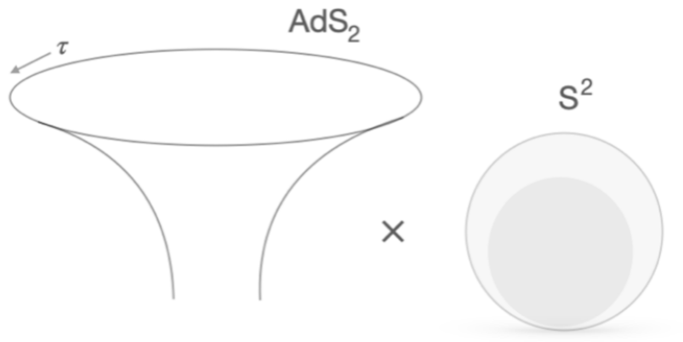
Chiara Toldo | Université Libre de Bruxelles

Near extremal black hole thermodynamics



2026 ENS Summer Institute

Based on D. Kapec, A. Sheta, A. Strominger, C. Toldo, Phys. Rev. Lett. 133 (2024),
M. Blacker, A. Castro, W. Sybesma, C. Toldo JHEP 08 (2025) 120
E. Despontin, S. Detournay, R. Mancilla, work in progress

Extremal black holes

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$$J \leqslant M^2 \qquad\qquad |Q| \leqslant M$$

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Violation of this bound results in a naked singularity.

Extremal black holes saturate the bound. They have zero temperature $T = 0$, while S_{BH} is nonzero. Inner and outer horizons coincide.

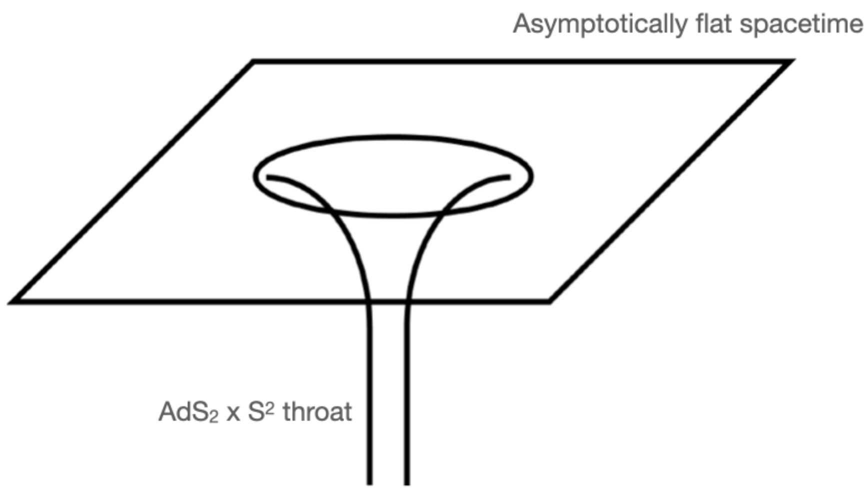
- huge degeneracy of ground states at $T = 0$
- Absence of symmetry protecting it (unless i.e. there is SUSY)

Extremal black holes

Geometry near the horizon develops an AdS_2 throat

e.g. for static black holes near horizon geometry is $AdS_2 \times S^2$. This happens also in spaces with cosmological constant

The region outside the horizon seems infinitely far away



Puzzle: black holes near extremality

Close to extremality, the energy accessible to system is [Preskill, Schwarz, Shapere, Trivedi, Wilczek, '91]

$$E_{BH} = 4\pi^2 Q^3 T^2 = \frac{T^2}{E_{gap}}$$

Typical energy of Hawking quantum is

$$E_{Hawk} \sim T$$

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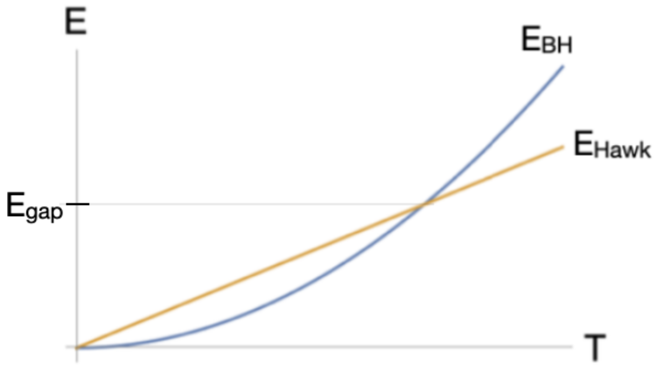
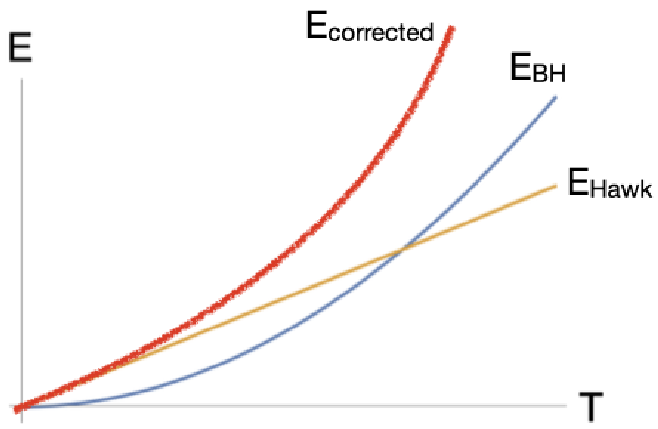


Below E_{gap} the energy available to the BH is not sufficient for the emission of a single Hawking quantum

At $E = E_{gap}$ temperature fluctuations become large compared to T itself

Puzzle: black holes near extremality

Lesson from [Iliesiu, Turiaci '19][Heydeman, Iliesiu, Turiaci, Zhao '20] for static charged black holes: adding quantum corrections, the energy is

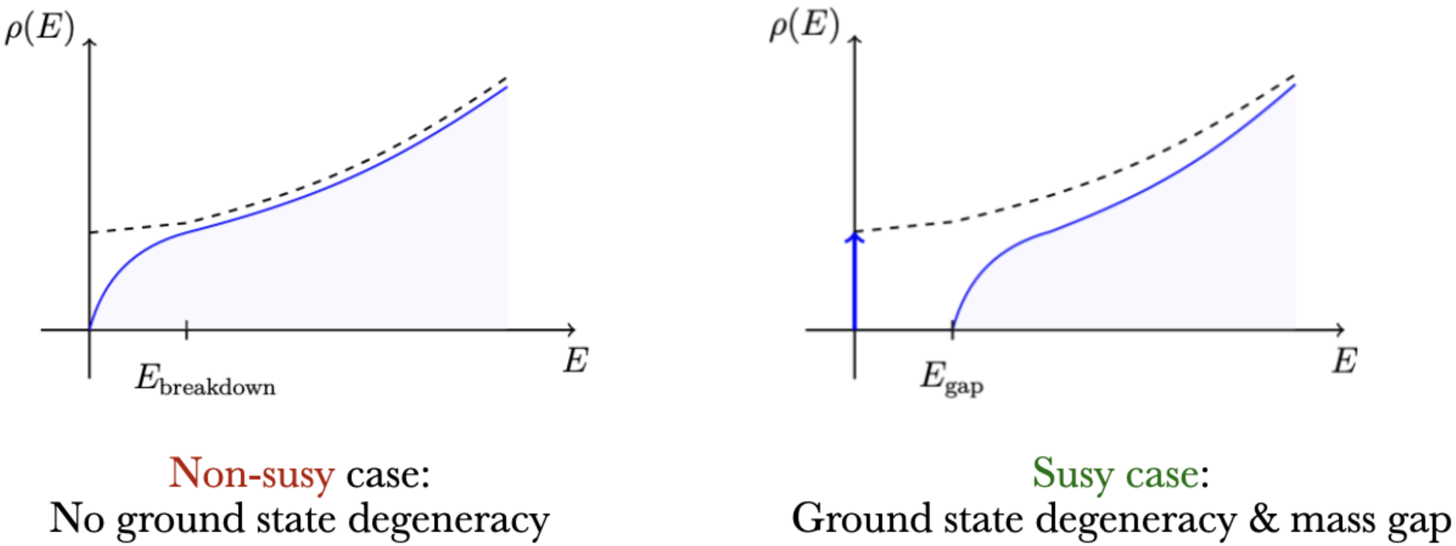


In addition, apparent degeneracy at $T = 0$ is lifted. Density of states vanishes at $E = 0$.

