

Thermal entropy & Entanglement transitions from Modular objects



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● [2209.03242] JHEP

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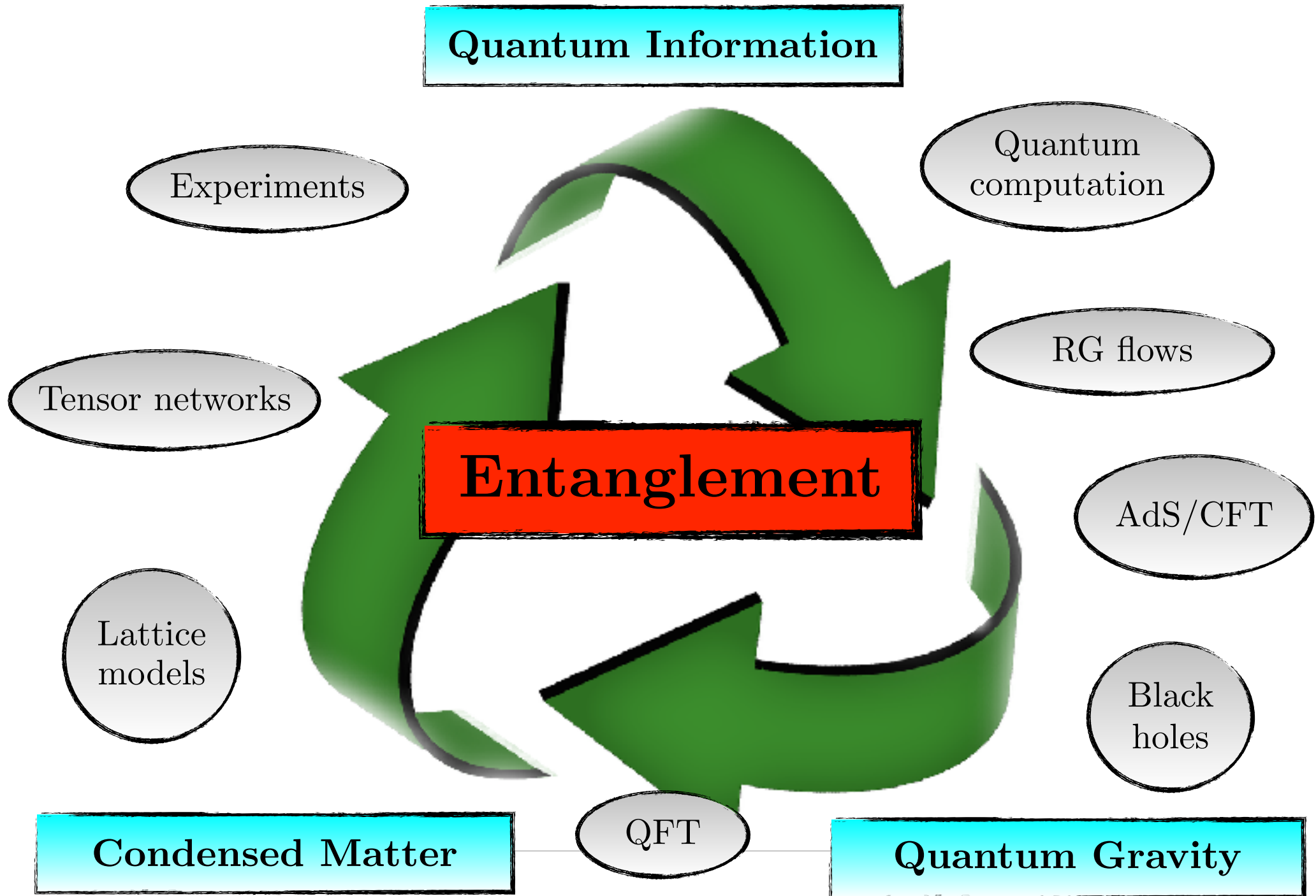
● [2403.03930] JHEP

Francesco Gentile, Andrei Rotaru
and E.T.

