomre disk of pure gas
- Euler-Poisson equation + linearization

$$\delta\widehat{\Sigma}_m(R, \omega) = -\frac{\Sigma_g}{\Delta_g} \frac{d^2\widehat{W}_m}{dR^2} - \frac{1}{R} \frac{d}{dR} \left[\frac{R\Sigma_g}{\Delta_g} \right] \frac{d\widehat{W}_m}{dR} + \left(\frac{m^2\Sigma_g}{R^2\Delta_g} + \frac{2m}{R\widetilde{\omega}_g} \frac{d}{dR} \left[\frac{\Sigma_g\Omega_g}{\Delta_g} \right] \right) \widehat{W}_m = \mathcal{L}_g[\widehat{W}_m](R, \omega).$$

Pure gas

- Toomre disk of pure gas
- Euler-Poisson equation + linearization

$$\delta\widehat{\Sigma}_m(R, \omega) = -\frac{\Sigma_g}{\Delta_g} \frac{d^2\widehat{W}_m}{dR^2} - \frac{1}{R} \frac{d}{dR} \left[\frac{R\Sigma_g}{\Delta_g} \right] \frac{d\widehat{W}_m}{dR} + \left(\frac{m^2\Sigma_g}{R^2\Delta_g} + \frac{2m}{R\widetilde{\omega}_g} \frac{d}{dR} \left[\frac{\Sigma_g\Omega_g}{\Delta_g} \right] \right) \widehat{W}_m = \mathcal{L}_g[\widehat{W}_m](R, \omega).$$

$$\delta\widehat{\Sigma}_m(R, \omega) = \sum_n \widehat{a}_n(\omega) \mathcal{D}_n^m(R),$$

$$\delta\widehat{\psi}_m(R, \omega) = \sum_n \widehat{a}_n(\omega) \mathcal{U}_n^m(R).$$

Pure gas

- Toomre disk of pure gas
- Euler-Poisson equation + linearization

$$\delta\widehat{\Sigma}_m(R, \omega) = -\frac{\Sigma_g}{\Delta_g} \frac{d^2\widehat{W}_m}{dR^2} - \frac{1}{R} \frac{d}{dR} \left[\frac{R\Sigma_g}{\Delta_g} \right] \frac{d\widehat{W}_m}{dR} + \left(\frac{m^2\Sigma_g}{R^2\Delta_g} + \frac{2m}{R\widetilde{\omega}_g} \frac{d}{dR} \left[\frac{\Sigma_g\Omega_g}{\Delta_g} \right] \right) \widehat{W}_m = \mathcal{L}_g[\widehat{W}_m](R, \omega).$$

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$$\widehat{\mathbf{a}} = (\mathbf{P} + \mathbf{L}) \cdot \widehat{\mathbf{a}}$$

Pure gas

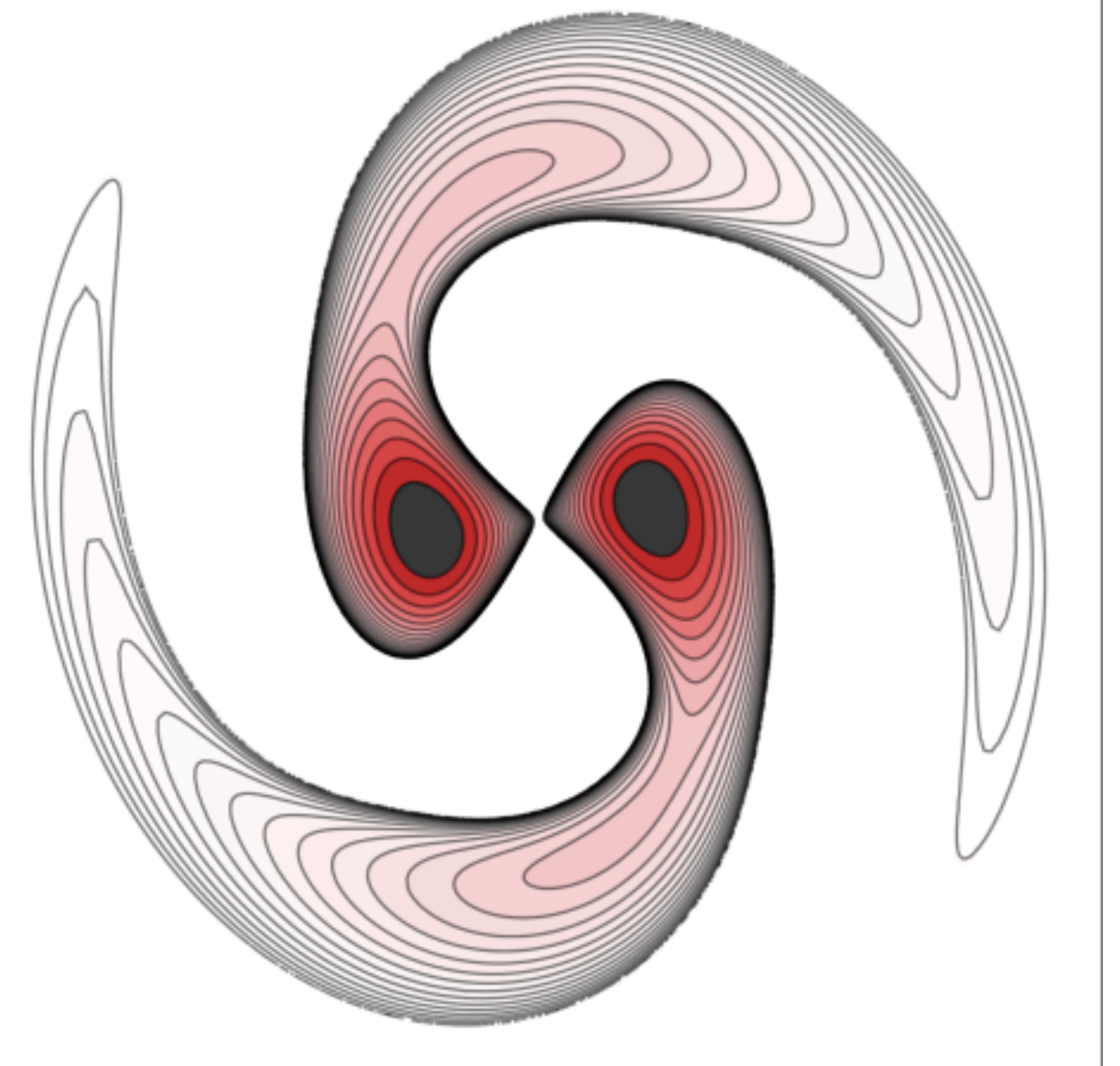
- Toomre disk of pure gas
- Euler-Poisson equation + linearization

$$\delta\widehat{\Sigma}_m(R, \omega) = -\frac{\Sigma_g}{\Delta_g} \frac{d^2 \widehat{W}_m}{dR^2} - \frac{1}{R} \frac{d}{dR} \left[\frac{R\Sigma_g}{\Delta_g} \right] \frac{d\widehat{W}_m}{dR} + \left(\frac{m^2 \Sigma_g}{R^2 \Delta_g} + \frac{2m}{R\widetilde{\omega}_g} \right)$$

$$\delta\widehat{\Sigma}_m(R, \omega) = \sum_n \widehat{a}_n(\omega) \mathcal{D}_n^m(R),$$

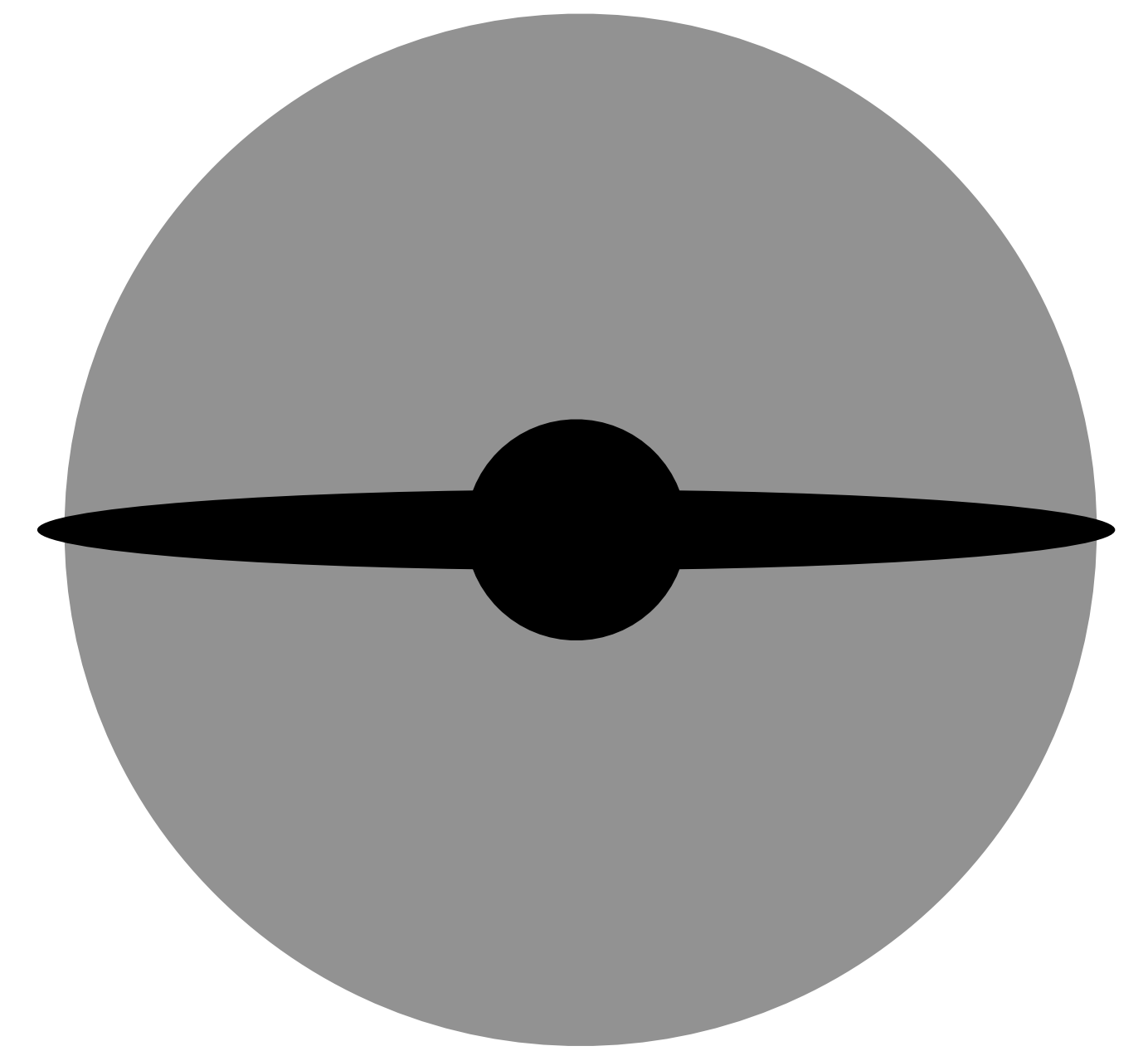
$$\delta\widehat{\psi}_m(R, \omega) = \sum_n \widehat{a}_n(\omega) \mathcal{U}_n^m(R).$$

$\widehat{\mathbf{a}}$

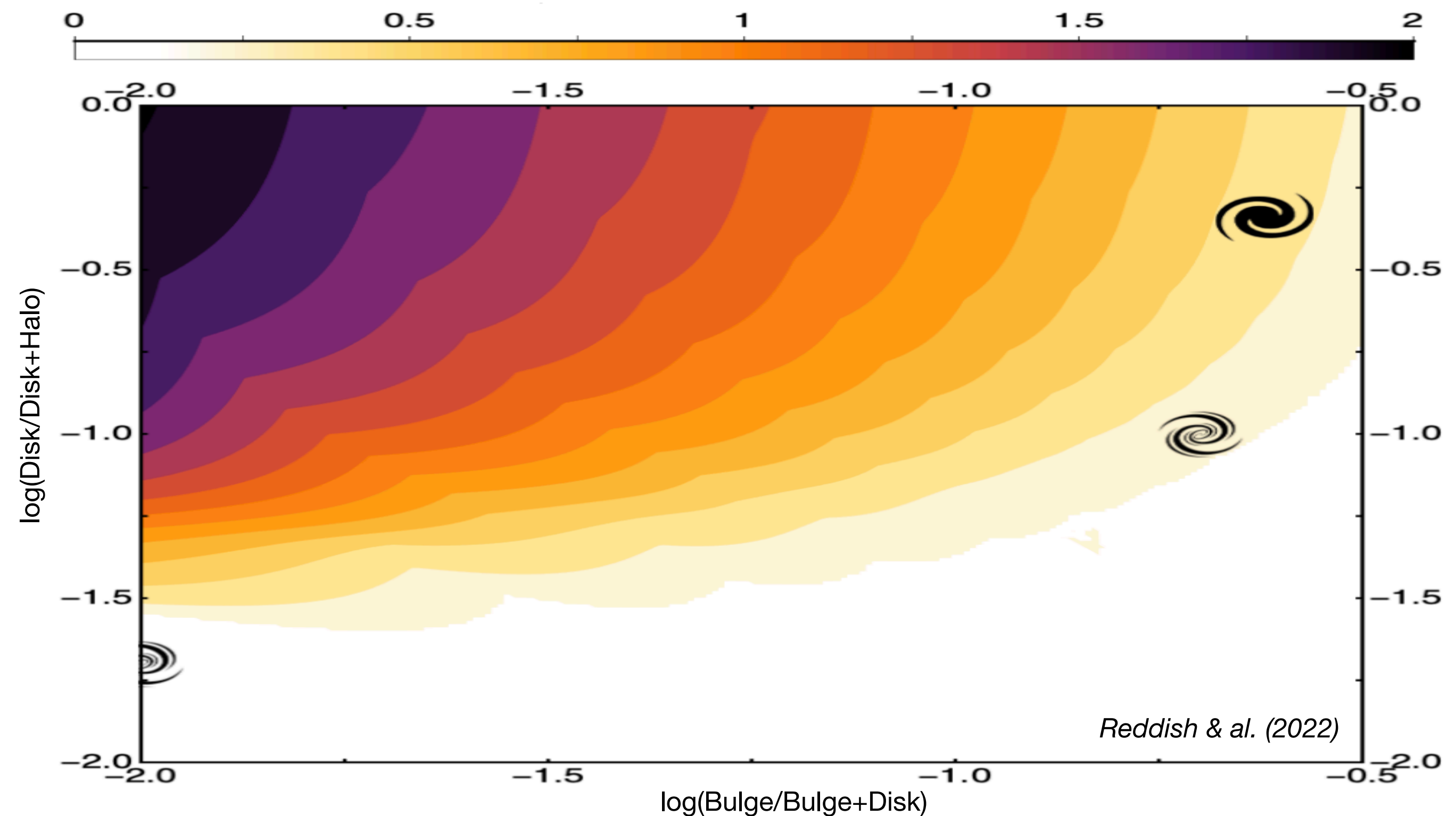


Application to NewHorizon

- Toomre disk of pure gas + bulge + DM halo
- Euler-Poisson equation + linearisation
- Global dispersion relation