Gravitational path integral coming from 2d reduced action. Excitation above extremality described by JT gravity, where quantum effects can be understood [Stanford, Witten '16], [Mertens, Turiaci, Verlinde '16]

Puzzle: black holes near extremality

Quantum-corrected density of black hole states [Heydeman, Iliesiu, Turiaci, Zhao '20]



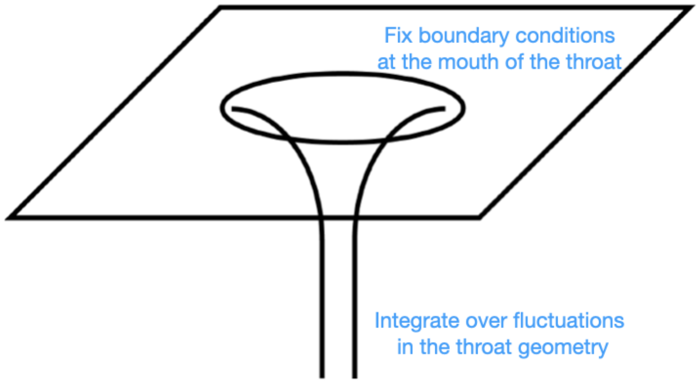
JT sector identified for near extremal Kerr [Castro, Godet '19] [Castro, Godet, Simon, Yu, '21] suggests that a similar mechanism should be in place

Quantum corrections from 4d

This also seen by taking into account certain *zero modes* in the near horizon gravitational path integral in 4d

Assumes that relevant part of the BH Hilbert space can be captured by gravitational dynamics near the throat

$$Z_{grav} = \int [Dg] e^{-S[g]} \qquad g \to \overline{g} \text{ at boundary}$$



Integrate over metrics subject to some boundary conditions fixed by the ensemble.
$\rightarrow$ Use saddle point approximation

Quantum corrections

The first correction to the value at saddle point comes from integrating over the quantum fluctuations about the saddle. This is where subtleties lie: divergencies due to zero modes appearing in the 1-loop computation [Sen '08]

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We need to compute a functional determinant. Schematically

$$\int_{-\infty}^{+\infty} e^{-\lambda x^2} dx = \sqrt{\frac{\pi}{\lambda}} \qquad\qquad \int \prod_i e^{-\lambda_i x_i^2} dx_i = \sqrt{\prod_i \frac{\pi}{\lambda_i}}$$

Approach: making use of the near-extremal configuration, which will act as a regulator for these divergencies [Iliesiu, Murthy, Turiaci '22]

Outline

- RN black holes
  - zero modes and log T corrections
  - de Sitter: three different limits, curious ultracold geometry

- Kerr black holes
  - Zero Modes for NHEK, subtleties due to superradiance

- (warped) AdS_3 black holes
  - Extension of the modes in the full geometry

Reissner–Nordström black holes

Extremal RN black holes

Black hole solution

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + \frac{dr^2}{(1 - \frac{2M}{r} + \frac{Q^2}{r^2})} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad F_{rt} = \frac{Q}{r^2}$$

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At extremality $M = |Q|$, zoom in the NH geometry with

$$\hat{t} = \frac{2r_0}{T}t, \qquad \hat{r} = r_+(T) + r_0 T(\cosh\eta - 1) \qquad T \to 0$$

obtain $AdS_2 \times S^2$

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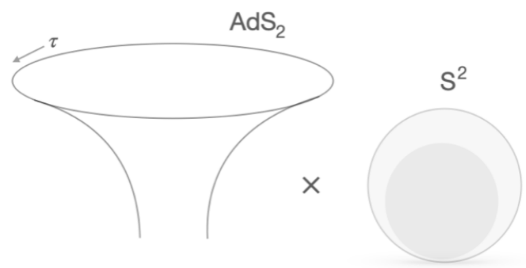
obtain $AdS_2 \times S^2$

$$ds^2 = (-\sinh^2\eta\, dt^2 + d\eta^2) + r_0^2(d\theta^2 + \sin^2\theta d\phi^2)$$



Extremal RN black holes

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$$ds^2 = -(1-\frac{2M}{r}+\frac{Q^2}{r^2})dt^2 + \frac{dr^2}{(1-\frac{2M}{r}+\frac{Q^2}{r^2})} + r^2(d\theta^2+\sin^2\theta d\phi^2) \qquad F_{rt} = \frac{Q}{r^2}$$

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Expansion close to extremality:

$$M = M_0 + \frac{T^2}{M_{gap}} \qquad S = S_0 + \frac{2T}{M_{gap}} \qquad M_{gap} = \frac{1}{4\pi^2 Q^3}$$

Quantum corrections and zero modes

First analytically continue $t = -i\tau$. Regularity at $\eta = 0$ requires periodicity $\tau \to \tau + 2\pi$

Partition function in the NH region is given by integral over metrics, subject to boundary conds

$$Z = \int [Dg] e^{-I[g]} \qquad I[g] = -\frac{1}{16\pi}\int_M d^4x\sqrt{g}R + I_{bdary}$$

$AdS_2 \times S^2$ is saddle point solution. Writing $g = \overline{g} + h$ where $\overline{g} = g_{NH}$ and expanding action at quadratic order

$$Z \approx \exp\left(-I[\overline{g}]\right) \int [Dh] \exp\left[-\int d^4x\sqrt{\overline{g}}\, h\mathcal{D}[\overline{g}]h\right] .$$

$\mathcal{D}$ is a 2nd-order differential operator. Path-integral computes $\int [Dh] e^{\left[-\int d^4x\, h\mathcal{D}[\overline{g}]h\right]} \sim \frac{1}{\det(\mathcal{D})}$

Quantum corrections and zero modes

For Einstein-Maxwell

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{g}(R - F_{\mu\nu}F^{\mu\nu})$$

Lichnerowitz operator $\mathcal{D}$ has form [Sen '09]

$$\begin{aligned}
&h^*_{\alpha\beta}\, \mathcal{D}^{\alpha\beta,\mu\nu}[\overline{g}]\, h_{\mu\nu} = \\
&-\frac{1}{16\pi} h^*_{\alpha\beta} \Big(\frac{1}{4}\overline{g}^{\alpha\mu}\overline{g}^{\beta\nu}\overline{\Box} - \frac{1}{8}\overline{g}^{\alpha\beta}\overline{g}^{\mu\nu}\overline{\Box} + \frac{1}{2}\overline{R}^{\alpha\mu\beta\nu} + \frac{1}{2}\overline{R}^{\alpha\mu}\overline{g}^{\beta\nu} \\
&- \frac{1}{2}\overline{R}^{\alpha\beta}\overline{g}^{\mu\nu} - \frac{1}{4}\overline{R}\overline{g}^{\alpha\mu}\overline{g}^{\beta\nu} + \frac{1}{8}\overline{R}\overline{g}^{\alpha\beta}\overline{g}^{\mu\nu} + \frac{1}{8}\overline{F}^2\left(2\overline{g}^{\alpha\mu}\overline{g}^{\beta\nu} - \overline{g}^{\alpha\beta}\overline{g}^{\mu\nu}\right) \\
&- \overline{F}^{\alpha\mu}\overline{F}^{\beta\nu} - 2\overline{F}^{\alpha\gamma}\overline{F}^{\mu}{}_{\gamma}\overline{g}^{\beta\nu} + \overline{F}^{\alpha\gamma}\overline{F}^{\beta}{}_{\gamma}\overline{g}^{\mu\nu} \Big) h_{\mu\nu}\,,
\end{aligned}$$

+ gauge fields

+ ghost part

Zero modes for Extremal RN black holes

$\mathcal{D}$ supports infinite family of normalizable zero modes on $\mathbb{H}^2$ [Camporesi, Higuchi '94]

$$h^{(Schw)}_{\mu\nu} dx^\mu dx^\nu \propto e^{in\tau} \frac{\tanh^{|n|}(\eta/2)}{\sinh^2(\eta)} \left(-\sinh^2(\eta) d\tau^2 + 2i \sinh(\eta) d\eta d\tau + d\eta^2 \right)$$

for $|n| > 1$

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for $|n| > 1$

$\rightarrow$ They are metric perturbations generated by large diffeomorphisms left unfixed by harmonic gauge. They obey

$$h^{(n)} \propto \mathcal{L}_{\xi^{(n)}} g_{\text{NHEK}}$$

with the vector field

$$\xi^{(n)} = e^{in\tau}\tanh^{|n|}\frac{\eta}{2}\left(\frac{in(|n| + \cosh\eta)}{\sinh\eta}\partial_\eta - \frac{|n|(|n| + \cosh\eta) + \sinh^2\eta}{\sinh^2\eta}\partial_\tau\right)$$