● [2503.19644] JSTAT

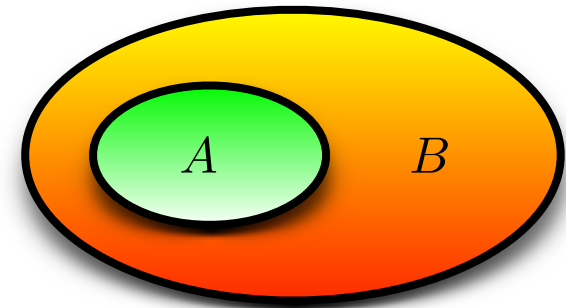
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Entanglement: a crossroad of interests



Entanglement: Hamiltonian, Spectrum and Entropy

- Quantum system in its ground state: $\rho = |\Psi\rangle\langle\Psi|$
Factorised Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



e.g.: spatial bipartition

- A's reduced density matrix

$$\rho_A = \text{Tr}_B \rho$$

(normalisation: $\text{Tr} \rho_A = 1$)

$$\rho_A \propto e^{-K_A}$$

Entanglement Hamiltonian (EH) K_A
(modular Hamiltonian)

- Entanglement Spectrum (ES): spectrum of the EH
- Entanglement Entropy (EE) $S_A \equiv -\text{Tr}_A(\rho_A \log \rho_A) = -\partial_n \text{Tr}_A(\rho_A^n) \big|_{n=1}$
- $\rho_A \propto e^{-K_A}$ generates the one-parameter family $\{\rho_A^{i\tau} : \tau \in \mathbb{R}\}$
of unitary operators providing an automorphism on the operator algebra
called *modular flow*

Motivation: Entanglement Entropy is not enough

Features of the bipartite entanglement

that are not captured by the entanglement entropy

Quantities based on
a single state

- Entanglement spectrum ←
- Entanglement entropy of disjoint regions
- Excited states
- Entanglement of purification
- ...

Quantities based on
two states

- Relative entropy
- Circuit complexity
- ...

The most natural way exploits the entanglement spectrum

Reduced
density matrix

$$\rho_A$$



Entanglement
spectrum

$$\{\lambda_j\}$$



Entanglement
entropy

$$S_A = - \sum_j \lambda_j \log \lambda_j$$

Motivation: Entanglement Entropy is not enough

Two features:

■ Entanglement entropy vs thermal entropy

E.g.: Entanglement entropy $\xrightarrow{\text{High temperature limit}}$ Thermal entropy