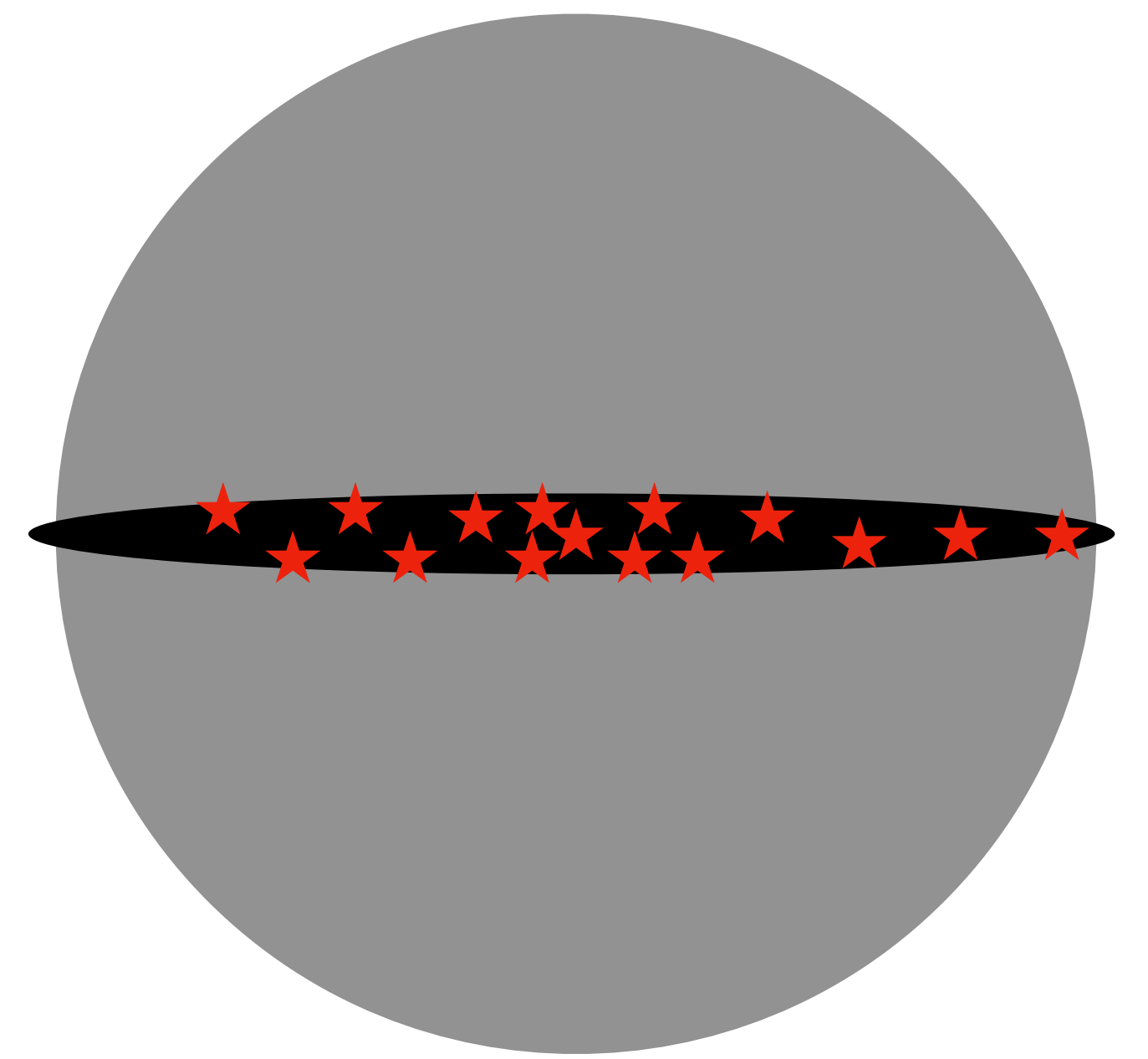


$$\hat{\mathbf{a}} = (\mathbf{P} + \mathbf{L}) \cdot \hat{\mathbf{a}} + \mathbf{L} \cdot \hat{\mathbf{b}},$$



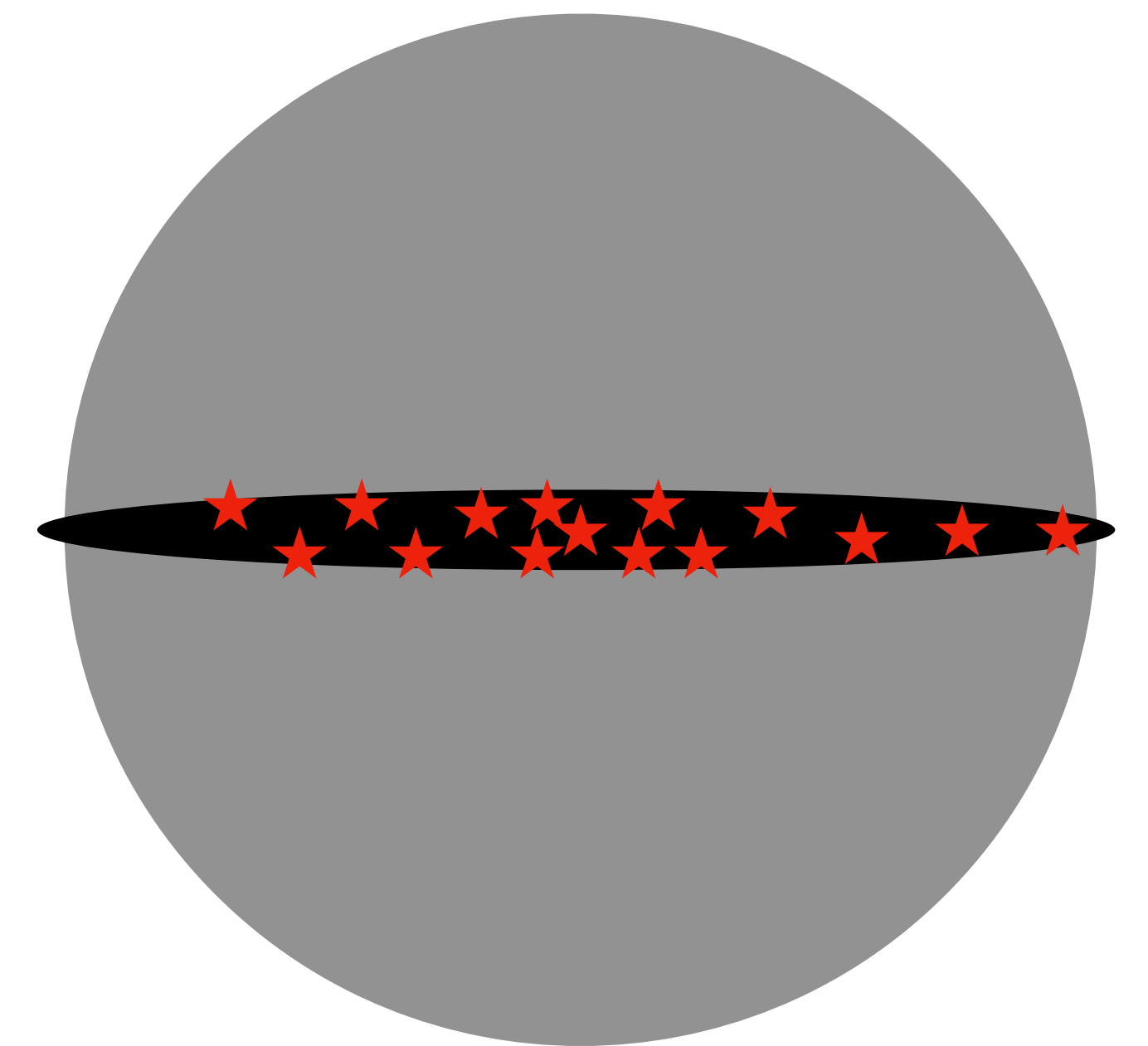
Coupled gas+stars

- Gas + stars + DM halo (external)



Coupled gas+stars

- Gas + stars + DM halo (external)



$$\widehat{\mathbf{a}} = (\mathbf{P} + \mathbf{L}) \cdot \widehat{\mathbf{a}}$$

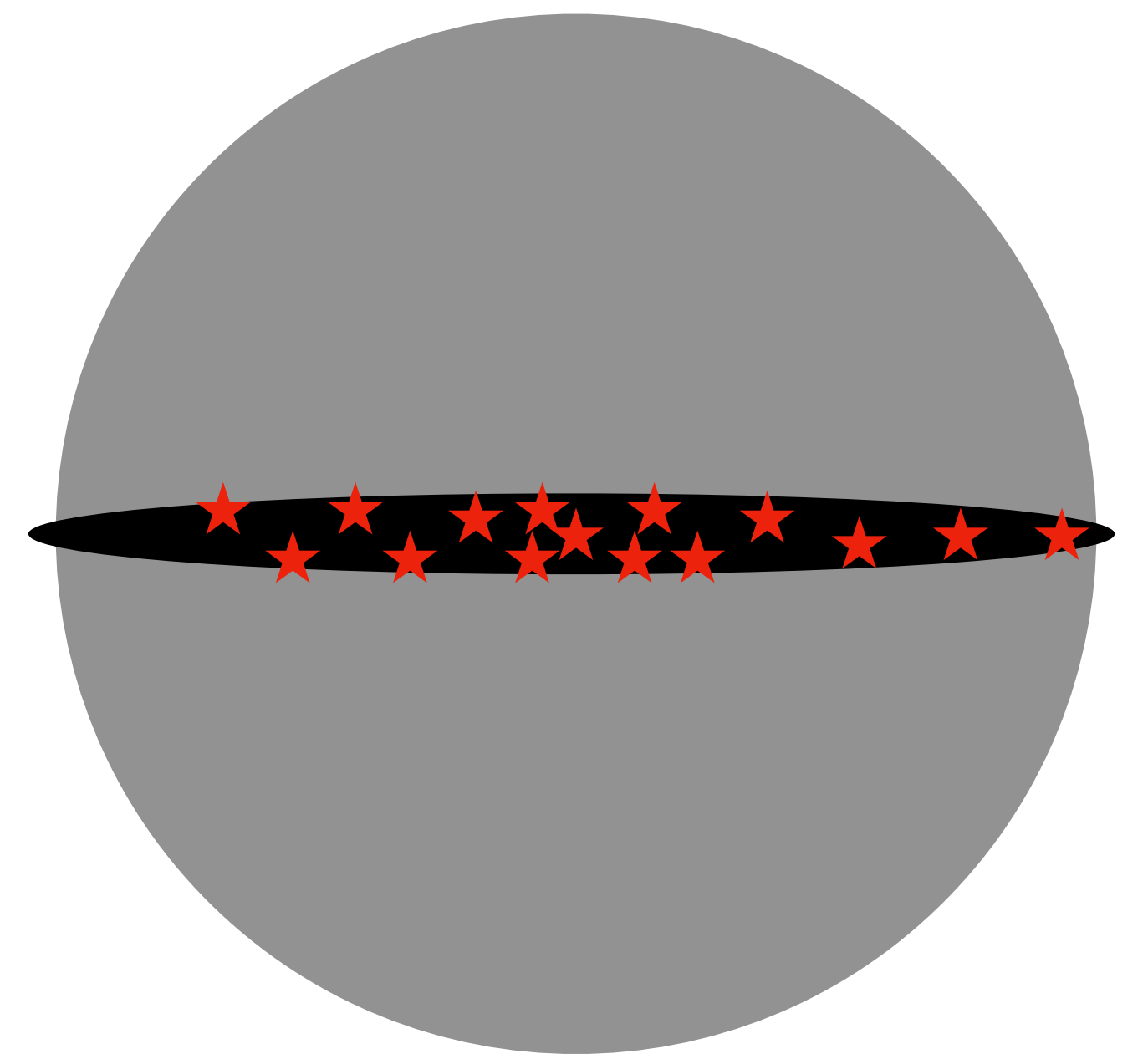
$$\widehat{\mathbf{b}} = \mathbf{H} \cdot \widehat{\mathbf{b}}.$$

$$\Sigma_{\text{gas}}(R, \phi) = \text{Re} \left[\sum_n \widehat{a}_n \mathcal{D}_n^m(R) e^{im\phi} \right],$$

$$\Sigma_{\text{stars}}(R, \phi) = \text{Re} \left[\sum_n \widehat{b}_n \mathcal{D}_n^m(R) e^{im\phi} \right].$$

Coupled gas+stars

- Gas + stars + DM halo (external)
- Coupling via gravity of respective disks



$$\widehat{\mathbf{a}} = (\mathbf{P} + \mathbf{L}) \cdot \widehat{\mathbf{a}} + \mathbf{L} \cdot \widehat{\mathbf{b}},$$