Relation to the Schwarzian

Repackaging these modes

$$\xi = \sum_n f_n \xi^{(n)}$$

Defining

$$f(\tau) = \sum_n f_n e^{in\tau}$$

Large η behavior

$$\xi \approx -f(\tau)\partial_\tau + f'(\tau)\partial_\eta \ .$$

Diffeos correspond to boundary time reparameterizations that send $\tau \to \tau - f(\tau), \eta \to \eta + f'(\tau)$

Modes are parameterized by element of $\mathrm{Diff}(S^1)/SL(2,\mathbb{R})$ ($n = -1, 0, 1$ are isometries hence h vanishes)

Recap

- $\mathcal{D}$ operator admits an infinite family of zero modes

$$\boxed{h^{(n)}_{\mu\nu}dx^\mu dx^\nu = \frac{1}{4\pi}\sqrt{\frac{3}{2}}\sqrt{|n|(n^2-1)}e^{in\tau}\frac{(\sinh\eta)^{|n|-2}}{(1+\cosh\eta)^{|n|}}\left(d\eta^2+2i\frac{n}{|n|}\sinh\eta\, d\eta d\tau-\sinh^2\eta\, d\tau^2\right)}$$

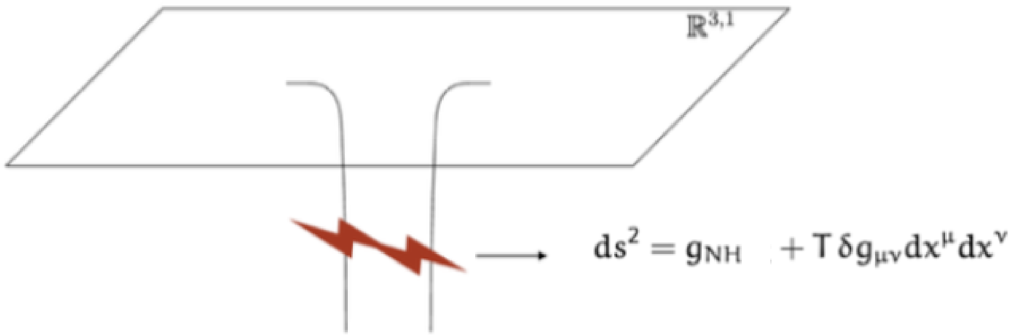
- these modes cost no action and have infinite volume, the one-loop approximation to the path integral therefore suffers from an infrared divergence

$$Z \propto \int_{\mathrm{Diff}(S^1)/SL(2,\mathbb{R})} [Df(\tau)] = \infty$$

- NH path integral is divergent and not well defined

How to fix this

Do not fully decouple geometry, keep O(T) term near horizon [Iliesiu, Murthy, Turiaci '22]



Use this geometry to regulate the computation

Compute the correction to eigenvalues via perturbation theory

How to fix this

Take the linear term in T in the decoupling limit

Zoom in the near-horizon geometry with change of coords

$$\hat{t} = \frac{2r_0}{\varepsilon(T)} t, \qquad \hat{r} = r_+(T) + r_0 \varepsilon(T)(\cosh\eta - 1), \ , \qquad \varepsilon(T) = 4\pi r_0 T,$$

Expansion in small T gives

$$ds^2 = g_{AdS_2 \times S^2} + T\,\delta g_{\mu\nu} dx^\mu dx^\nu$$

Eigenvalue correction

Expanding everything to first order in T

$$(\overline{\mathcal{D}} + \delta\mathcal{D})(h_n^0 + \delta h_n) = (\Lambda_n^0 + \delta\Lambda_n)(h_n^0 + \delta h_n)$$

$h^0 =$ extremal eigenfunctions with eigenvalues Λ^0

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Isolating $O(T)$ terms, we get

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Taking the inner product with h_m^0, using orthonormality

$$\delta\Lambda_n = \int d^4x \sqrt{g} (h_n^0)_{\alpha\beta} \delta\mathcal{D}^{\alpha\beta,\mu\nu} (h_n^0)_{\mu\nu} \ .$$

Therefore corrected one loop determinant* is $\log Z = -\frac{1}{2}\sum_n \log(\Lambda_n^0 + \delta\Lambda_n)$

*Modes with nonzero eigenvalues produce subleading corrections T-dependent terms

$\log T$ Corrections to the Entropy

After integration it simplifies to

$$\delta\Lambda_n = \frac{nT}{16Q}\,, \qquad n \geqslant 2\,.$$

hence correction is

$$\delta \log Z = 2 \cdot (-1/2) \sum_{n\geqslant 2} \log \delta\Lambda_n = \log\left(\prod_{n\geqslant 2} \frac{16Q}{nT}\right)$$

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Using zeta function regularization $\prod_{n \geqslant 2} \frac{\alpha}{n} = \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha^{3/2}}$ we obtain

$$\delta \log Z = \log \left(\frac{T^{3/2}}{128\sqrt{2\pi} Q^{3/2}} \right) = \boxed{\frac{3}{2} \log T} + \dots\,.$$

log T Corrections to the Entropy

At low temperatures

$$Z[T]_{\text{Black Hole}} \sim T^{3/2} exp[S_0 + 8\pi^2 Q^3 T]$$

Few remarks:

- Approximation is not valid when T is too small, i.e. $T < Q^\beta e^{-\alpha S_0}$, where $\alpha, \beta \sim O(1)$. Below this temperature, the partition function is so small that other saddles will begin competing

- When $T \sim 1/Q$ the linear term competes with the leading S_0 term: small-T approximation of the geometry not valid (correction term becomes as large as the NH metric).

log T Corrections to the Entropy

One finds $\rho(E) \to 0$ as $E \to 0$: no exponential ground state degeneracy or thermodynamic mass gap. Ground states are spread out over a dense energy band above the vacuum.

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There are additional, ensemble dependent zero modes arising from $U(1)$ gauge symmetry and isometries of S^2 [Sen '11]

$$h_{i\mu} = \frac{1}{\sqrt{2}} \epsilon_{ij} \partial^j Y_l^m(\theta, \phi) v_\mu \qquad v_\mu = \partial_\mu \Phi_n(\tau, \eta)$$

Each contribute $\sim \frac{1}{2} \log T$

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Each contribute $\sim \frac{1}{2}\log T$

... Opens the way to computations for different black holes

de Sitter black holes

[w M. Blacker, A. Castro, W. Sybesma, & F. Mariani]

de Sitter black holes

Reissner-Nordström de Sitter black hole:

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad F_{rt} = \frac{Q}{r^2}$$

with

$$V(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2\right) = -\frac{\Lambda}{3}\frac{(r-r_{c_2})(r-r_c)(r-r_+)(r-r_-)}{r^2}$$

Positive roots of $V(r)$:

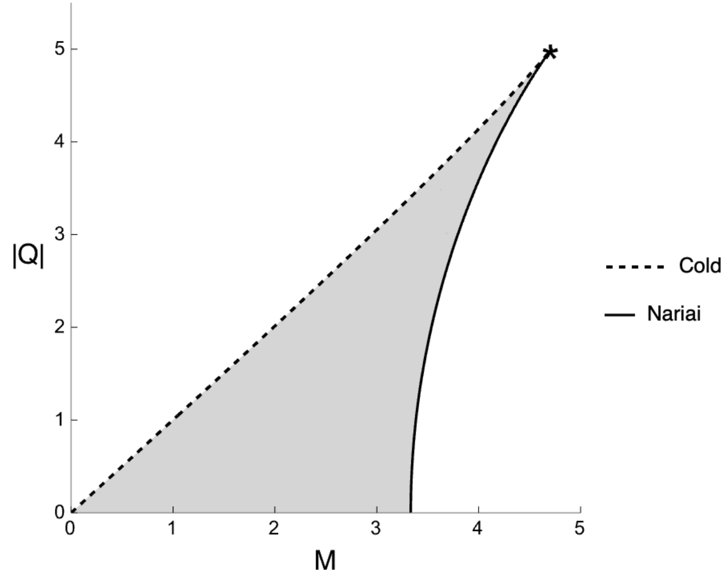
- $r = r_+$ outer horizon
- $r = r_-$ inner horizon
- $r = r_c$ cosmological horizon

Extremality is when two (or more) horizons coincide

de Sitter black holes

Presence of cosmological horizon:
3 different limits [Romans '92]

1. Cold black hole: $r_+ = r_-$ $AdS_2 \times S^2$
2. Nariai solution $r_+ = r_c$ $dS_2 \times S^2$
3. Ultracold solution $r_+ = r_- = r_c$ $Mink_2 \times S^2$