○ This talk: CFT_2 and AdS/CFT

■ Entanglement transitions

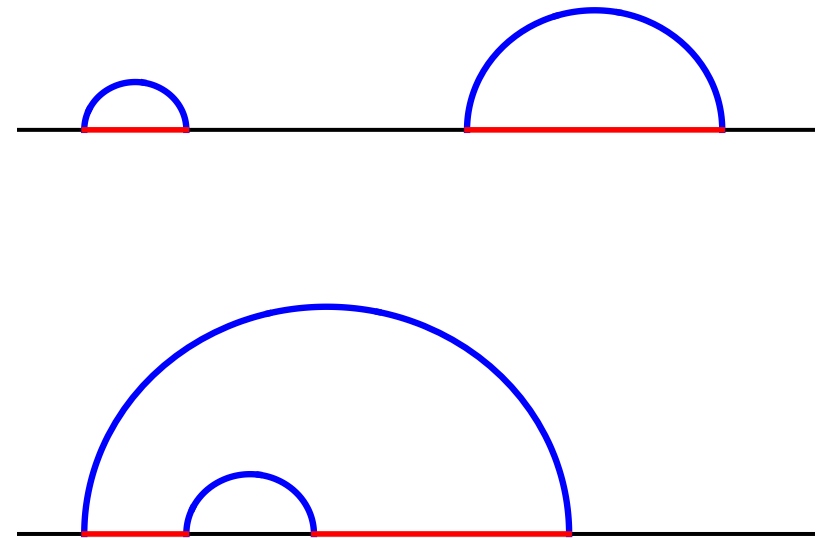
E.g.: AdS/CFT

Large c

Two disjoint regions

Critical distance

○ This talk: Harmonic chain



Outline

- ➔ Modular Hamiltonians of an interval in CFT_2
 - Ground state & thermal state
 - Contours for the entanglement entropies
- ➔ Modular conjugations in CFT_2 [[Mintchev, E.T. \(2022\)](#)]
 - Ground state & thermal state
 - Thermal entropy of an interval
- ➔ Modular conjugations & Holographic bit threads
 - AdS_3 and BTZ black brane [[Mintchev, E.T. \(2022\)](#)]
 - Schwarzschild AdS_{d+2} black brane
[[Caggioli, Gentile, Seminara, E.T. \(2024\)](#)]
- ➔ Entanglement Hamiltonian of two blocks in the harmonic chain
 - Transitions in the large mass regime
[[Gentile, Rotaru, E.T. \(2025\)](#)]

Modular Hamiltonian & Modular Conjugation

■ Von Neumann algebra $\mathcal{A}$ of localised observables acting on $\mathcal{H}$ which contains a cyclic and separating vector Ω

■ $S : \mathcal{A}\Omega \rightarrow \mathcal{A}\Omega$ linear $S \mathcal{O}\Omega = \mathcal{O}^*\Omega \quad \forall \mathcal{O} \in \mathcal{A}$

→ $S = J \Delta^{1/2}$ unique polar decomposition

$\Delta \equiv e^{-K}$ self-adjoint and positive
unbounded in general modular operator of $(\mathcal{A}, \Omega)$
 K (full) modular Hamiltonian

J anti-unitary modular conjugation of $(\mathcal{A}, \Omega)$

→ $\Delta\Omega = \Omega \quad J\Omega = \Omega \quad J = J^* = J^{-1}$

■ $\sigma_\tau \mathcal{O} \equiv \Delta^{-i\tau} \mathcal{O} \Delta^{i\tau} \quad \tau \in \mathbb{R}$ modular evolution (flow) of $\mathcal{O}$
generated by K

○ Tomita-Takesaki theorem $\Delta^{-i\tau} \mathcal{A} \Delta^{i\tau} = \mathcal{A}$
 $\Delta^{-i\tau} \mathcal{A}' \Delta^{i\tau} = \mathcal{A}'$ $J \mathcal{A} J = \mathcal{A}'$

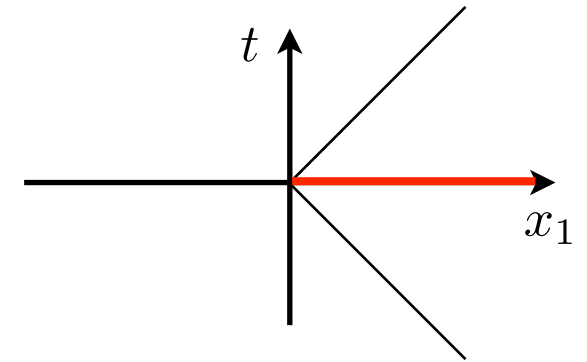
→ Typically, σ_τ and J do not have a geometric action

Modular conjugations in 2D CFT
&
Thermal entropy

EH in field theories: Bisognano-Wichmann & CFT

- QFT in its ground state, A is the (right) half-space $x_1 > 0$ in $\mathbb{R}^{1,d-1}$

$$K_A = \int_A x_1 T_{00} d^{d-1}x$$

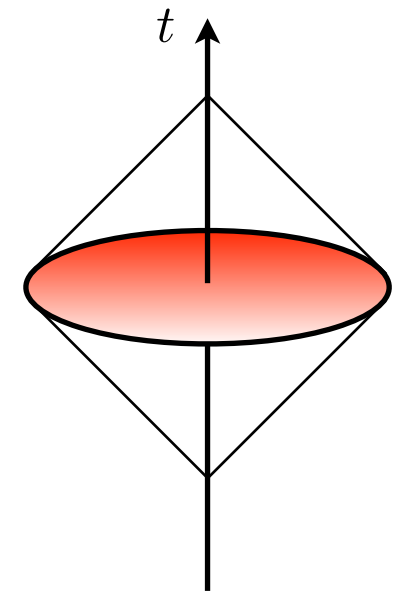


K_A is the generator of the Lorentz boosts
in the (right) Rindler wedge [Bisognano, Wichmann, (1975)]