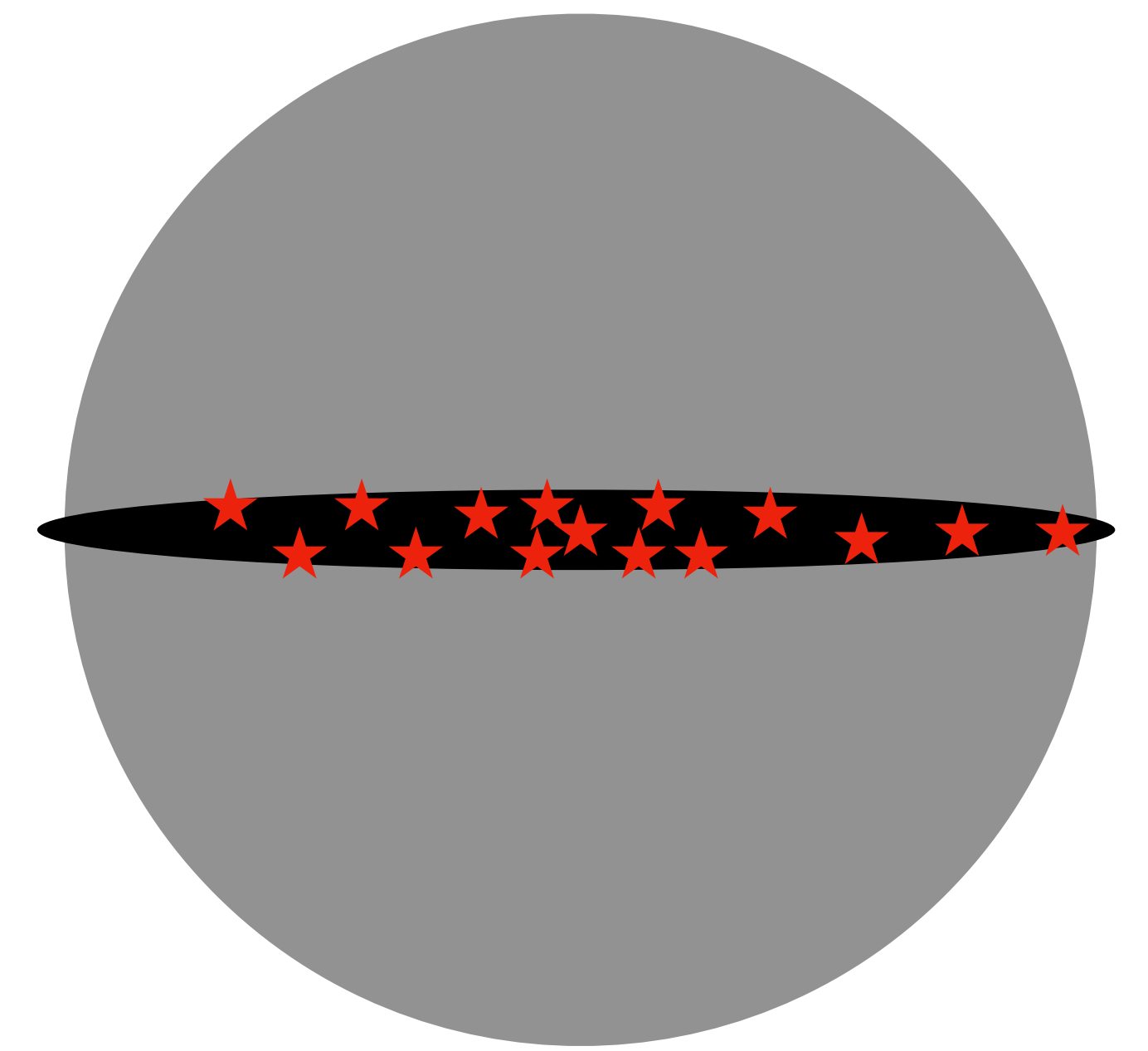
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Coupled gas+stars

- Gas + stars + DM halo (external)
- Coupling via gravity of respective disks
- Global dispersion relation



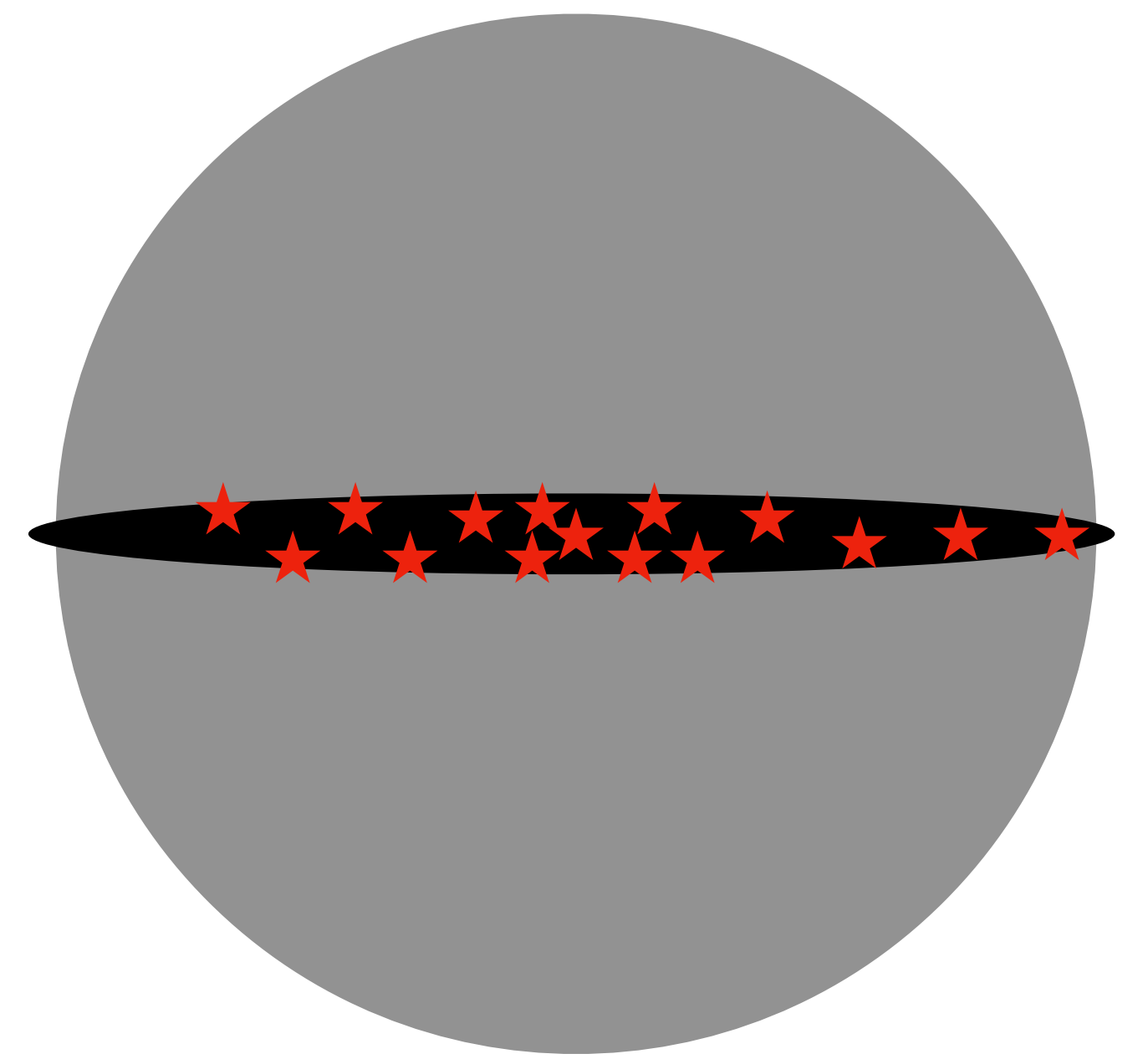
$$\begin{aligned}\widehat{\mathbf{a}} &= (\mathbf{P} + \mathbf{L}) \cdot \widehat{\mathbf{a}} + \mathbf{L} \cdot \widehat{\mathbf{b}}, \\ \widehat{\mathbf{b}} &= \mathbf{H} \cdot \widehat{\mathbf{a}} + \mathbf{H} \cdot \widehat{\mathbf{b}}.\end{aligned}$$

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$$\begin{bmatrix} \widehat{\mathbf{a}} \\ \widehat{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \mathbf{P} + \mathbf{L} & \mathbf{L} \\ \mathbf{H} & \mathbf{H} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{a}} \\ \widehat{\mathbf{b}} \end{bmatrix} = \mathbf{M} \begin{bmatrix} \widehat{\mathbf{a}} \\ \widehat{\mathbf{b}} \end{bmatrix}.$$

Coupled gas+stars

- Gas + stars + DM halo (external)
- Coupling via gravity of respective disks
- Global dispersion relation



$$\begin{aligned}\widehat{\mathbf{a}} &= (\mathbf{P} + \mathbf{L}) \cdot \widehat{\mathbf{a}} + \mathbf{L} \cdot \widehat{\mathbf{b}}, \\ \widehat{\mathbf{b}} &= \mathbf{H} \cdot \widehat{\mathbf{a}} + \mathbf{H} \cdot \widehat{\mathbf{b}}.\end{aligned}$$

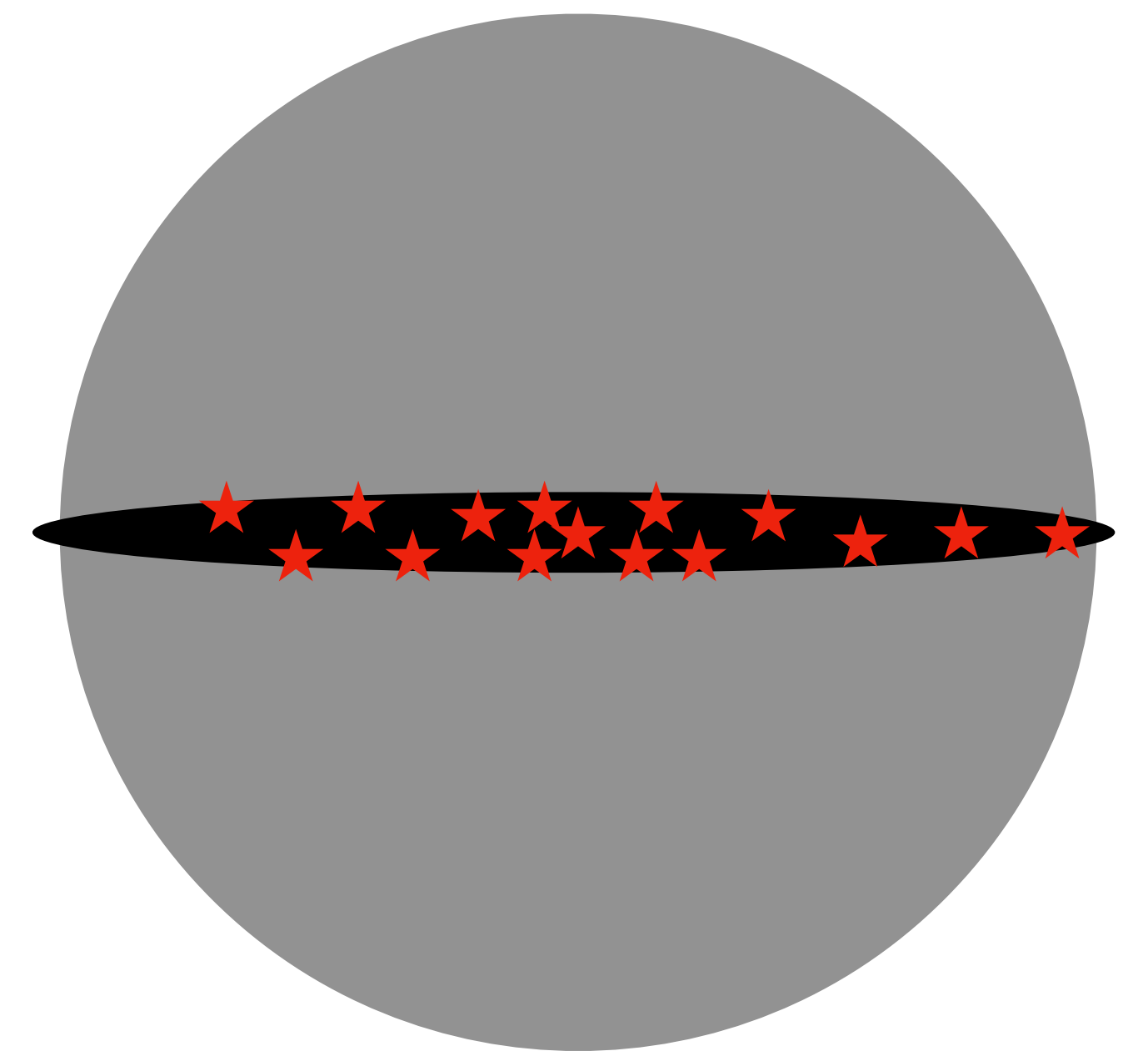
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$$\det[\mathbf{I} - \mathbf{M}(\omega)] = \det \begin{bmatrix} \mathbf{I} - \mathbf{P}(\omega) - \mathbf{L}(\omega) & -\mathbf{L}(\omega) \\ -\mathbf{H}(\omega) & \mathbf{I} - \mathbf{H}(\omega) \end{bmatrix} = 0.$$

Coupled gas+stars

- Gas + stars + DM halo (external)
- Coupling via gravity of respective disks
- Global dispersion relation



$$\begin{aligned}\widehat{\mathbf{a}} &= (\mathbf{P} + \mathbf{L}) \cdot \widehat{\mathbf{a}} + \mathbf{L} \cdot \widehat{\mathbf{b}}, \\ \widehat{\mathbf{b}} &= \mathbf{H} \cdot \widehat{\mathbf{a}} + \mathbf{H} \cdot \widehat{\mathbf{b}}.\end{aligned}$$