Goals: Analyze different excitations above $T = 0$: gives JT gravity on $\mathcal{M}_2$
& compute quantum (Log T) corrections to BH entropy in the three cases
[Blacker, Castro, Sybesma, CT '25]
[Maulik, Mitra, Mukherjee, Ray '25]

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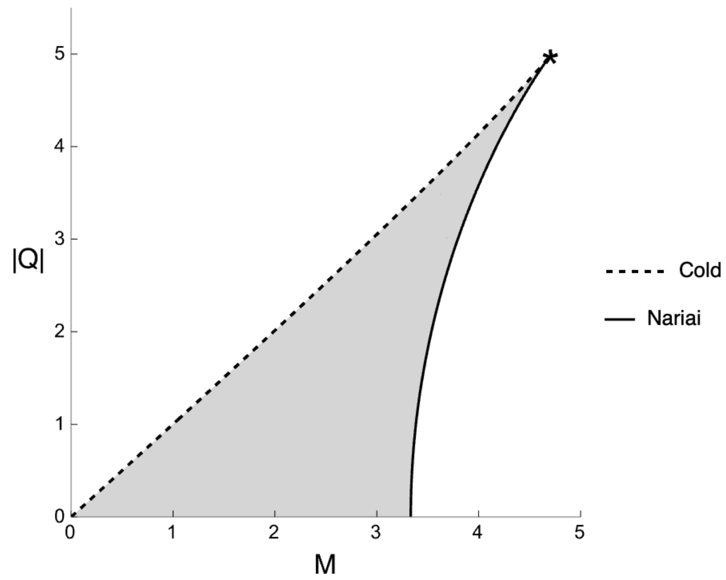

Subtlety: Near extremality the Euclidean geometry has conical singularities. Here only analysis in the NH geometry

Cold de Sitter black holes $AdS_2 \times S^2$

Expanding around extremality

$$S_+ = S_0 + \frac{2T}{M_{gap}} \qquad M = M_0 + \frac{T^2}{M_{gap}} \qquad \text{with} \qquad M_{gap} = \frac{1 - 2r_0^2\Lambda}{2\pi^2 r_0^3} > 0$$

Compute eigenvalue correction in perturbation theory $\quad \delta\Lambda_n = \frac{nT_+}{16 r_0} \quad |n| > 1$

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Hence

$$\delta \log Z = \frac{3}{2} \log \frac{T}{r_0} - \log 64 (2\pi)^{1/2} \qquad l_{AdS_2} = \frac{r_0^2 \ell_4^2}{(\ell_4^2 - 6r_0^2)}$$

Need to include contributions from

- rotational zero modes (3) $\sim \frac{3}{2} \log \frac{T}{r_0}$
- gauge field mode (1) $\sim \frac{1}{2} \log \frac{T}{r_0}$

... Total correction is $\delta \log Z = \left(\frac{3}{2} + \frac{3}{2} + \frac{1}{2}\right) \log \frac{T}{r_0} = \frac{7}{2} \log \frac{T}{r_0}$

de Sitter black holes

- **Nariai** has near horizon $dS_2 \times S^2$. Solution has negative heat capacity

$$M = M_0 + \frac{T_c^2}{M_{gap}^N} \qquad\qquad M_{gap}^N = \frac{1 - 2r_0^2\Lambda}{2\pi^2 r_0^3} < 0$$

Several pathologies, including tensor modes with negative eigenvalue correction, vector modes with negative norm (plus subtleties in Nariai thermodynamics and evaporation in [Bousso,Hawking '96 '97])

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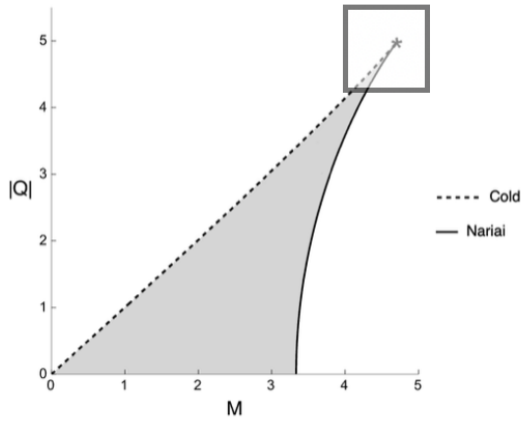
- **Ultracold** has near horizon $Mink_2 \times S^2$ still elusive. Working in transverse traceless gauge

$$h_\mu{}^\mu = 0 \qquad \nabla^\nu h_{\nu\mu} = 0 \qquad \mathcal{L}_\xi g_{\mu\nu} = h_{\mu\nu}$$

with $ds^2 = \rho^2 d\phi^2 + d\rho^2$ we find no normalizable solution.

Ultracold configuration [w.i.p.]



Work with Axial gauge: find modes that reproduce the full Warped Witt algebra

Reduction to 2d gives $\widehat{\text{CGHS}}$ models (flat dilaton gravity)

- Proof of principle that (normalizable) modes cannot exist?
- Results of [Arnaudo, Bonelli, Tanzini '25] show a 3/2 logT correction for Ultracold Kerr-dS. Can it be checked via these modes?

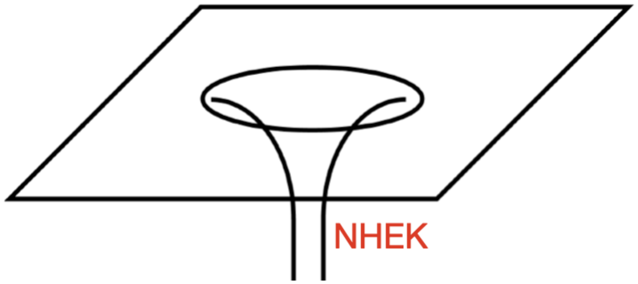
Kerr black holes

[Kapec, Sheta, Strominger, **CT** '23] [Rakic, Rangamani, Turiaci '23]

Near Horizon Extremal Kerr

For $T \to 0$ the geometry is the Near Horizon Extremal Kerr (NHEK) geometry [Bardeen, Horowitz '99]

$$ds^2 = J\left(1+\cos^2\theta\right)\left(-\sinh^2\eta\, dt^2 + d\eta^2 + d\theta^2\right) + J\frac{4\sin^2\theta}{1+\cos^2\theta}\left(d\phi + (\cosh\eta - 1)\, dt\right)^2$$



The NHEK metric has $SL(2,\mathbb{R}) \times U(1)$ symmetry with generators

$$L_{\pm 1} = \frac{e^{\mp t}}{\sinh\eta}\left(\cosh\eta\, \partial_t \pm \sinh\eta\, \partial_\eta + (\cosh\eta - 1)\partial_\phi\right)\,,$$

$$L_0 = \partial_t + \partial_\phi\,, \qquad W = \partial_\phi.$$

Quantum corrections and zero modes

Quadratic fluctuation operator term for Einstein–Hilbert action in NHEK is

$$h_{\alpha\beta} D^{\alpha\beta,\mu\nu}_{\mathrm{NHEK}} h_{\mu\nu} = -\frac{1}{16\pi} h_{\alpha\beta} \left(\frac{1}{4} \bar{g}^{\alpha\mu} \bar{g}^{\beta\nu} \bar{\Box} - \frac{1}{8} \bar{g}^{\alpha\beta} \bar{g}^{\mu\nu} \bar{\Box} + \frac{1}{2} \bar{R}^{\alpha\mu\beta\nu} \right) h_{\mu\nu}$$

It supports infinite family of normalizable zero modes

Quantum corrections and zero modes

Quadratic fluctuation operator term for Einstein–Hilbert action in NHEK is

$$h_{\alpha\beta} D^{\alpha\beta,\mu\nu}_{\mathsf{NHEK}} h_{\mu\nu} = -\frac{1}{16\pi} h_{\alpha\beta} \left(\frac{1}{4} \overline{g}^{\alpha\mu} \overline{g}^{\beta\nu} \overline{\Box} - \frac{1}{8} \overline{g}^{\alpha\beta} \overline{g}^{\mu\nu} \overline{\Box} + \frac{1}{2} \overline{R}^{\alpha\mu\beta\nu} \right) h_{\mu\nu}$$