- σ_τ and J have geometric action (Lorentz boost and reflection)

- CFT $_d$ and A is a ball of radius R

$$K_A = 2\pi \int_A \frac{R^2 - r^2}{2R} T_{00} d^{d-1}x$$



The Rindler wedge can be conformally mapped
into the causal diamond of A

[Hislop, Longo, (1982)] [Casini, Huerta, Myers, (2011)]

Modular Hamiltonians in 1+1 conformal field theories

- CFT₂ on the line in the ground state and $A = (0, \ell)$

$$K_A = 2\pi \int_0^\ell \frac{x(\ell - x)}{\ell} T_{00} dx$$

- Various static configurations where K_A is local and has the form
[Wong, Klich, Pando-Zayas, Vaman, (2013)]

$$K_A = 2\pi \int_A \beta_{\text{loc}}(x) T_{00} dx$$

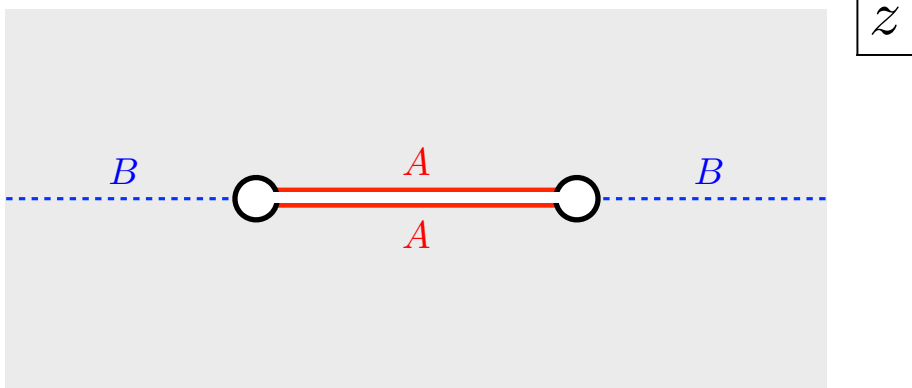
with the proper weight function $\beta_{\text{loc}}(x)$, which can be understood through a conformal map in BCFT [Cardy, E.T., (2016)]

- In many important cases K_A is non local (in different ways) and typically it is model dependent
 - The simplest case is K_A for a massless Dirac field on the line in the ground state, when A is the union of two disjoint intervals
[Casini, Huerta, (2009)] [Longo, Martinetti, Rehren, (2009)]

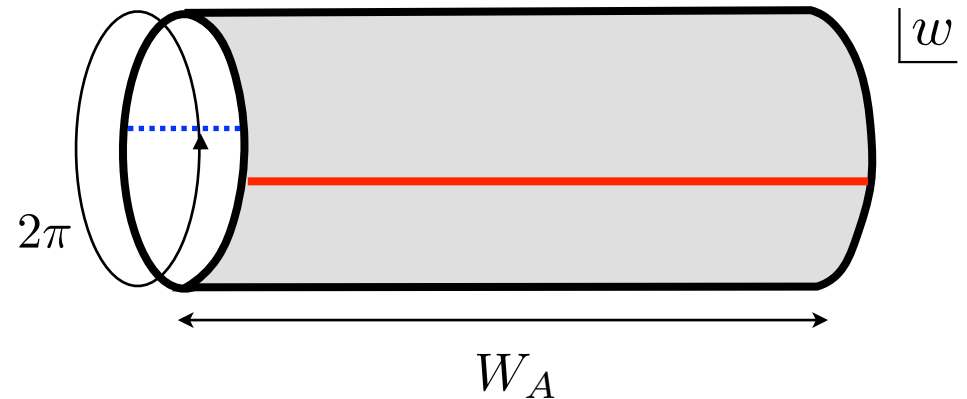
Mapping to the annulus: EE & Entanglement Spectrum