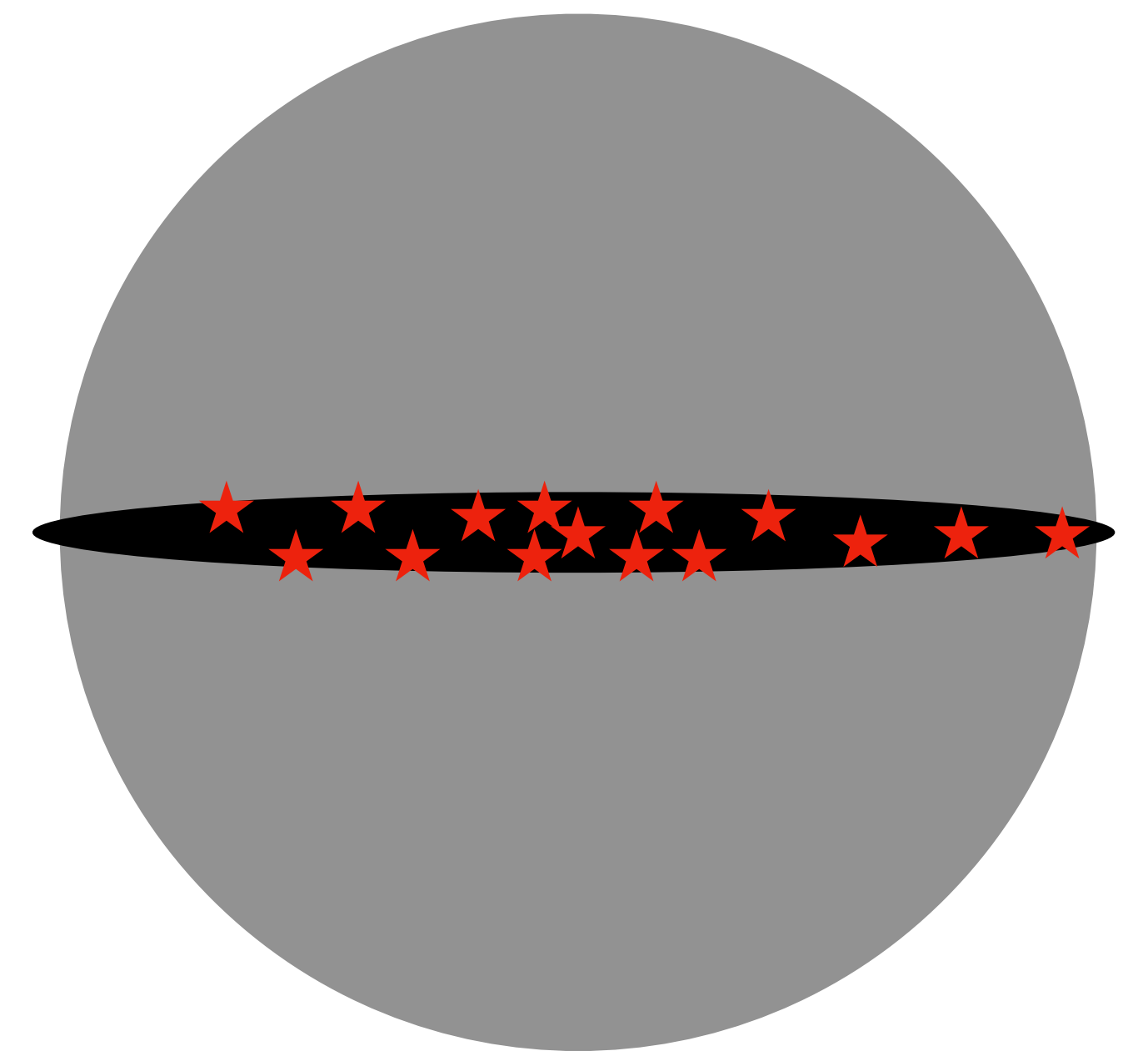
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$$\det[\mathbf{I} - \mathbf{M}] = \det \left[(\mathbf{I} - \mathbf{M}_g)(\mathbf{I} - \mathbf{M}_\star) - \mathbf{L}_g \mathbf{M}_\star \right] = 0.$$

Coupled gas+stars

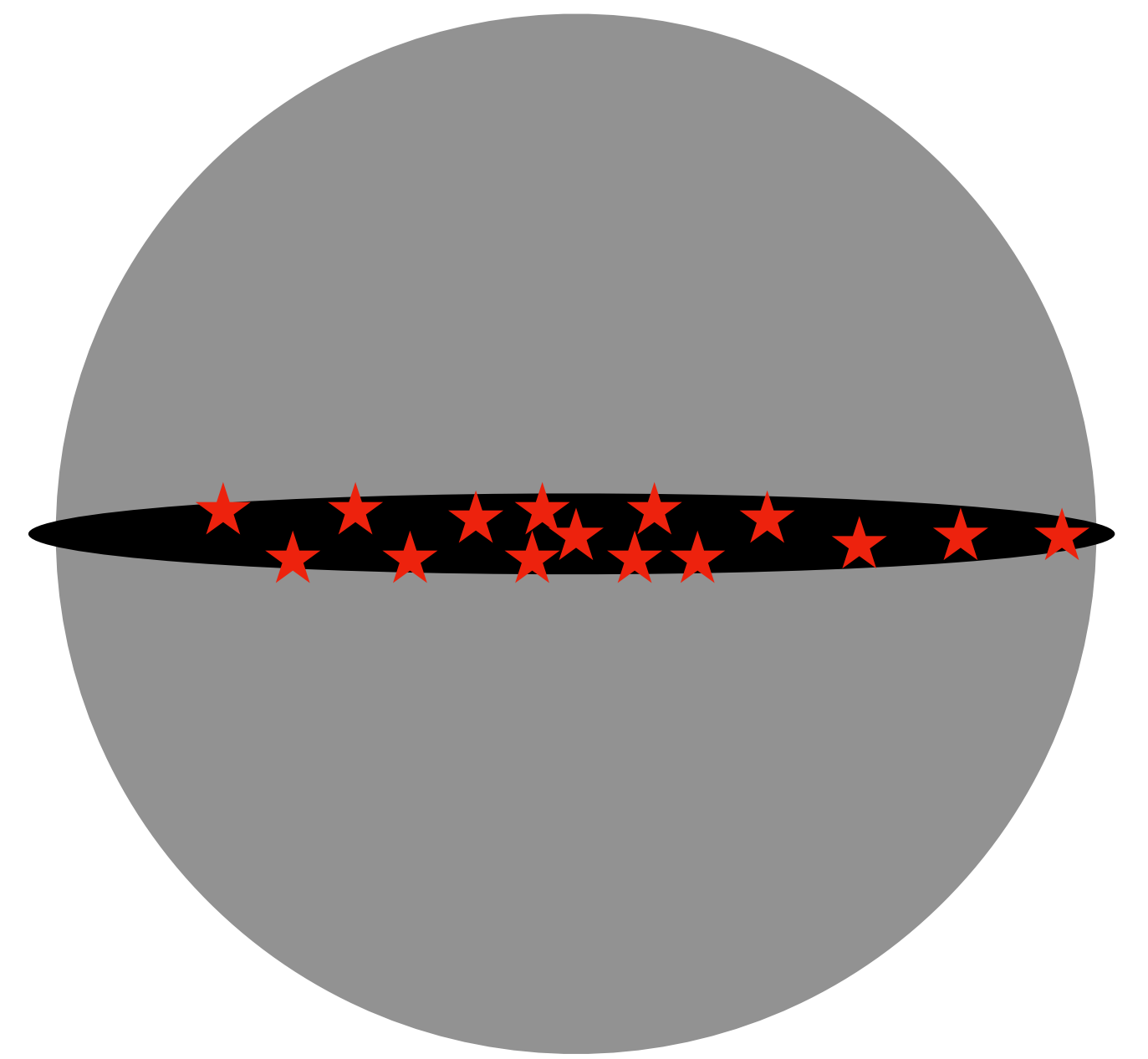
- Gas + stars + DM halo (external)
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Coupled gas+stars

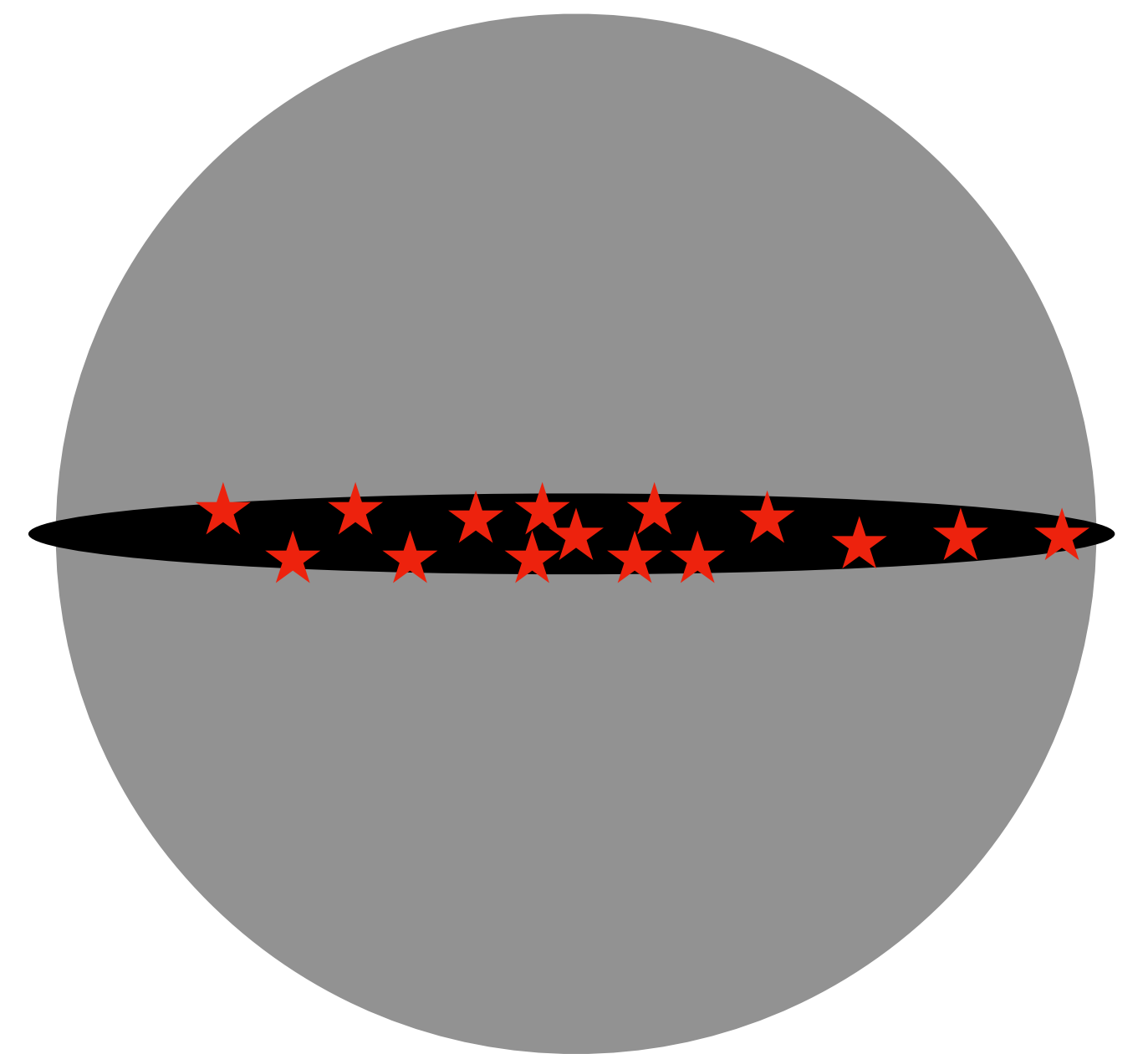
- Gas + stars + DM halo (external)
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- Global dispersion relation



$$\det[\mathbf{I} - \mathbf{M}] = \det [\mathbf{I} - (\mathbf{M}_g + \mathbf{M}_\star) - \mathbf{P}_g \mathbf{M}_\star] = 0$$

Coupled gas+stars

- Gas + stars + DM halo (external)
- Coupling via gravity of respective disks
- Global dispersion relation



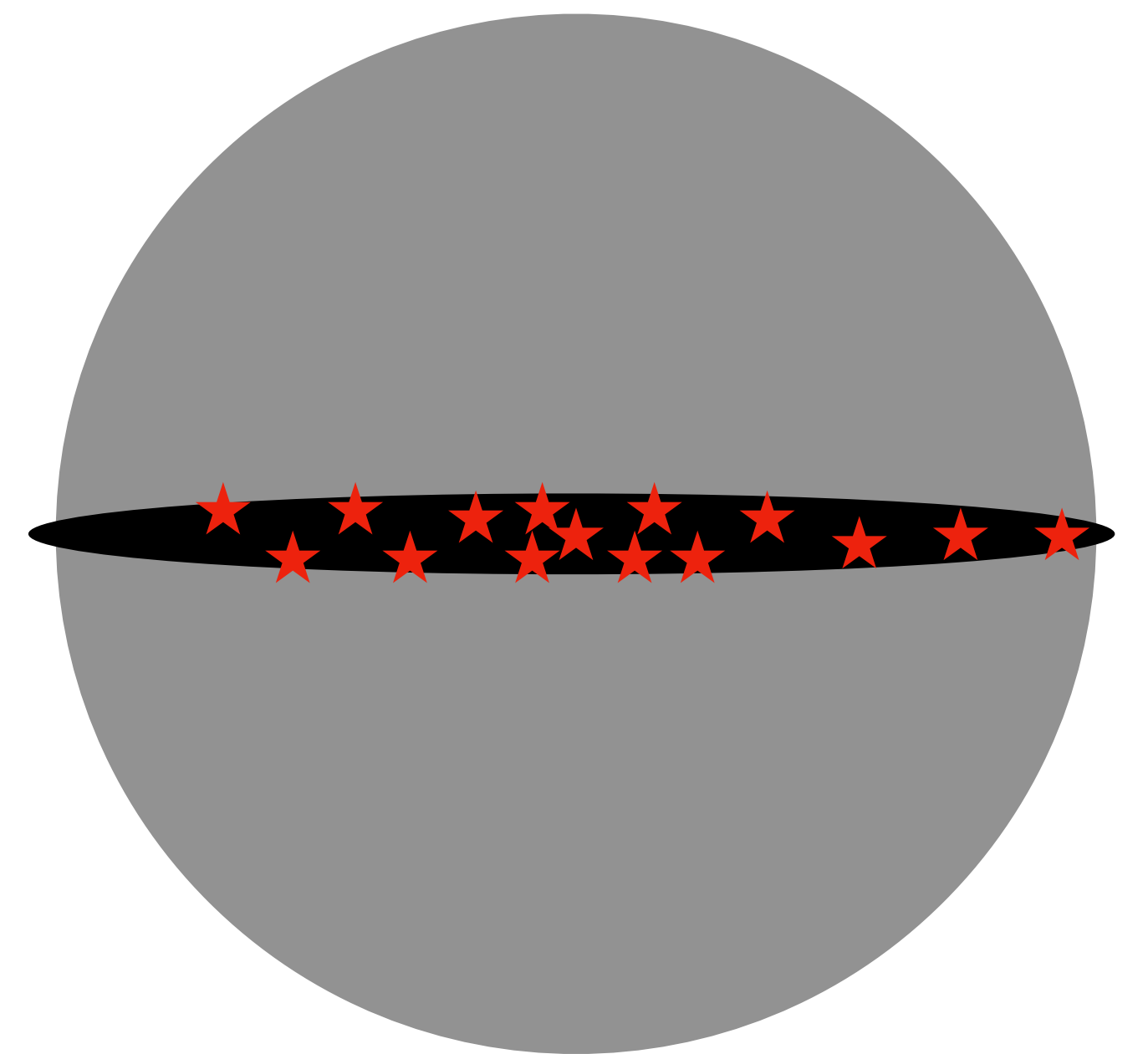
$$\det[\mathbf{I} - \mathbf{M}] = \det [\mathbf{I} - (\mathbf{M}_g + \mathbf{M}_\star) - \mathbf{P}_g \mathbf{M}_\star] = 0$$