It supports infinite family of normalizable zero modes

$$h^{(n)}_{\mu\nu} dx^\mu dx^\nu = \frac{1}{4\pi} \sqrt{\frac{3}{2}} \sqrt{|n|(n^2-1)} (1+\cos^2\theta) e^{in\tau} \frac{(\sinh\eta)^{|n|-2}}{(1+\cosh\eta)^{|n|}} (d\eta^2 + 2i \frac{n}{|n|} \sinh\eta \, d\eta \, d\tau - \sinh^2\eta \, d\tau^2)$$

$$\text{for } |n| > 1$$

Also generalized to higher dimensions [Pando Zayas, Zhang '26]. Again generated by

$$h^{(n)} \propto \mathcal{L}_{\xi^{(n)}} g_{\mathsf{NHEK}}$$

Quantum corrections and zero modes

Use near extremal horizon expansion

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + T\delta g_{\mu\nu} + \mathcal{O}(T^2) \,,$$

and eigenvalue correction in perturbation theory

$$\delta\Lambda_n = \int d^4x \sqrt{\bar{g}} (h_n^0)_{\alpha\beta} \delta D^{\alpha\beta,\mu\nu} (h_n^0)_{\mu\nu}$$

Quantum corrections and zero modes

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Using

$$\delta D^{\alpha\beta,\mu\nu} = \delta D^{\alpha\beta,\mu\nu}_{\mathrm{Box}} + \delta D^{\alpha\beta,\mu\nu}_{\mathrm{Riemann}}$$

where

$$\delta D^{\alpha\beta,\mu\nu}_{\mathrm{Box}} = -\frac{1}{16\pi} \delta \left(\frac{1}{4} g^{\alpha\mu} g^{\beta\nu} \Box - \frac{1}{8} g^{\mu\nu} g^{\alpha\beta} \Box \right) \qquad \delta D^{\alpha\beta,\mu\nu}_{\mathrm{Riemann}} = -\frac{1}{32\pi} \delta \left(R^{\alpha\mu\beta\nu} \right)$$

with $g = \overline{g} + \delta g$.

Quantum corrections and zero modes

Use near extremal horizon expansion

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + T\delta g_{\mu\nu} + \mathcal{O}(T^2)\ ,$$

and eigenvalue correction in perturbation theory

$$\delta\Lambda_n = \int d^4x\sqrt{\bar{g}}(h_n^0)_{\alpha\beta}\delta D^{\alpha\beta,\mu\nu}(h_n^0)_{\mu\nu}$$

Inserting form of zero-modes

$$\int d^4x\sqrt{\bar{g}}(h_n^0)_{\alpha\beta}\delta D^{\alpha\beta,\mu\nu}_{\text{Riemann}}(h_n^0)_{\mu\nu} = -\frac{3n(n^2-1)T}{128J^{1/2}}\int_0^\infty d\eta\Big[16(\pi-2)\coth\eta\,\mathrm{csch}^2\eta\ \tanh^{2n}\left(\frac{\eta}{2}\right)\Big]$$

$$\int d^4x\sqrt{\bar{g}}(h_n^0)_{\alpha\beta}\delta D^{\alpha\beta,\mu\nu}_{\text{Box}}(h_n^0)_{\mu\nu} = -\frac{3n(n^2-1)T}{128J^{1/2}}\int_0^\infty d\eta\Big[\mathrm{csch}^3\eta\,\mathrm{sech}^4\left(\frac{\eta}{2}\right)\Big((\pi-2)\cosh 3\eta +$$

$$+(4(n-2)n+7\pi-30)\cosh\eta - 2(n-2\pi+4)\cosh 2\eta + 2n(4n+7)+4\pi\Big)\tanh^{2n}\left(\frac{\eta}{2}\right)\Big]$$

Quantum corrections and zero modes

Final eigenvalue correction in perturbation theory

$$\delta\Lambda_n = \frac{3nT}{64J^{1/2}}\,, \qquad n \geqslant 2\,.$$

Hence

$$Z[T]_{\text{Black Hole}} \sim T^{3/2} exp[S_0 + 8\pi^2 J^{3/2} T]$$

[Kapec, Sheta, Strominger, **CT** '23] [Rakic, Rangamani, Turiaci '23]

Corrections likely to be unobservable due to **superradiance**: extract J and takes BH out of extremality*

*Communications with J. Maldacena and S. Murthy

Quantum corrections and zero modes

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[Kapec, Sheta, Strominger, **CT** ’23] [Rakic, Rangamani, Turiaci ’23]

$\rightarrow$ We argue that Kerr states are *resonances* in the complex energy plane: widths (imaginary part) require full Kerr geometry, throat calculation gives info about the distribution along real axis

Subtleties & further developments

- Other sources of instability [Horowitz, Remmen, Santos '24]

- NHEK should have a $U(1)$ rotational mode. There are subtleties in Kerr for imposing transverse traceless gauge [Rakic, Rangamani, Turiaci '23]. Rotational modes $\leftrightarrow$ ghost zero modes [Kolanowski, Marolf, Rangamani, Rakic, Turiaci '24]

- The $\frac{3}{2}\log T$ corrections can be retrieved for BTZ via Quasinormal modes [Kapec,Law, CT '24], and also for Kerr [Arnaudo, Bonelli, Tanzini '24] via DHS formula

(Warped) AdS$_3$ black holes

collaboration with E. Despontin, S. Detournay, R. Mancilla

Further developments

Can modes be extended in full geometry?

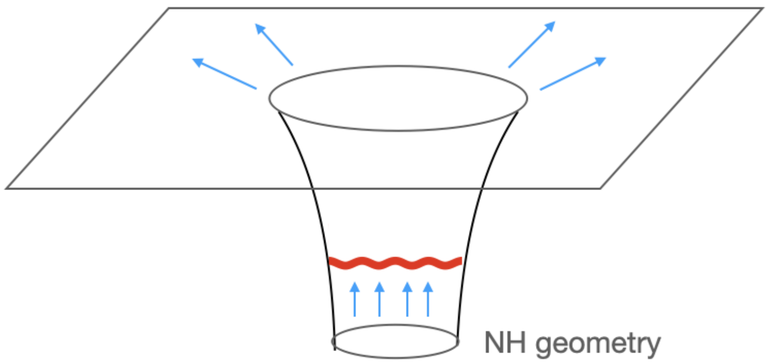
In BTZ this can be done [Kolanowski, Marolf,Rakic,Rangamani, Turiaci '24], [Acito, Sempe '25], [Bac, Castro, Jain '26]. Finite number of normalizable modes for finite T, become infinite in NH extremal limit

Further developments

Can modes be extended in full geometry?

In BTZ this can be done [Kolanowski, Marolf,Rakic,Rangamani, Turiaci '24], [Acito, Sempe '25], [Bac, Castro, Jain '26]. Finite number of normalizable modes for finite T, become infinite in NH extremal limit

- reduce to extremal modes for $T \to 0$
- connect near and far region



$\to$ Subtleties in boundary conditions

Warped AdS$_3$ and TMG

- obtained from NHEK at fixed θ
- $SL(2,\mathbb{R}) \times U(1)$ symmetry

Tractable toy model for Kerr

Warped AdS$_3$ and TMG

- obtained from NHEK at fixed θ
- $SL(2,\mathbb{R}) \times U(1)$ symmetry

Tractable toy model for Kerr

Warped black holes are solutions of TMG (topologically massive gravity)

$$S_{TMG} = S_{EH} + S_{CS},$$

$$S_{EH} = \frac{1}{2\kappa^2}\int d^3x\sqrt{-g}\left(R + \frac{2}{\ell^2}\right) \qquad S_{CS} = \frac{1}{4\mu\kappa^2}\int d^3x\sqrt{-g}\;\epsilon^{\mu\nu\delta}\left(\Gamma^{\alpha}_{\mu\beta}\partial_\nu\Gamma^{\beta}_{\delta\alpha} + \frac{2}{3}\Gamma^{\chi}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\Gamma^{\alpha}_{\delta\chi}\right)$$

with EOMs

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{\ell^2}g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0$$

Warped AdS$_3$ and TMG

- obtained from NHEK at fixed θ
- $SL(2,\mathbb{R}) \times U(1)$ symmetry

$\Big\}$ *Tractable* toy model for Kerr

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with EOMs

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{\ell^2}g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0$$

Solutions with vanishing Cotton tensor $C_{\mu\nu} = 0$ are locally AdS$_3$ (such as BTZ).