[Cardy, E.T., (2016)]

Reduced density matrix



Partition function on the annulus



■ Rényi entropies

$$\text{Tr} \rho_A^n = \frac{\mathcal{Z}_{n\text{-annulus}}}{(\mathcal{Z}_{\text{annulus}})^n}$$

Modular parameter of the annulus

$$q \equiv e^{-2\pi^2/W_A}$$

■ $W_A \gg 1$

$$S_A^{(n)} \sim \frac{c}{12} \left(1 + \frac{1}{n}\right) W_A + g_a + g_b$$

boundary entropies
 $g_{a,b} = -\log \langle a, b | 0 \rangle$

■ Entanglement spectrum

$\Rightarrow$

Entanglement gaps $\mathfrak{g}_r \equiv \log \lambda_{\max} - \log \lambda_r$

$$-\log \lambda_j = \frac{c}{12} W_A + \log(\langle a|0\rangle \langle 0|b\rangle) + \left(\Delta_j - \frac{c}{24}\right) \frac{2\pi^2}{W_A} + O(\epsilon^p)$$

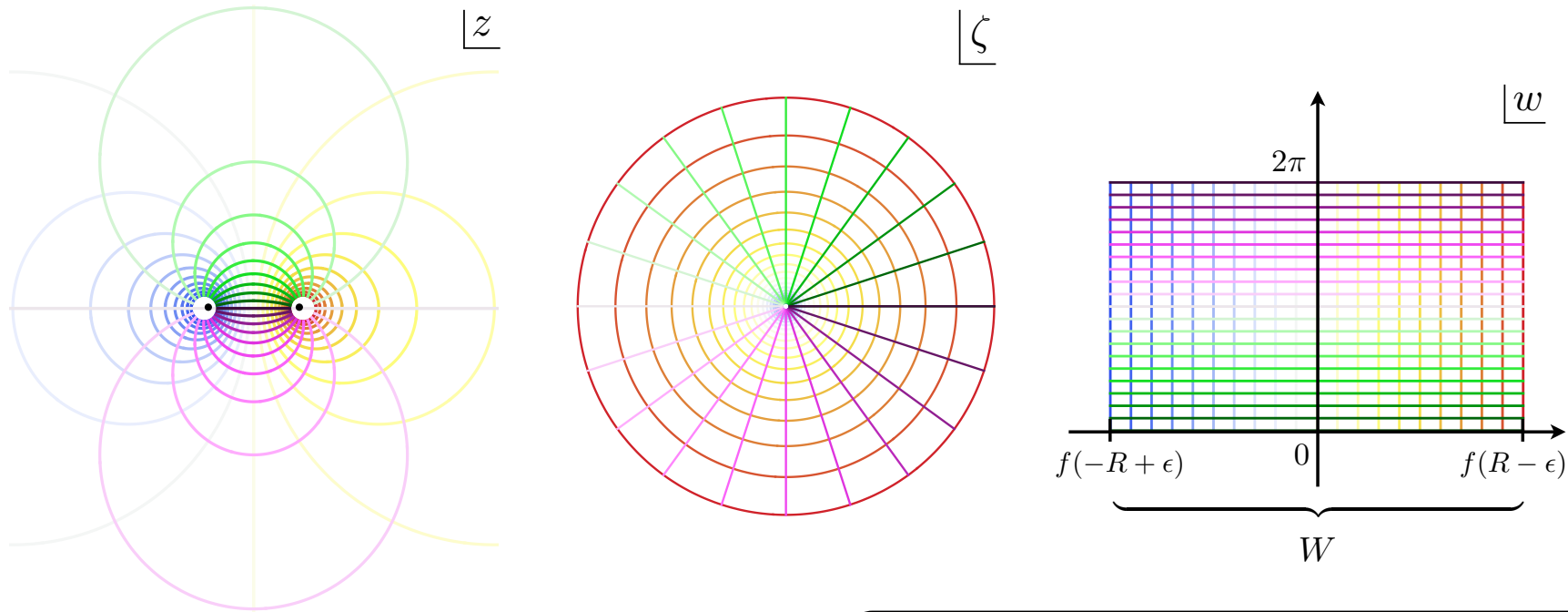
$$\frac{\mathfrak{g}_r}{\mathfrak{g}_1} = \frac{\Delta_r}{\Delta_1}$$