Two stellar components:

$$\det[\mathbf{I} - \mathbf{M}] = \det [\mathbf{I} - (\mathbf{M}_1 + \mathbf{M}_2)] = 0$$

Coupled gas+stars

- Gas + stars + DM halo (external)
- Coupling via gravity of respective disks
- Global dispersion relation



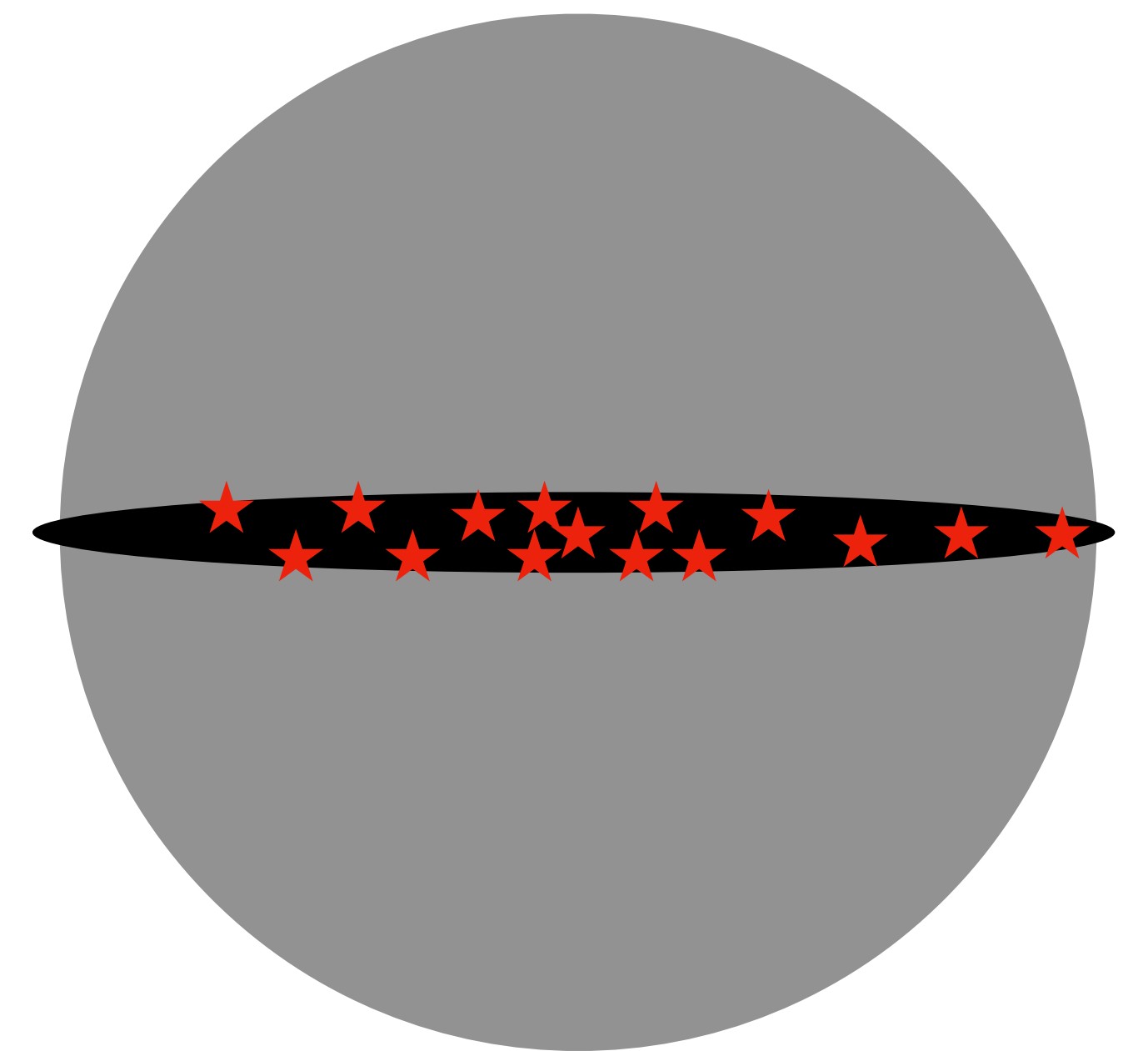
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Two stellar components:

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Coupled gas+stars

- Gas + stars + DM halo (external)
- Coupling via gravity of respective disks
- Global dispersion relation

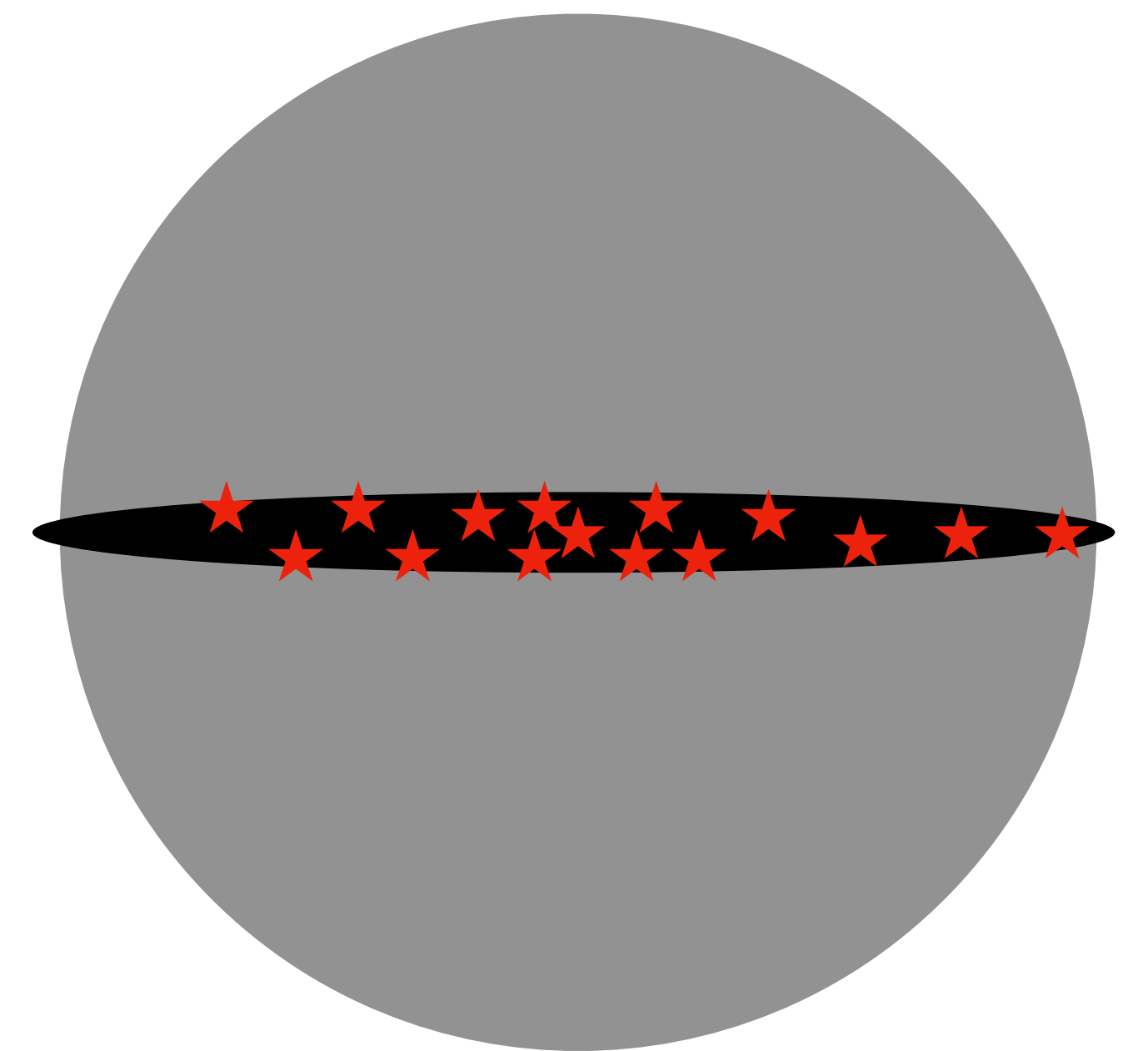


$$\det[\mathbf{I} - \mathbf{M}] = \det [\mathbf{I} - (\mathbf{M}_g + \mathbf{M}_\star) - \mathbf{P}_g \mathbf{M}_\star] = 0$$

Tightly wound + cool disk limit: reduction to the local dispersion relation

Coupled gas+stars

- Gas + stars + DM halo (external)
- Coupling via gravity of respective disks
- Global dispersion relation



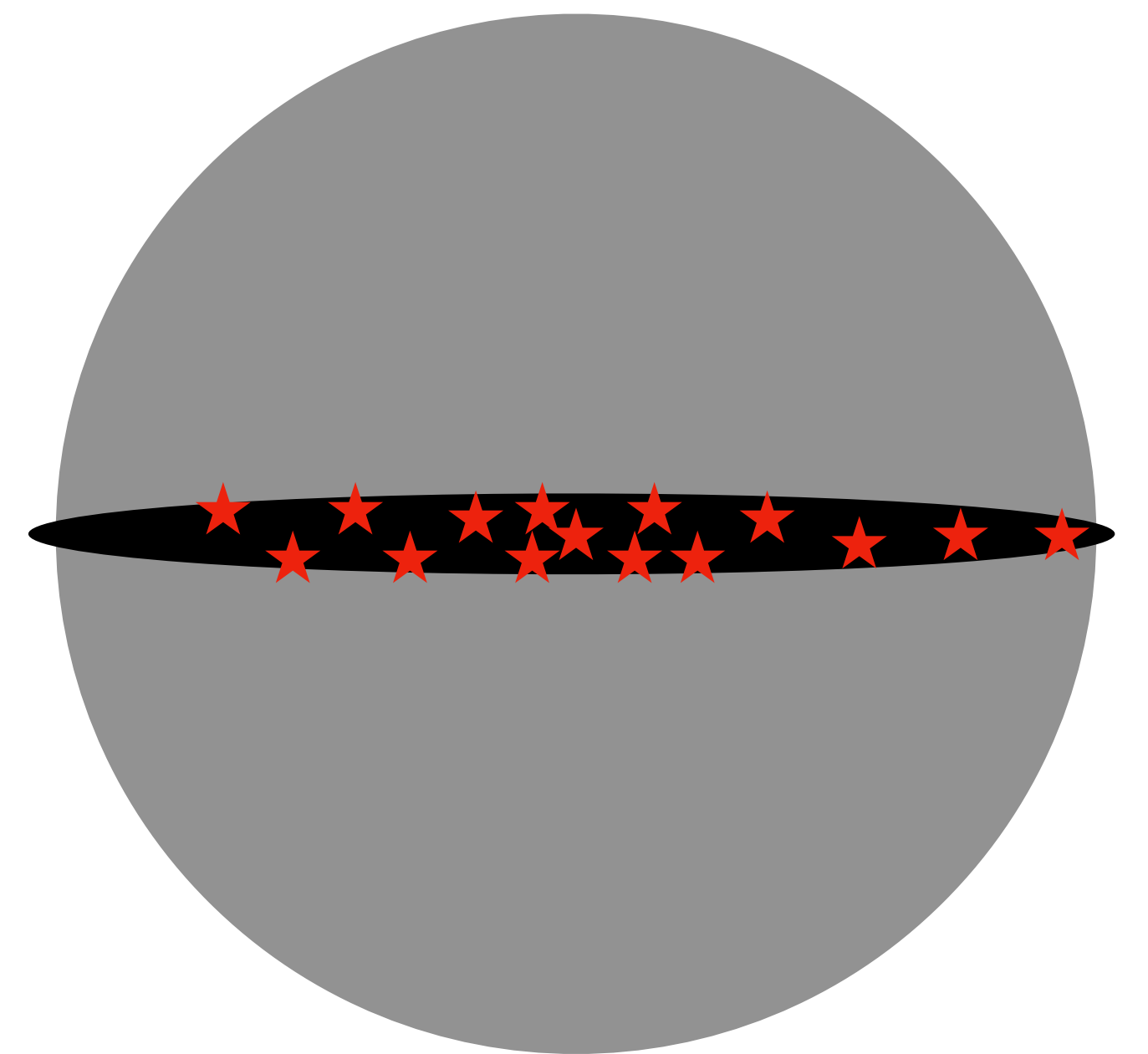
$$\det[\mathbf{I} - \mathbf{M}] = \det [\mathbf{I} - (\mathbf{M}_g + \mathbf{M}_\star) - \mathbf{P}_g \mathbf{M}_\star] = 0$$