Solutions with $C_{\mu\nu} \neq 0$ include warped AdS$_3$ spaces (and quotients thereof - *warped BHs*)

Warped AdS$_3$ black holes

Metric

$$ds^2 = ds^2_{\mathsf{BTZ}}(L) + \frac{\nu^2+3}{2\nu^2} H^2 P \otimes P$$

$$\vec{P} = \frac{\ell}{\nu(r_- + r_-)}\left(i d\tau + \frac{d\phi}{L}\right)$$

$$H^2 = -\frac{3}{2}\frac{(\nu^2-1)}{\nu^2+3}\,, \quad L = \frac{2\ell}{\sqrt{\nu^2+3}}\,, \quad \nu = \frac{\mu\ell}{3}$$

Warped AdS$_3$ black holes

Metric

$$ds^2 = f(r)dt^2 + \frac{(1-2H^2)r^2}{f(r)\mathcal{R}(r)^2}dr^2 + \mathcal{R}(r)^2\Big(d\phi + iN^\phi(r)dt\Big)^2$$

$$f(r) = \frac{(1-2H^2)(r^2-r_+^2)(r^2-r_-^2)}{\mathcal{R}(r)^2L^2}\,, \qquad \mathcal{R}(r)^2 = (1-2H^2)r^2 - 2H^2\frac{(r^2-r_+^2)(r^2-r_-^2)}{(r_-+r_+)^2}$$

$$N^\phi = -\frac{1}{\mathcal{R}^2(r)L}\left((1-2H^2)r_-r_+ + 2H^2\frac{(r^2-r_+^2)(r^2-r_-^2)}{(r_-+r_+)^2}\right)\,,$$

Warped AdS$_3$ black holes

Metric

$$ds^2 = f(r)dt^2 + \frac{(1-2H^2)r^2}{f(r)\mathcal{R}(r)^2}dr^2 + \mathcal{R}(r)^2\Big(d\phi + iN^\phi(r)dt\Big)^2$$

$$f(r) = \frac{(1-2H^2)(r^2-r_+^2)(r^2-r_-^2)}{\mathcal{R}(r)^2L^2}\,, \qquad \mathcal{R}(r)^2 = (1-2H^2)r^2 - 2H^2\frac{(r^2-r_+^2)(r^2-r_-^2)}{(r_-+r_+)^2}$$

$$N^\phi = -\frac{1}{\mathcal{R}^2(r)L}\left((1-2H^2)r_-r_+ + 2H^2\frac{(r^2-r_+^2)(r^2-r_-^2)}{(r_-+r_+)^2}\right)\,,$$

Near horizon geometry

$$ds_E^2 = \ell_2^2\big(\sinh^2(\eta)d\tau_E^2 + d\eta^2\big) + R_0^2\Big(d\varphi + \frac{i\ell_2}{R_0}C(\nu)(\cosh(\eta)-1)d\tau_E\Big)^2 \qquad C(\nu) = \frac{2\nu}{\sqrt{\nu^2+3}}$$

For $\nu = 1$ we retrieve BTZ black hole

Zero modes in the near horizon geometry

Tensor (Schwarzian) modes and **Rotational modes** in NH geometry: same for BTZ

$$\delta\Lambda_n^{Schw} = \frac{\left(5\nu^2+3\right)|n|T}{192L\nu^2 r_0}, \quad |n|>1 \qquad\qquad \delta\Lambda_n^{Rot} = \frac{(\nu^2-3)|n|T}{96L\nu^2 r_0}, \quad |n|>0$$

[Despontin, Detournay, Mancilla, CT, wip]

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[Despontin, Detournay, Mancilla, CT, wip]

Idea: Newman Penrose approach, useful when studying BH radiation and perturbations. Use null frames K, L for NH geometry

$$P = \left(R_0 d\varphi + i\ell_2 C(\nu)(\cosh(\eta)-1)d\tau_E\right)$$

$$K = \frac{\ell_2}{\sqrt{2}}(i\sinh(\eta)d\tau_E + d\eta) \qquad L = \frac{\ell_2}{\sqrt{2}}(-i\sinh(\eta)d\tau_E + d\eta)$$

Satisfying $\quad K^2 = L^2 = 0, \quad K\cdot L = 1\,, \quad P\cdot L = P\cdot K = 0$

Newman Penrose approach

Decompose perturbations as

$$h_{\mu\nu} = h_{kk} K_\mu K_\nu + h_{pk}(K_\mu P_\nu + P_\mu K_\nu)$$

where we imposed transverse traceless condition

Tensor (Schwarzian): can be rewritten as

$$h_{\mu\nu}{}^{Schw} = h_{kk}{}^{Schw} K_\mu K_\nu$$

Rotational modes:

$$h_{\mu\nu}{}^{Rot} = h_{kk}{}^{Rot} K_\mu K_\nu + h_{pk}{}^{Rot}(K_\mu P_\nu + P_\mu K_\nu) \; ,$$

with coefficients uniquely determined.

From NH geometry to Full geometry

$\rightarrow$ We can do the same in the full WAdS$_3$ geometry and find extended modes!

Find triads for warped AdS$_3$ BH full geometry

$$P = \frac{\ell}{\nu(r_- + r_-)}\left(i d\tau + \frac{d\phi}{L}\right)$$
$$K = \frac{\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}}{\sqrt{2}(r_+ + r_-)}\left(\frac{1}{L} i d\tau + \frac{Lr}{(r^2 - r_+^2)(r^2 - r_-^2)} dr + d\phi\right)$$
$$L = -\frac{\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}}{\sqrt{2}(r_+ + r_-)}\left(\frac{1}{L} i d\tau - \frac{Lr}{(r^2 - r_+^2)(r^2 - r_-^2)} dr + d\phi\right)$$

Use same ansatz as NH modes. Full geometry modes satisfy

$$\mathcal{D}_{\mu\nu}{}^{\alpha\beta,TMG} h_{\alpha\beta} = \lambda(T) h_{\mu\nu}$$

Gluing to the boundary

Take inspiration from modes in the NH geometry we can solve

$$\boxed{h_{kk}{}^{Schw} = N_1 e^{iE_n\tau}(r^2 - r_+^2)^{n/2-1}(r^2 - r_-^2)^{-\frac{n}{2}\frac{r_-}{r_+}-1}\,, \quad h_{pk}{}^{Schw} = 0\,, \qquad E_n = \frac{2n\pi T}{L}}$$

with eigenvalue matching the NH geometry one

$$\lambda(T) \xrightarrow{T\to 0} \frac{\left(5\nu^2 + 3\right)|n|T}{192 L \nu^2 r_0}$$

Gluing to the boundary

Take inspiration from modes in the NH geometry we can solve

$$\boxed{{h_{kk}}^{\color{blue}{Schw}} = N_1 e^{iE_n\tau}(r^2-r_+^2)^{n/2-1}(r^2-r_-^2)^{-\frac{n}{2}\frac{r_-}{r_+}-1}\,, \quad {h_{pk}}^{Schw} = 0\,, \qquad E_n = \frac{2n\pi T}{L}}$$

with eigenvalue matching the NH geometry one

$$\lambda(T) \xrightarrow{T\to 0} \frac{\left(5\nu^2+3\right)|n|T}{192L\nu^2 r_0}$$

Only modes with

$$1 < |n| < \frac{3r_+}{r_+ - r_-}$$

are normalizable.

$\rightarrow$ In a similar way we can find the extended rotational zero mode

Gluing to the boundary

For the rotational mode the eigenvalue

$$\lambda^{\text{Rot.}} \xrightarrow{T\to 0} \frac{\left(\nu^2-3\right)|n|T}{96\nu^2 r_0} + O(T^2)$$

Matches NH geometry one *provided* one is careful in imposing the transverse traceless gauge and consider the correction to the eigenfunctions $\delta h_{\mu\nu}$

$$\delta\lambda = \langle \overline{h}^{(-n)} \Big| \delta\mathcal{L}\ \overline{h}^{(n)} \rangle + \langle \overline{h}^{(-n)} \Big| \overline{\mathcal{L}}\ \delta h^{(n)} \rangle$$

For BTZ [Bac, Castro, Jain '26]: subtleties in matching the correction to the rotational eigen-mode when gluing. Follows a similar story

Gluing to the boundary

Boundary conditions (for BTZ)

$$h_{ab}^{\text{Schw.}} \sim O(r^{m}), \qquad h_{ab}^{\text{Rot.}} \sim O(r^{2+m}), \qquad \text{with } m = |n|\left(1 - \frac{r_-}{r_+}\right).$$

- Schwarzian and rotational modes generally violate both Brown–Henneaux and CSS boundary conditions, since they grow too rapidly near the AdS boundary

- for $T \to 0$, Schwarzian mode compatible with Brown Henneaux and CSS, rotational mode only CSS. Consistent with $\frac{3+1}{2}\log(T)$ correction from dual WCFT picture of [Aggarwal, Castro, Detournay, Muhlmann '22]

$\rightarrow$ warped BTZ: in progress

Conclusions and perspectives

Framework for quantum corrections to near extremal BH entropy.

- New light on extremal BH physics
- It predicts lifting of the ground state degeneracy of extremal non-susy black holes. How to account for superradiance?
- Sharpens the distinction between near- and far-region dynamics
- NP formalism might be useful for higher dimensions. Embedding horizon picture into boundary?

the end. Thank you!