Δ_j conformal dimensions (primaries and descendants) of the BCFT

Entanglement hamiltonian of an interval in the line

[Läuchli, (2013)] [Wong, Klich, Pando-Zayas, Vaman, (2013)] [Ohmori, Tachikawa, (2015)] [Cardy, E.T., (2016)]

■ Ground state and single interval $A = (-R, +R)$ on the infinite line



$$w = f(z) = \log \zeta = \log \left(\frac{z + R}{R - z} \right)$$

$$K_A = \int_A \frac{R^2 - x^2}{2R} T_{00}(x) dx$$

$$S_A^{(n)} = \frac{c}{6} \left(1 + \frac{1}{n} \right) \log(\ell/\epsilon) + g_a + g_b + \text{corrections}$$

Entanglement hamiltonian in the thermal state

[Wong, Klich, Pando-Zayas, Vaman, (2013)] [Cardy, E.T., (2016)]

- Finite interval $(-R, R)$
in an infinite system
at finite temperature

$$f(z) = \log \left(\frac{e^{2\pi z/\beta} - e^{-2\pi R/\beta}}{e^{2\pi R/\beta} - e^{2\pi z/\beta}} \right)$$

$$K_A = \frac{\beta}{\pi} \int_A \frac{\sinh[\pi(R-x)/\beta] \sinh[\pi(x+R)/\beta]}{\sinh(2\pi R/\beta)} T_{00}(x) dx$$

$$S_A^{(n)} = \frac{c}{6} \left(1 + \frac{1}{n} \right) \log \left(\frac{\beta}{\pi\epsilon} \sinh(\pi\ell/\beta) \right) + g_a + g_b + \text{corrections}$$

➔ Entanglement entropy at high temperature [Calabrese, Cardy, (2004)]

$$S_A = \frac{c}{3} \log \left(\frac{\beta}{\pi\epsilon} \sinh(\pi\ell/\beta) \right) \longrightarrow S_A \simeq \frac{\pi c}{3\beta} \ell$$

Gibbs entropy in CFT_2 of an isolated system
[Affleck, (1986)] [Blöte, Cardy, Nightingale, (1986)]

$$s_{\text{th}} = \frac{\pi c}{3\beta}$$

Contours for the entanglement entropies

- The contour function for $S_A^{(n)}$ is a map $s_A^{(n)} : A \rightarrow \mathbb{R}$ such that

$$S_A^{(n)} = \sum_{i \in A} s_A^{(n)}(i)$$

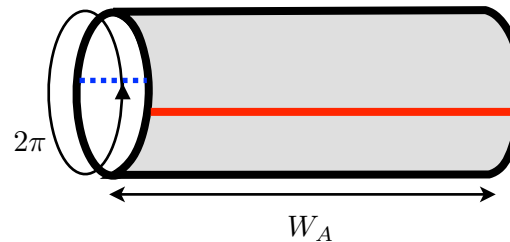
$$s_A^{(n)}(i) \geq 0$$

- ➔ This $s_A^{(n)}(i)$ is just a density for $S_A^{(n)}$. Further constraints should be imposed.
- ➔ The entanglement contour should provide information about the spatial structure of entanglement. [Botero, Reznik, (2004)] [Chen, Vidal, (2014)]