Tightly wound + cool disk limit: reduction to the local dispersion relation

$$\frac{2\pi G|k|\Sigma_g}{\kappa_\star^2 - \bar{\omega}_\star^2 + c_\star^2 k^2} + \frac{2\pi G|k|\Sigma_\star \mathcal{F}}{\kappa_\star^2 - \bar{\omega}_\star^2} = 1. \quad \text{Rafikov (2001)}$$

Coupled gas+stars

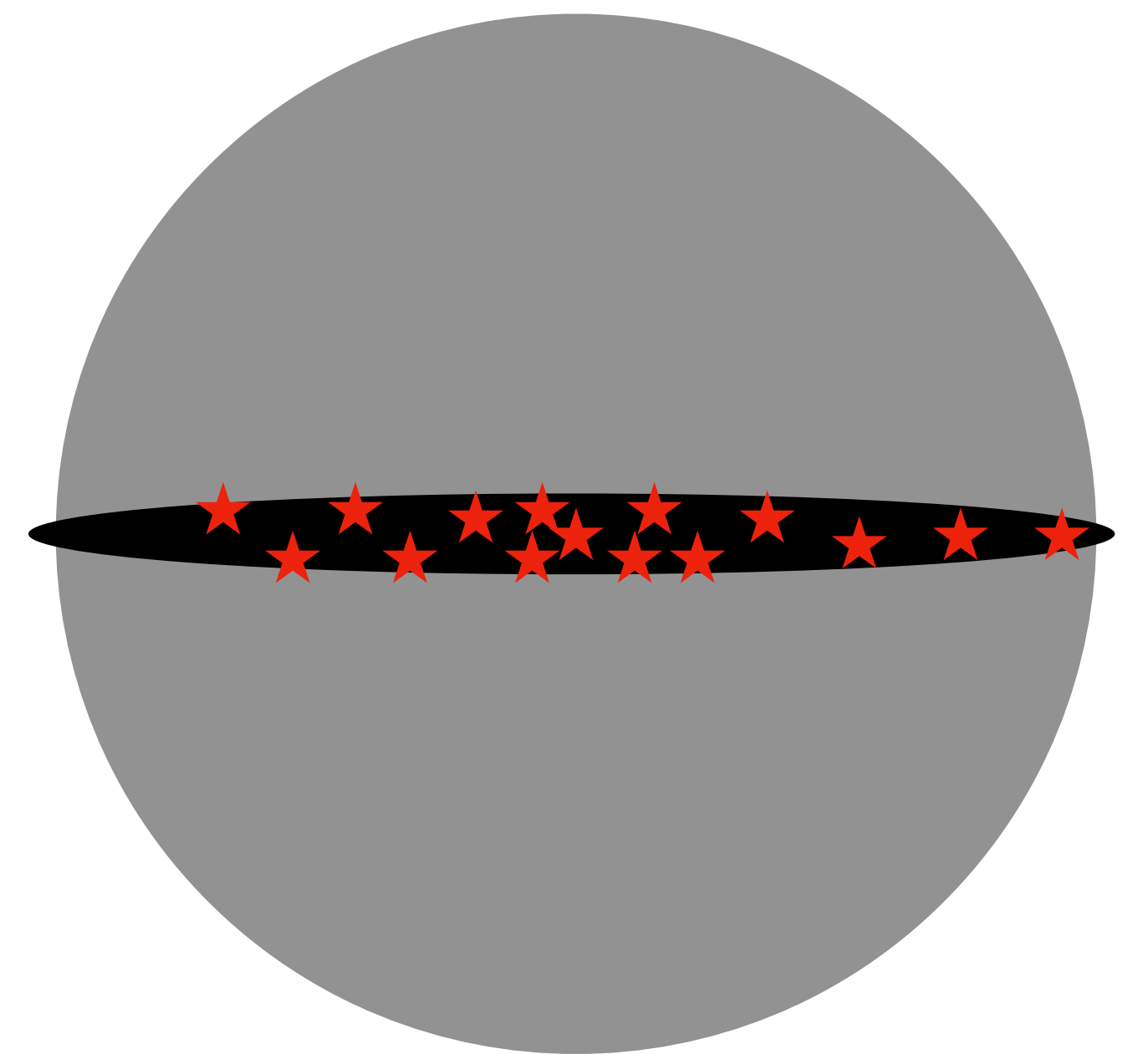
- Gas + stars + DM halo (external)
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- Global dispersion relation



Parameter exploration : how do each disk component impact the global stability ?

Coupled gas+stars

- Gas + stars + DM halo (external)
- Coupling via gravity of respective disks
- Global dispersion relation



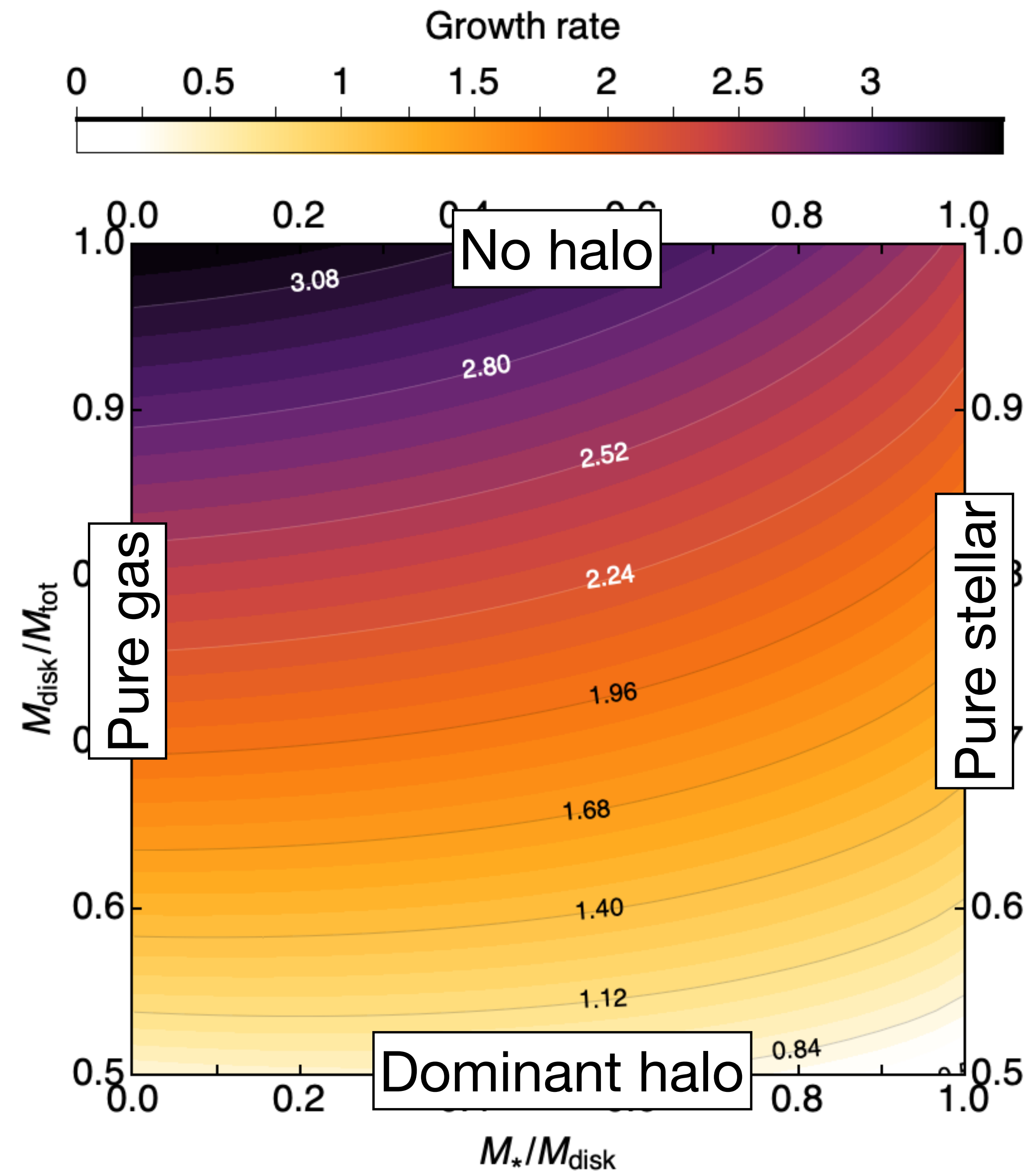
Parameter exploration : how do each disk component impact the global stability ?

-> Evolution of the location of the dominant mode

Coupled gas+stars

- Gas + stars + DM halo (external)
- Coupling via gravity of respective disks
- Global dispersion relation

Parameter space map (slice) :