Additional slides

Corrected one loop determinant* is $\log Z = -\frac{1}{2}\sum_n \log(\Lambda_n^0 + \delta\Lambda_n)$

Classical contribution:

$$\log Z_{tree} = S_0 + 2\pi^2 J^{3/2} T$$

1-loop contribution (T-dependent)

$$\log Z_{1-loop} = \frac{1}{180}(2n_S - 26n_V + 7n_F + 154)\log S_0 + \frac{3}{2}\log\left(\frac{T}{J^{3/2}}\right)$$

Total free energy (neglecting $\log S_0$ terms)

$$-\beta F = \log Z_{tree} + \log Z_{1-loop} = \log(Z_{tree} * Z_{1-loop}) = S_0 + 2\pi^2 J^{3/2} T + \frac{3}{2}\log\left(\frac{T}{J^{3/2}}\right)$$

Hence

$$Z_{tot} \propto T^{3/2} exp[S_0 + 2\pi^2 J^{3/2} T]$$

*Modes with nonzero eigenvalues produce subleading corrections T-dependent terms

Additional slides

$$Z(\beta) = \int dE \rho(E) e^{-\beta E}$$

Free energy $F(T) = -T \log Z$. Thermodynamic entropy is $S = -\frac{\partial F}{\partial T}$, equivalently

$$S = \log Z + \beta E \qquad E = -\frac{\partial \log Z}{\partial \beta}$$

aka Legendre transform relating canonical and microcanonical ensemble

Also $\log \rho(E) = S(E)$

Black holes and JT gravity

JT gravity: 2d dilaton gravity theory describing dynamics of spherically symmetric fluctuations of AdS_2 and transverse sphere S^2

$$S = \int d^2x \sqrt{g}\varphi \left(R + \frac{2}{l_{AdS_2}} \right)$$

Boundary time reparameterizations $t \to f(t)$ are the only dynamical degrees of freedom

$$S = \int dt\{f(t), t\} \qquad \{f(t), t\} = \frac{f'''(t)}{f'(t)} - \frac{3}{2}\left(\frac{f''(t)}{f'(t)}\right)^2 \qquad \text{Schwarzian action}$$

Symmetry breaking from $\mathrm{Diff}(S^1)$ to $SL(2, R)$ [Maldacena, Stanford, Yang '16]

Recent developments in JT gravity explain how to quantize it exactly [Stanford, Witten '16], [Mertens, Turiaci, Verlinde '16]

TMG formulas

The equations of motion are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{\ell^2} g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0 \,, \tag{1}$$

$$C_{\mu\nu} = D_\mu^{\ \alpha} \Big(R_{\alpha\nu} - \frac{1}{4} g_{\alpha\nu} R \Big) \,, \tag{2}$$

where ϵ^{abc} is the Levi-Civita tensor, and $D_\mu^{\ \beta}$ stands for the curl covariant derivative

$$D_\mu^{\ \beta} \equiv \varepsilon_\mu^{\ \alpha\beta} \, \nabla_\alpha \tag{3}$$

Some useful properties of the Cotton tensor are

$$C = 0 \,, \quad \nabla^\mu C_{\mu\nu} = 0 \,, \quad C_{\mu\nu} = C_{\nu\mu} \,. \tag{4}$$

Despite the presence of the Cotton tensor, all TMG solutions have negative Ricci scalar $R = -6/\ell^2$.

For general background, the TMG Lichnerowicz operator is

$$E^{(1)}_{\mu\nu}+\frac{1}{\mu}D_{\mu}{}^{\alpha}E^{(1)}_{\alpha\nu}+\frac{\epsilon_{\mu}{}^{\alpha\beta}}{\mu}\left(-\frac{1}{2}\overline{E}_{\beta}{}^{\lambda}(\nabla_{\alpha}h_{\lambda\nu}+\nabla_{\nu}h_{\lambda\alpha}-\nabla_{\lambda}h_{\alpha\nu})-h_{\alpha\lambda}\nabla^{\lambda}\overline{E}_{\beta\nu}-h_{\beta\lambda}\nabla^{\alpha}\overline{E}_{\lambda\nu}\right)=\lambda E_{\mu\nu} \quad (5)$$

WARPED WITT

$$[L_m, L_n] = (m-n)L_{m+n}, \quad (6)$$

$$[L_m, P_n] = -nP_{m+n}, \quad (7)$$

$$[P_m, P_n] = 0. \quad (8)$$

de Sitter black holes

- **Cold** black hole $AdS_2 \times S^2$. Total correction:

$$\delta \log Z = \left(\frac{3}{2} + \frac{3}{2} + \frac{1}{2}\right) \log \frac{T}{r_0} = \frac{7}{2} \log \frac{T}{r_0}$$

Contributions from tensor modes, rotational zero modes, gauge field mode

- **Nariai** has near horizon $dS_2 \times S^2$. Solution has negative heat capacity

$$M = M_0 + \frac{T_c^2}{M_{gap}^N} \qquad M_{gap}^N = \frac{1 - 2r_0^2\Lambda}{2\pi^2 r_0^3} < 0$$

Several pathologies, including tensor modes with negative eigenvalue correction, vector modes with negative norm

- **Ultracold** has near horizon $Mink_2 \times S^2$ still elusive: non normalizable modes

Nariai configuration $dS_2 \times S^2$

Negative heat capacity:

$$M = M_0 + \frac{T_c^2}{M_{gap}^N} \qquad\qquad M_{gap}^N = \frac{1 - 2r_0^2\Lambda}{2\pi^2 r_0^3} < 0$$

- Tensor modes have negative eigenvalue correction $\delta\Lambda_n = -nT_c/(16r_0)$
- Vector modes have negative norm $\langle h|h\rangle < 0$ and negative correction $\delta\Lambda_n < 0$
- Gauge field modes have negative norm and positive correction $\delta\Lambda_n > 0$

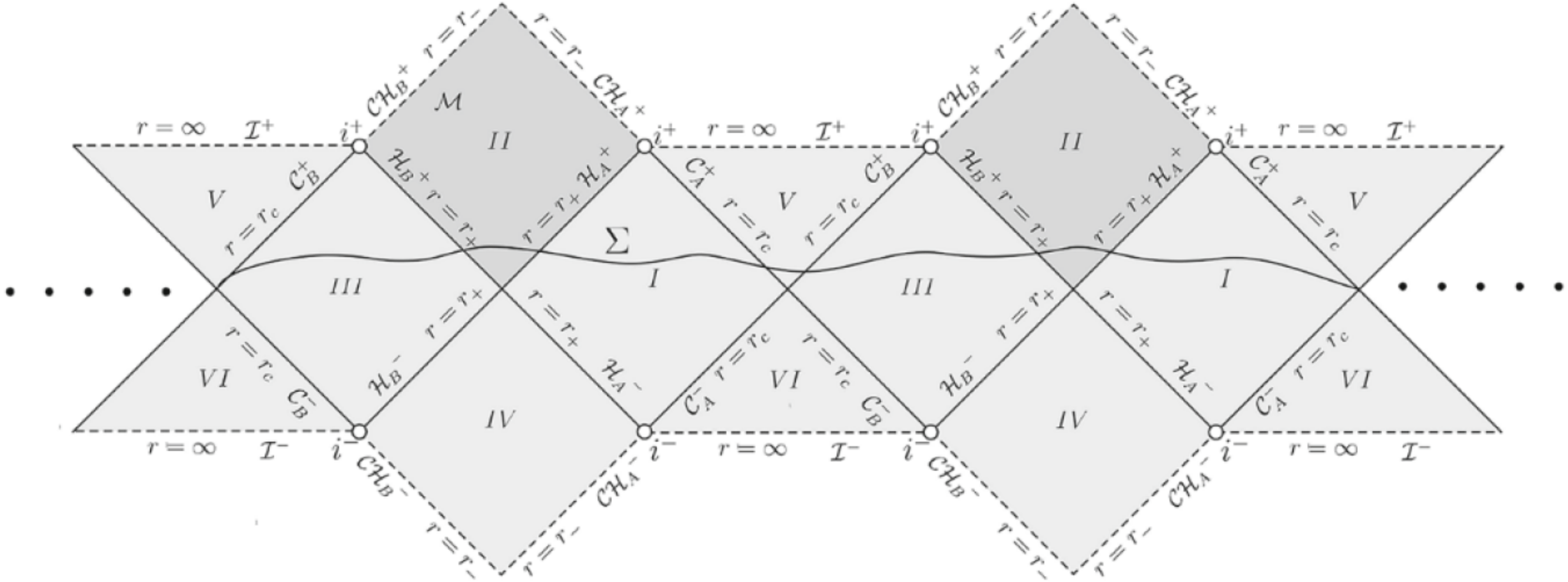
One can consider a complexified solution

$$r \to i\rho \qquad (r_N \to i\rho_N) \qquad ds^2 = -(EAdS_2 \times S^2)$$

gauge field modes have negative norm and negative correction $\delta\Lambda_n < 0$