- Density of entanglement entropy constructed through a differential operator acting on S_A . [Bhattacharya, Hubeny, Rangamani, Takayanagi, (2017)]

- The entanglement hamiltonians in 2D CFTs suggest a function for the contour in the continuum. [Wong, Klich, Pando-Zayas, Vaman, (2013)]
[Cardy, E.T., (2016)] [Coser, De Nobili, E.T., (2017)]

$$K_A = \int_A \frac{T_{00}(x)}{f'(x)} dx$$



$$S_A^{(n)} = \frac{c}{12} \left(1 + \frac{1}{n} \right) W_A + C$$

$$W_A = \int_{A_\epsilon} f'(x) dx$$

$\Rightarrow$

$$s_A^{(n)}(x) = \frac{c}{12} \left(1 + \frac{1}{n} \right) f'(x) + \frac{C}{\ell}$$

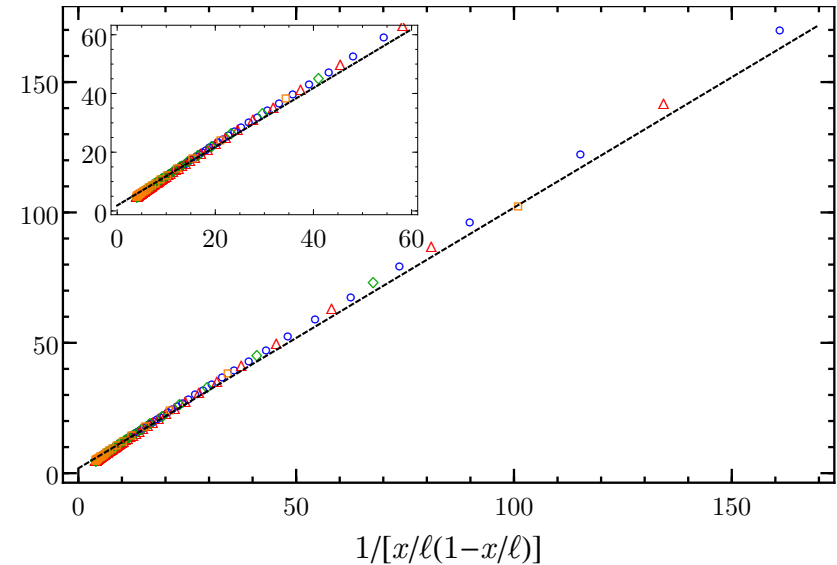
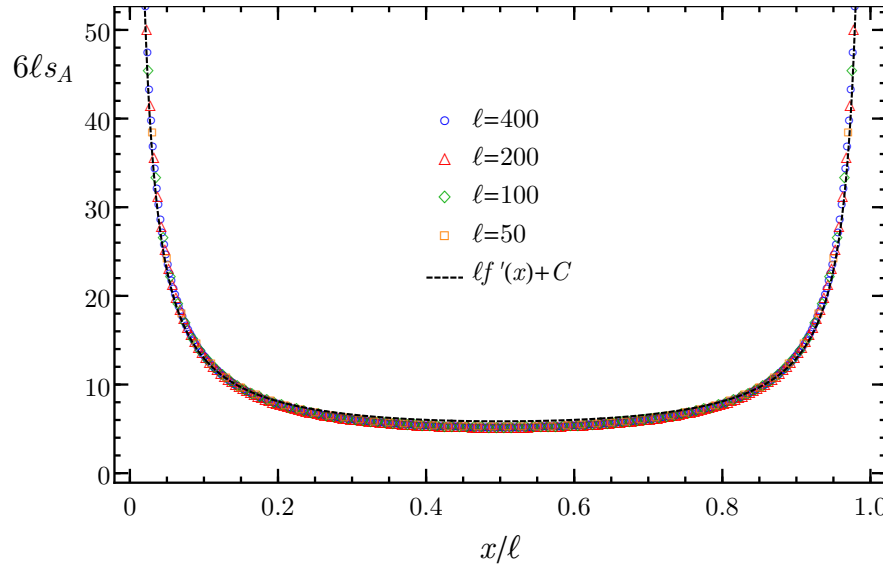
Contour function: interval in the infinite harmonic chain