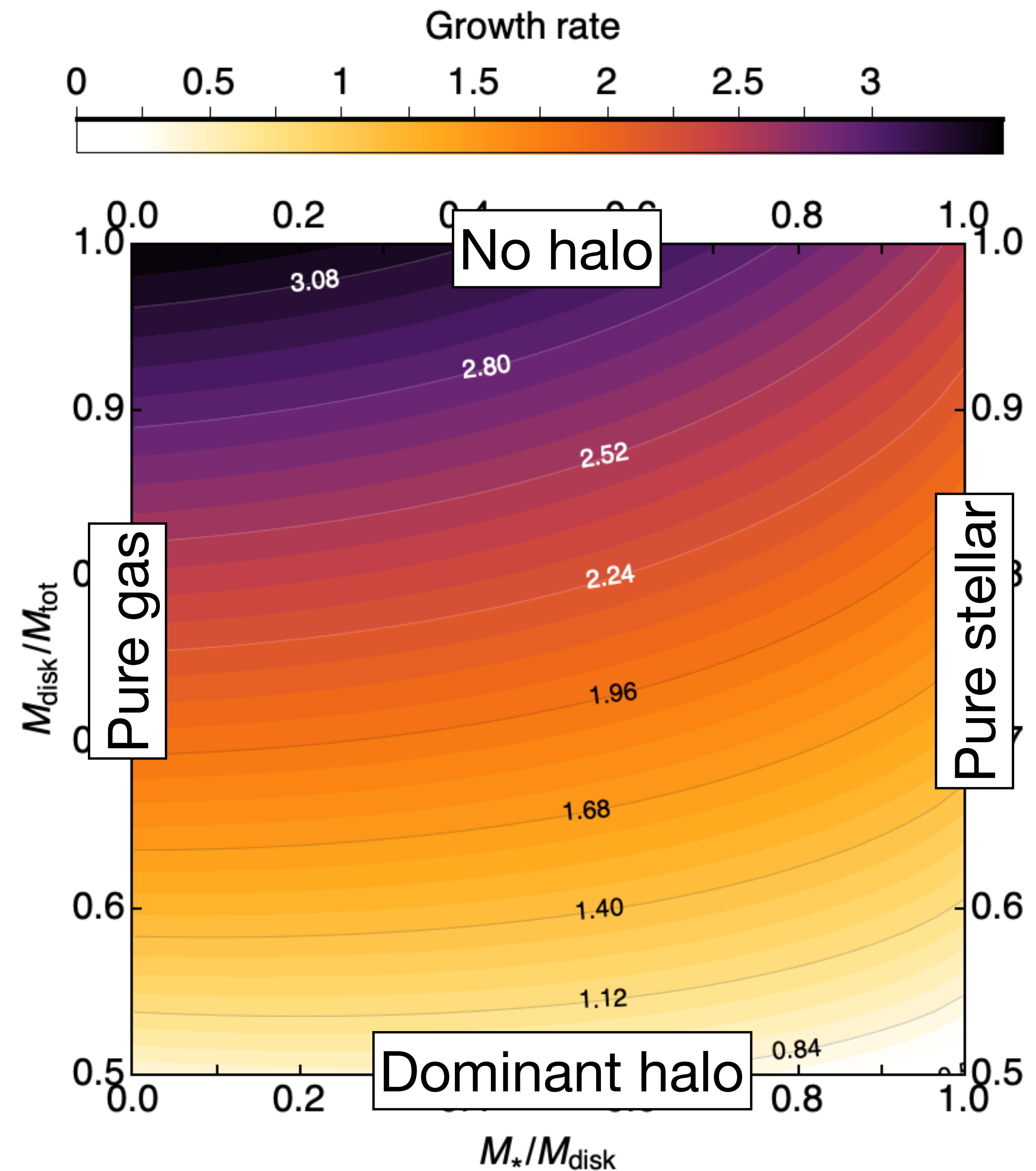


Coupled gas+stars

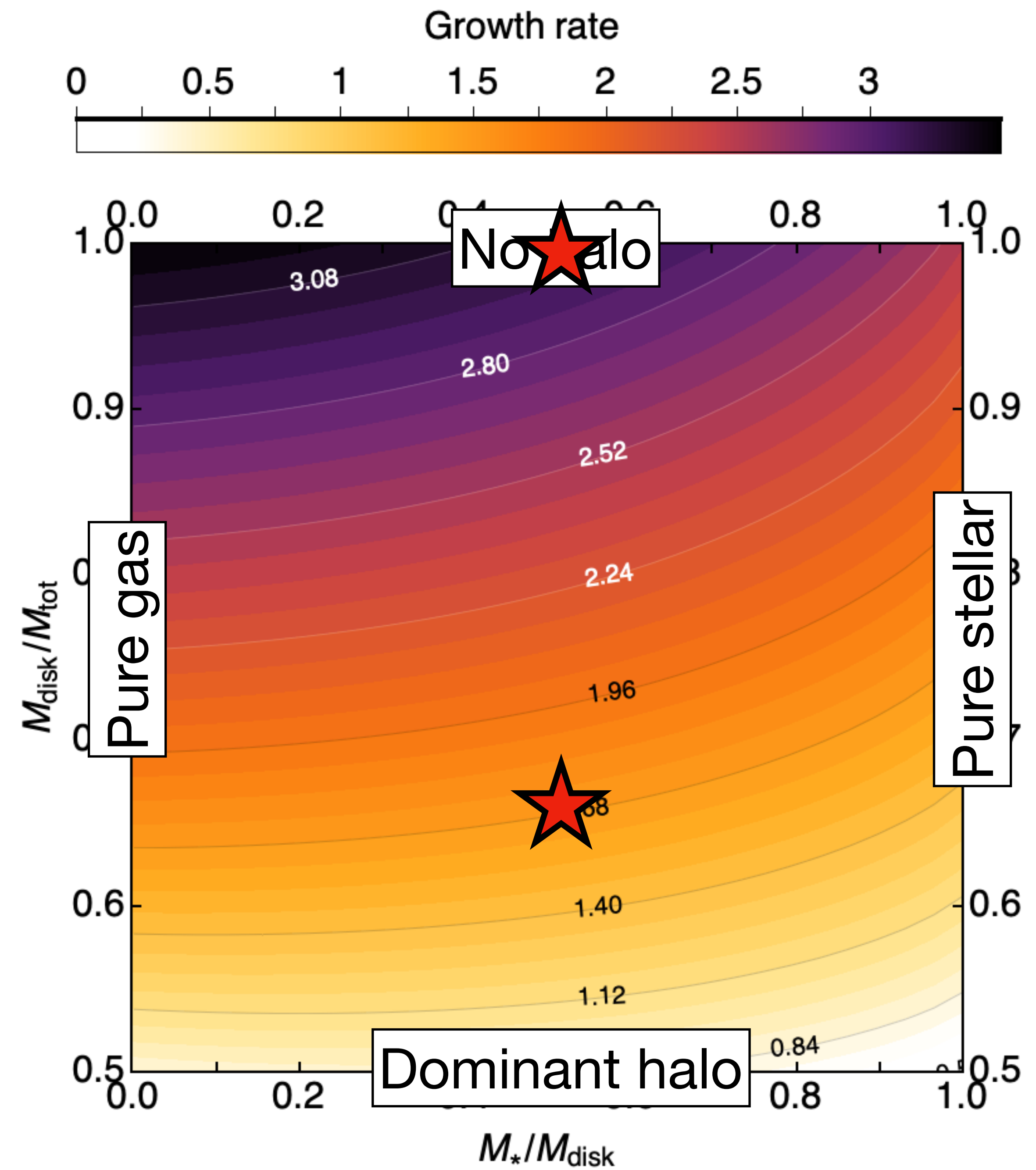
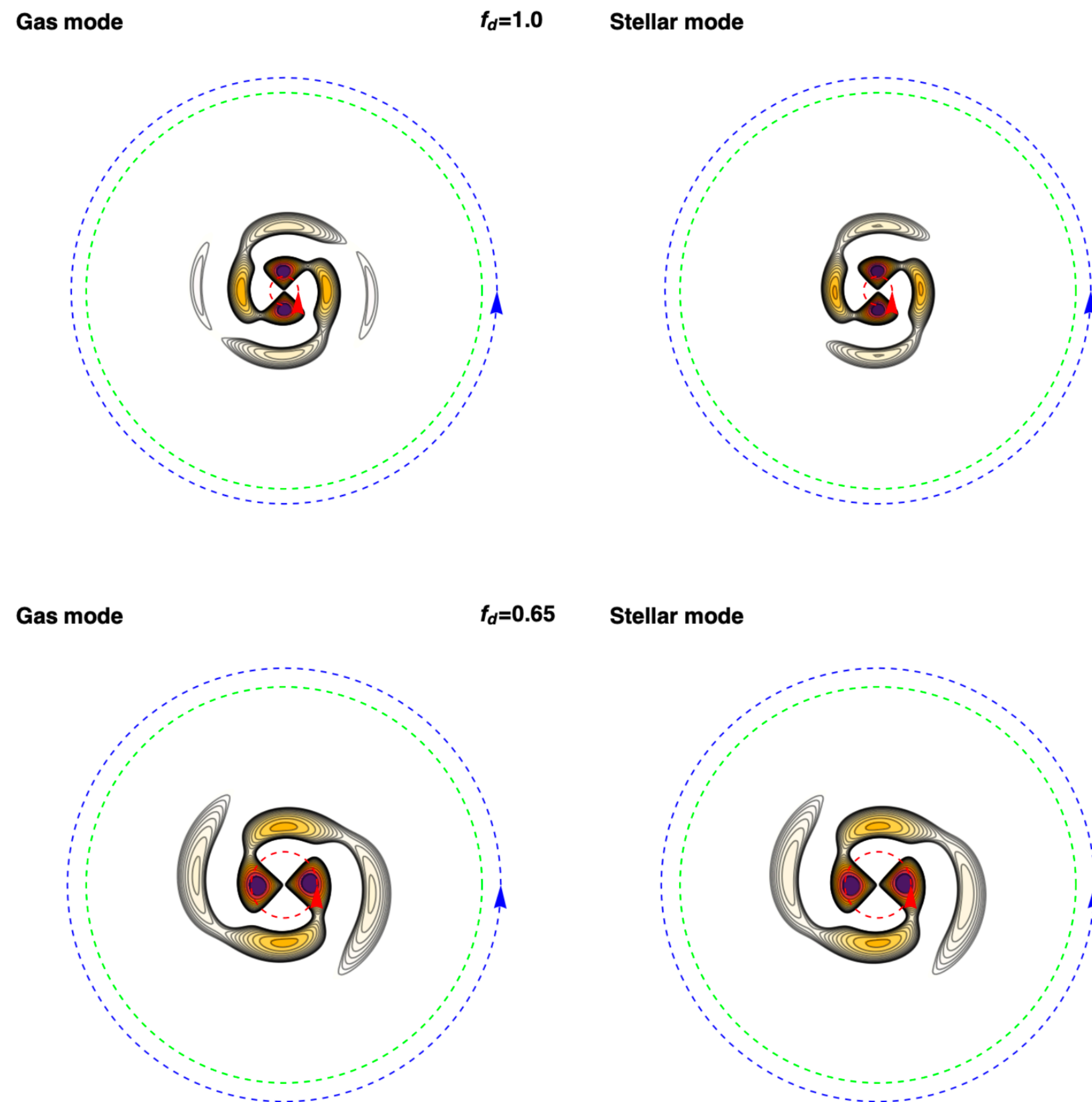
- Gas + stars + DM halo (external)
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- Global dispersion relation

Parameter space map (slice) :

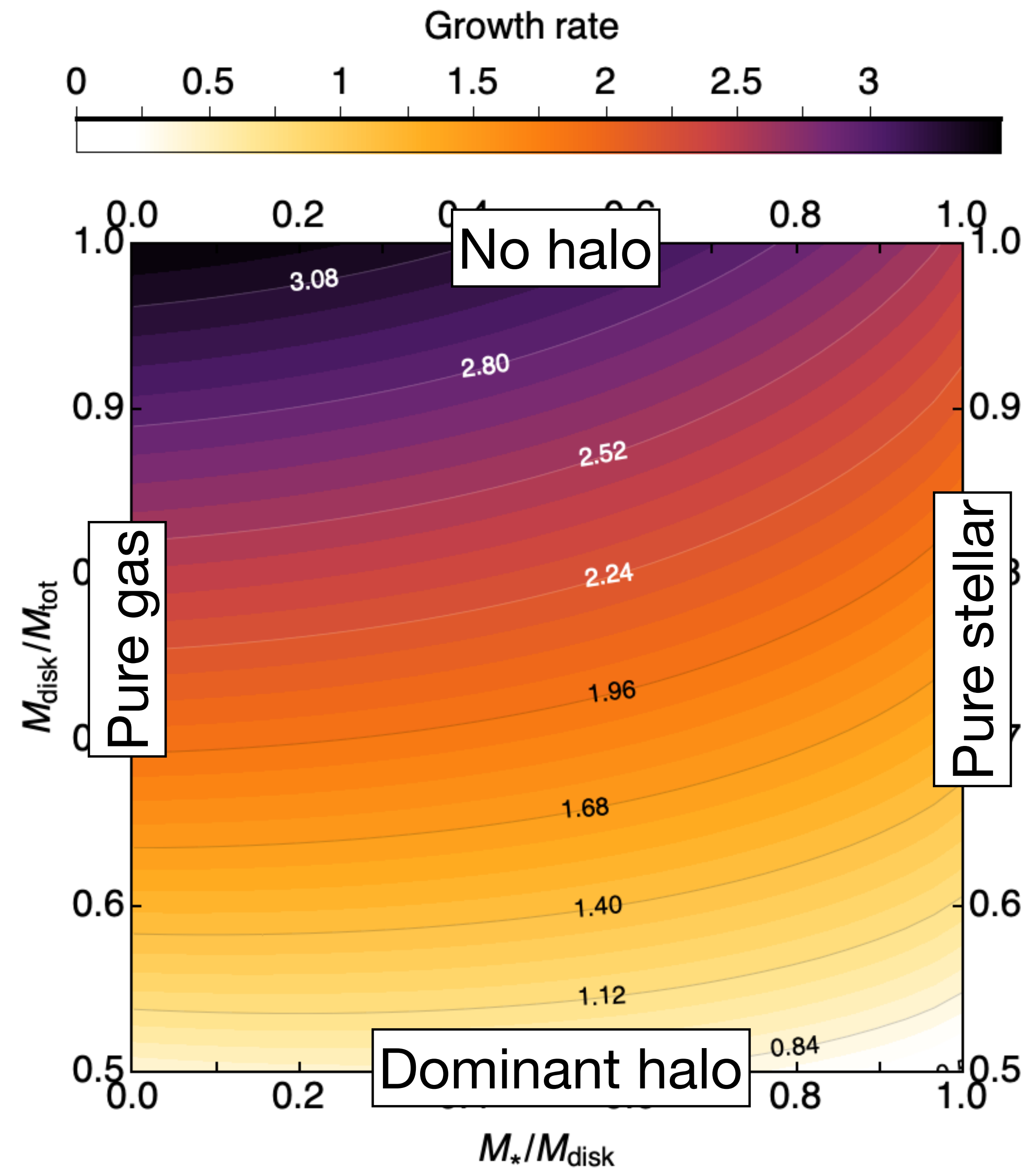
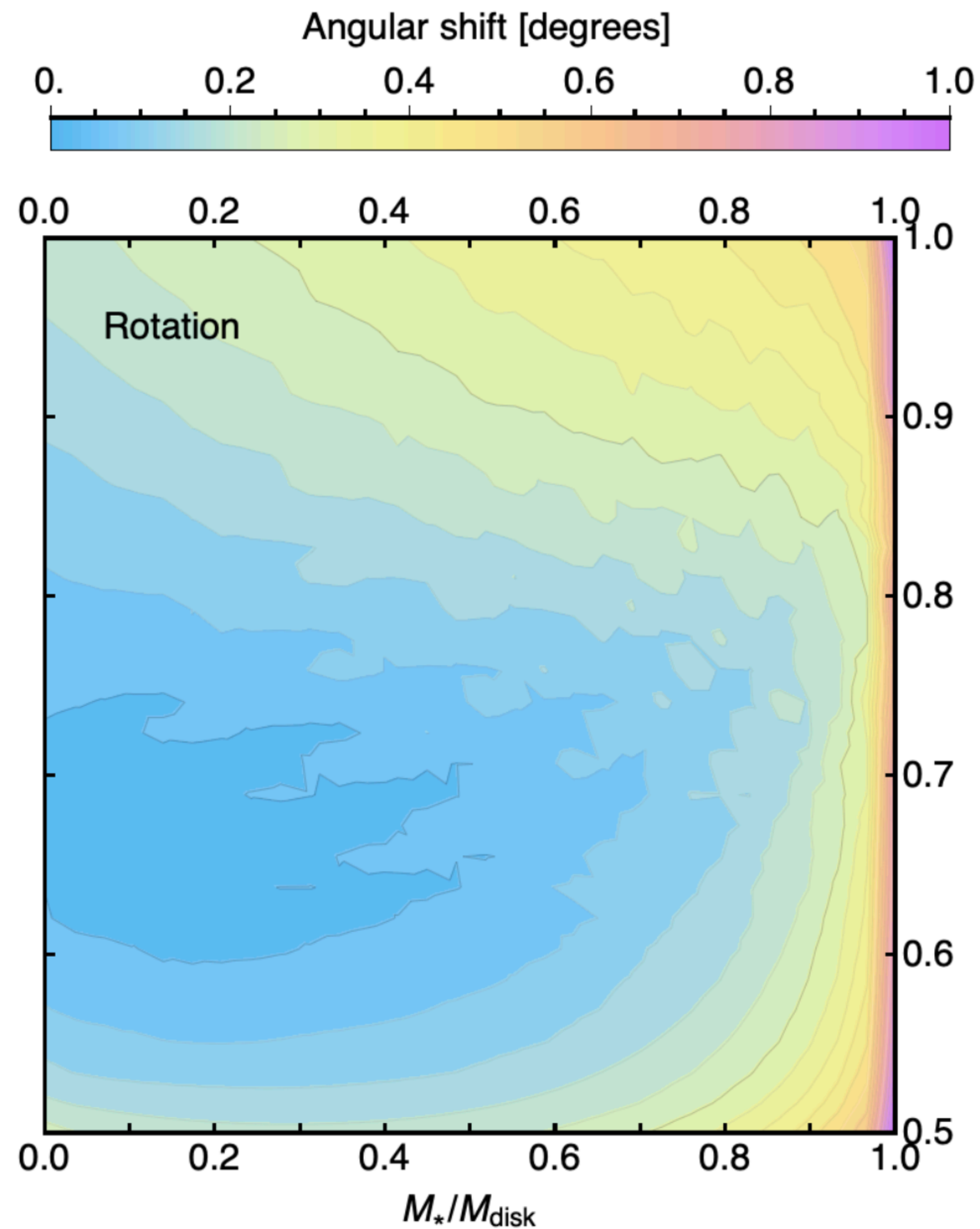
-> Fit galaxies and place it on the
(multidimensional) map !