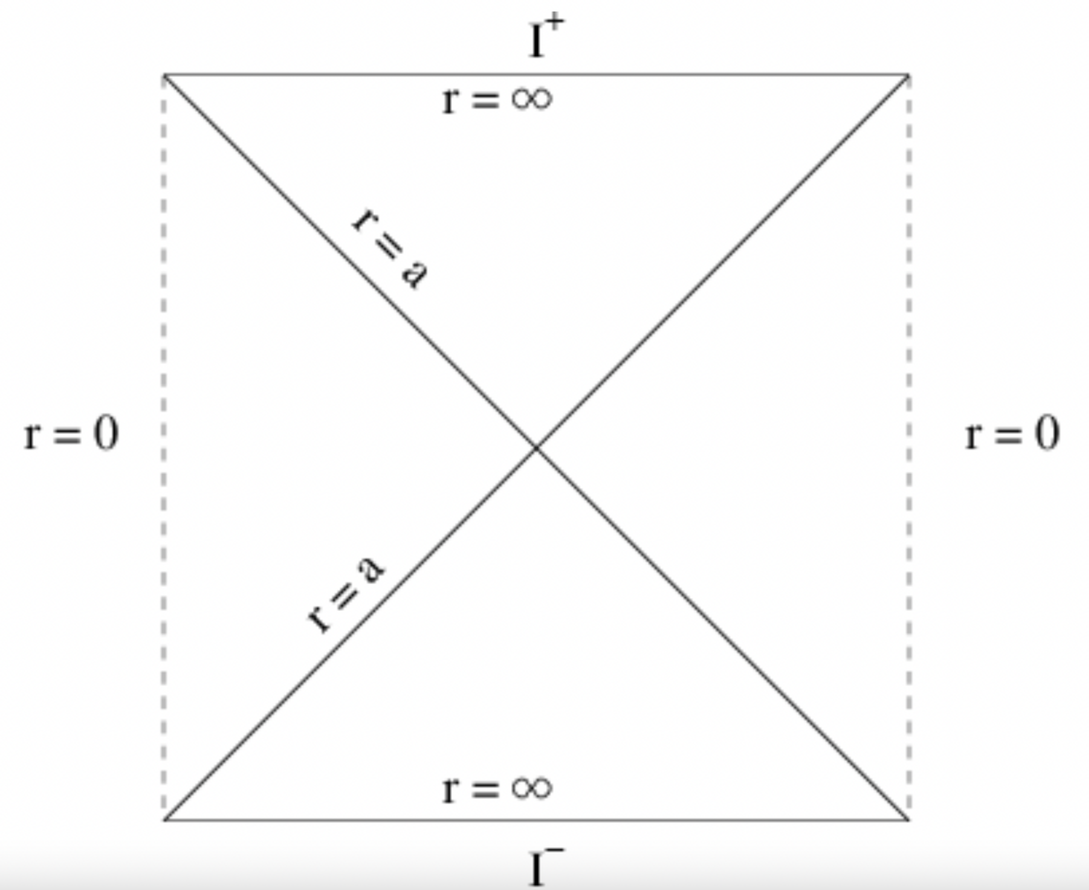
Backup slides

Penrose diagram of RN de Sitter



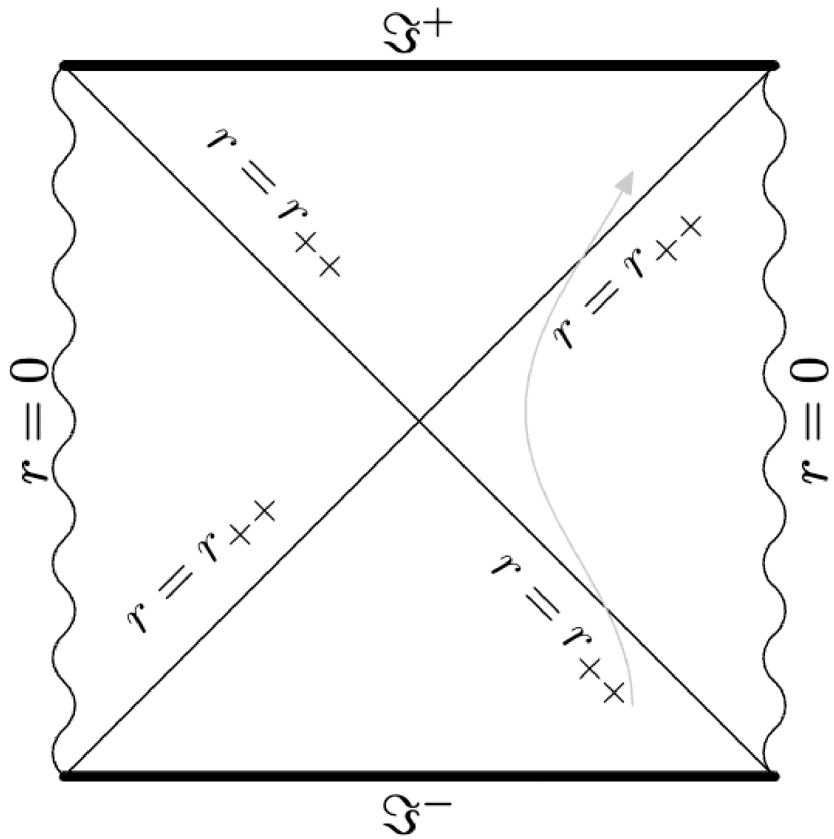
Backup slides

Penrose diagram of de Sitter space



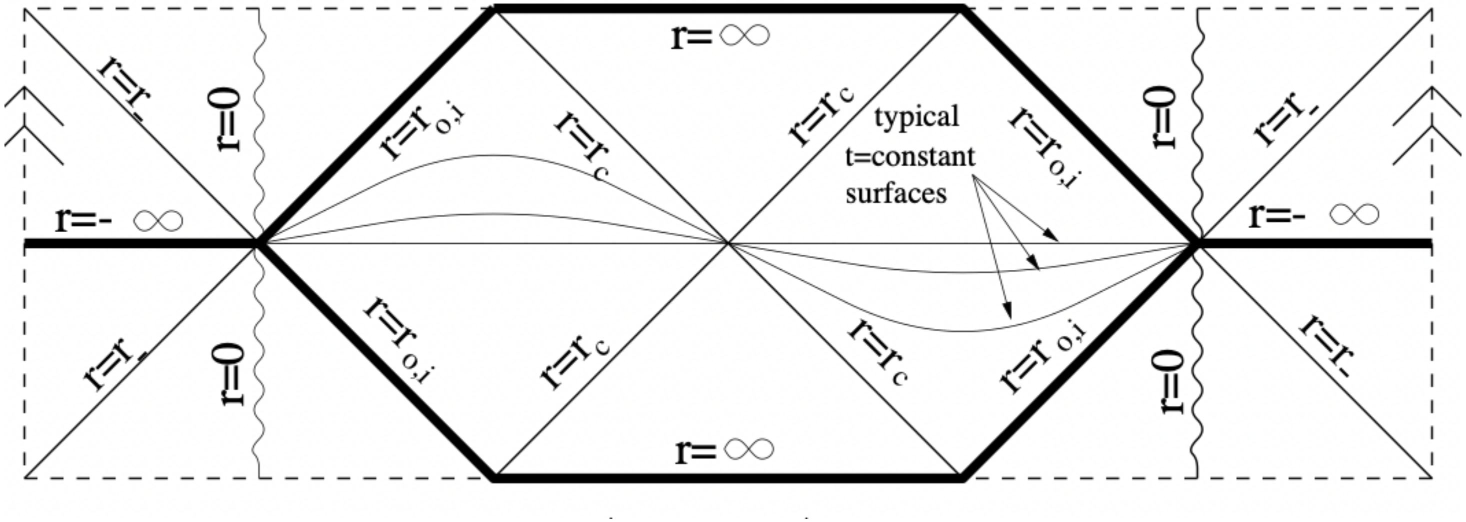
Backup slides

Penrose diagram of Ultracold space



Backup slides

Penrose diagram of Cold



Backup slides

Penrose diagram of Nariai