Coupled gas+stars



Coupled gas+stars

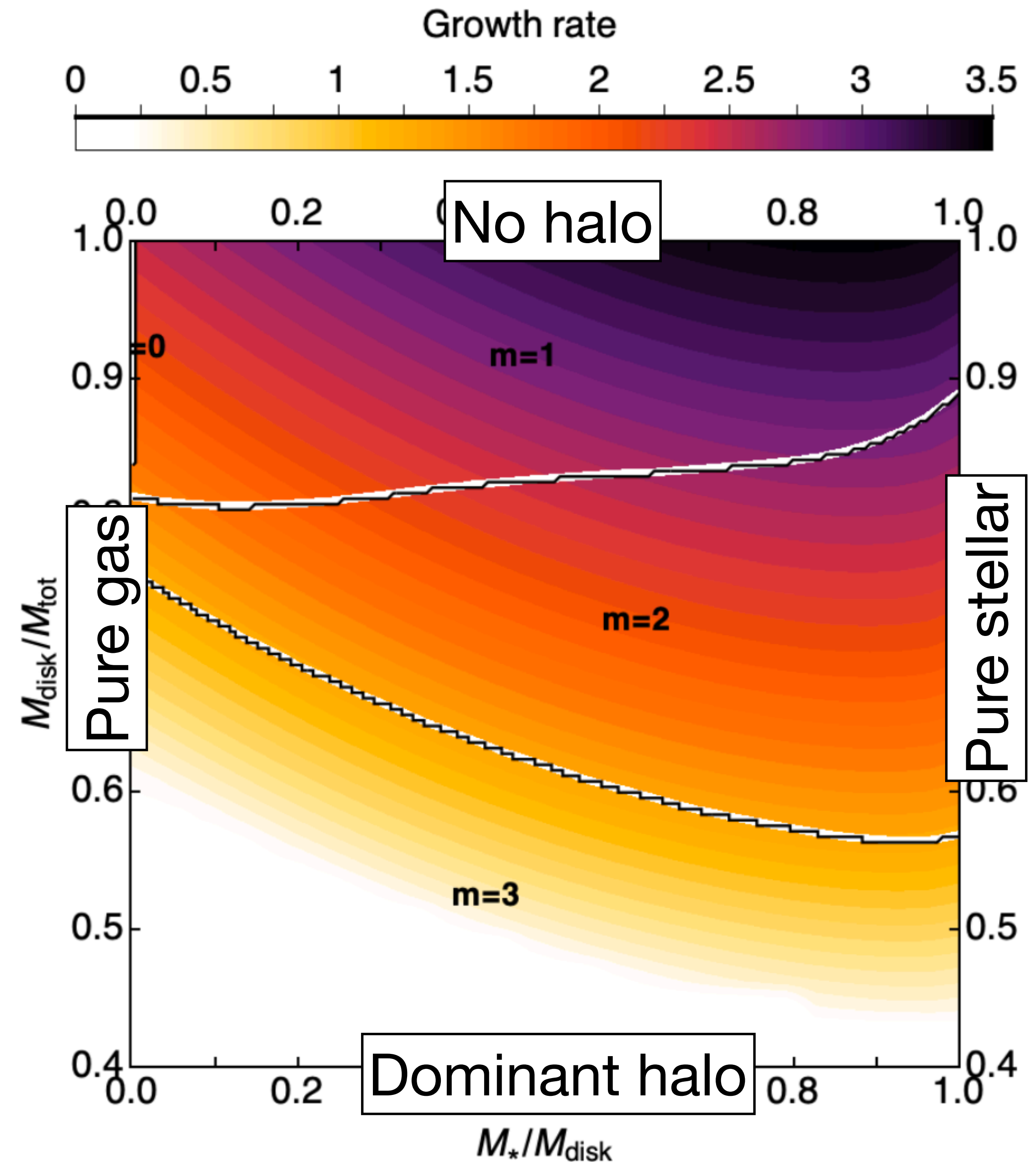


Coupled gas+stars

- Gas + stars + DM halo (external)
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Parameter space map (slice) :

-> Which one is the dominant mode ?



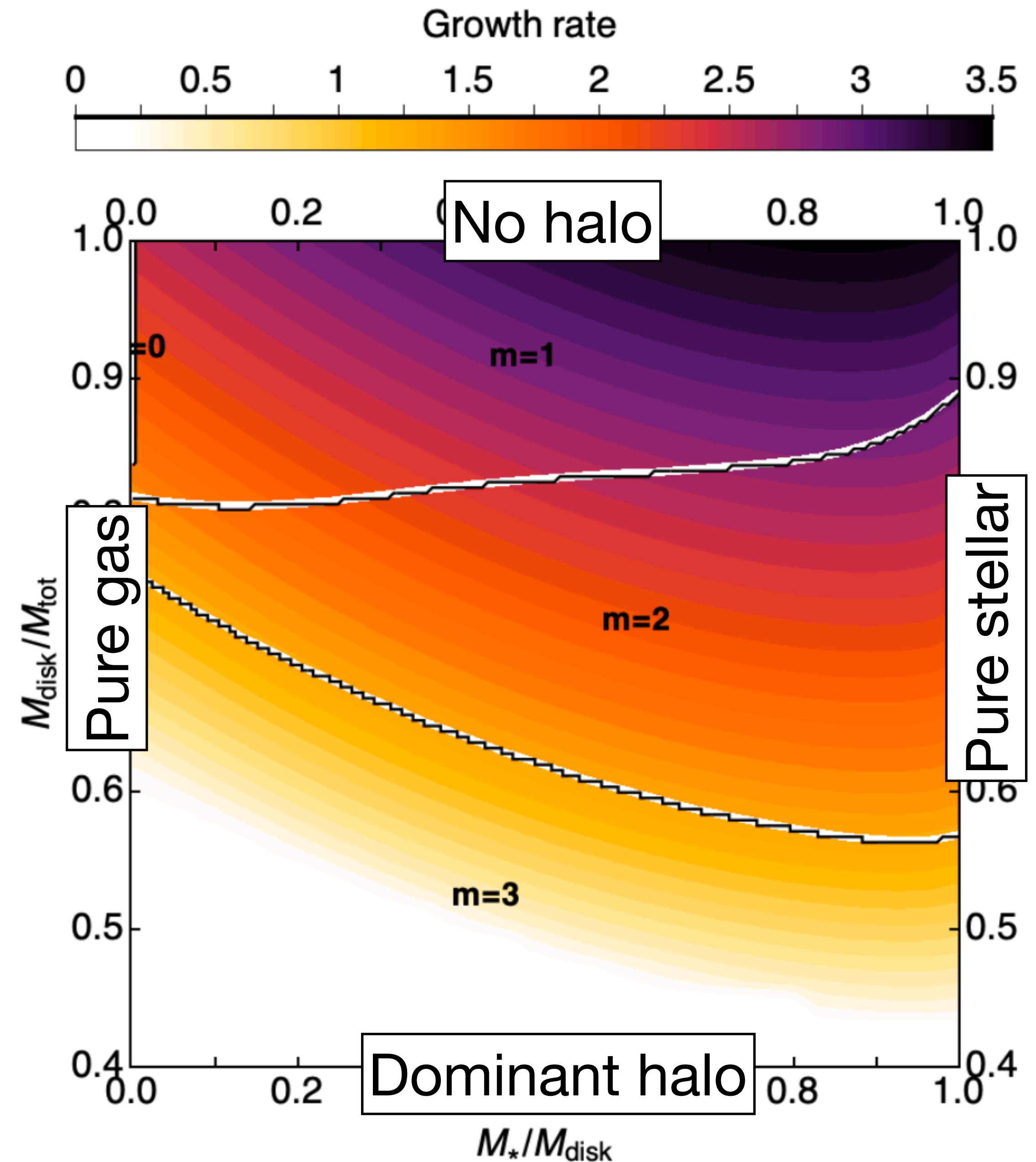
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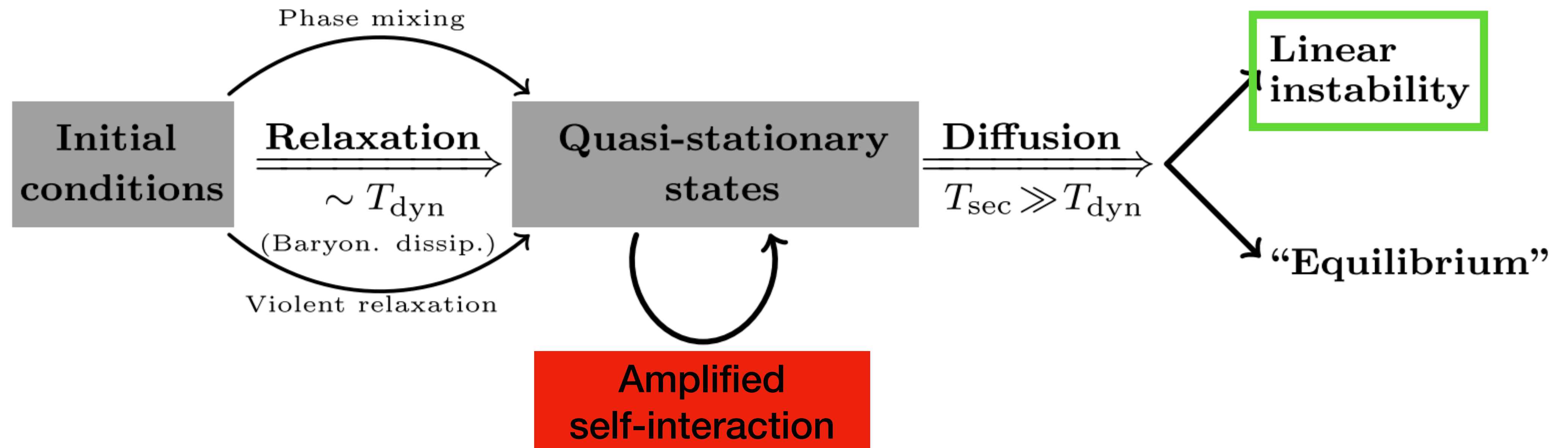
$$\frac{M_{\text{disk}}}{M_{\text{disk}} + M_{\text{halo}}} \propto \frac{1}{m} \quad \text{Dobbs \& Baba (2014)}$$



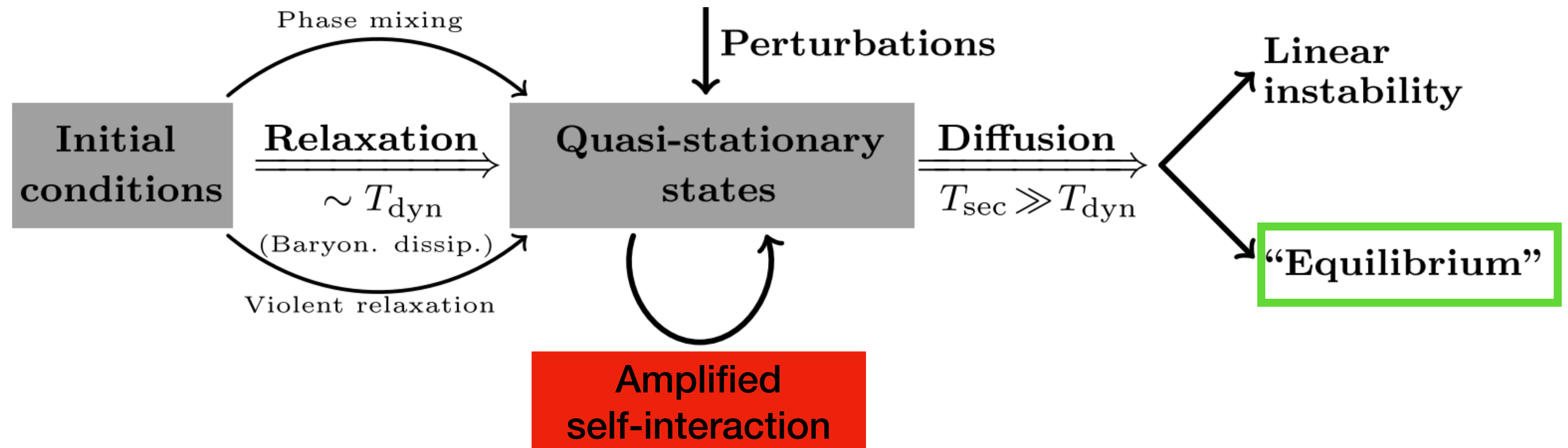
Contents

- 1) Context
- 2) Dynamics of self-gravitating systems
- 3) Linear response theory : the response matrix
- 4) Application
- 5) Future

The coupled time response



The coupled time response



The coupled time response

- Time response to perturbation
 - Zang (1976) and Toomre (1981)
 - Weinberg (1998)
 - Dootson et Magorrian (2022)
 - Rozier et al. (2022) : Coupled star disk/DM
 - Rozier et al. (2024)

The coupled time response

- Time response to perturbation

$$\frac{\partial \delta \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} [R \Sigma_g \delta v_R] + \frac{\Sigma_g}{R} \frac{\partial}{\partial \phi} [\delta v_\phi] + \Omega_g \frac{\partial}{\partial \phi} [\delta \Sigma] = 0,$$

$$\frac{\partial \delta v_R}{\partial t} + \Omega_g \frac{\partial \delta v_R}{\partial \phi} - 2\Omega_g \delta v_\phi = -\frac{\partial W}{\partial R},$$

$$\frac{\partial \delta v_\phi}{\partial t} + \frac{\kappa_g^2}{2\Omega_g} \delta v_R + \Omega_g \frac{\partial \delta v_\phi}{\partial \phi} = -\frac{1}{R} \frac{\partial W}{\partial \phi},$$

The coupled time response

- Time response to perturbation

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$$\mathbf{v}(R, t) = \int_{-\infty}^t dt' e^{-im\Omega_0[R](t-t')} e^{-\mathbf{A}[R](t-t')} \mathbf{S}(R, t')$$

The coupled time response

- Time response to perturbation
- Very small modifications to the equation : Volterra-like system

$$\Sigma(t) = - \int_{-\infty}^t dt' e^{-im\Omega_0(t-t')} \left(\frac{1}{R} \frac{d}{dR} [R\Sigma_0 v_R(R, t')] + \frac{im\Sigma_0}{R} v_\phi(R, t') \right)$$

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Generalization of the gas- and stellar-only analysis

Near and far future

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