[Coser, De Nobili, E.T. (2017)]

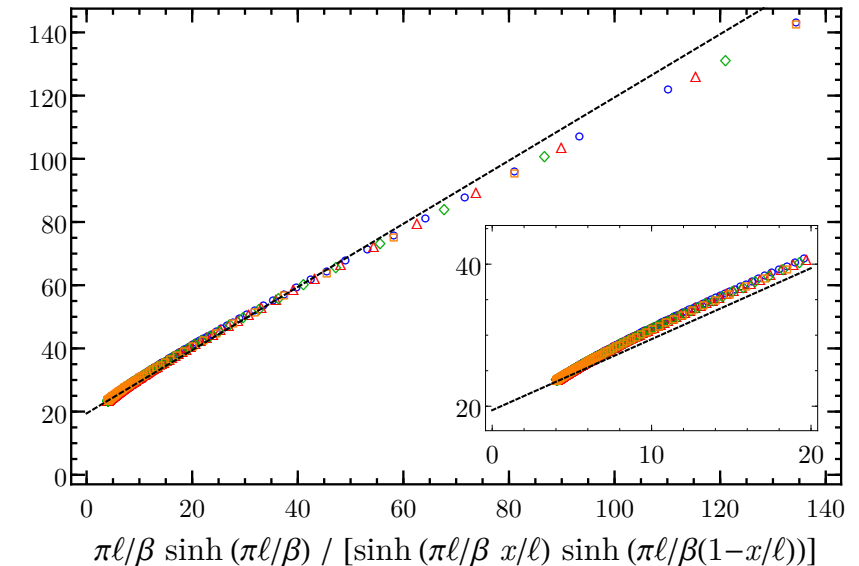
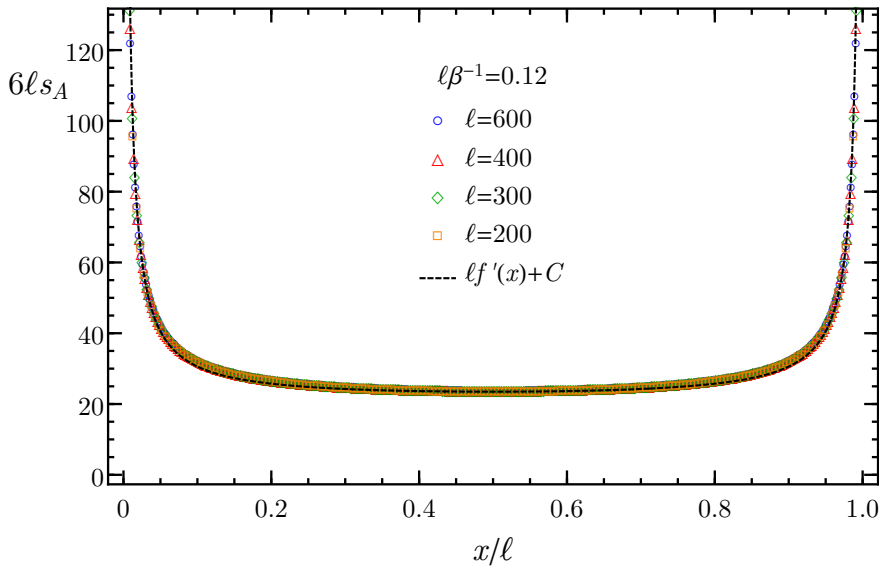
A

Translation invariance $\Rightarrow \omega \neq 0$. Massless regime ($\omega\ell = 4 \times 10^{-4}$)

$T = 0$

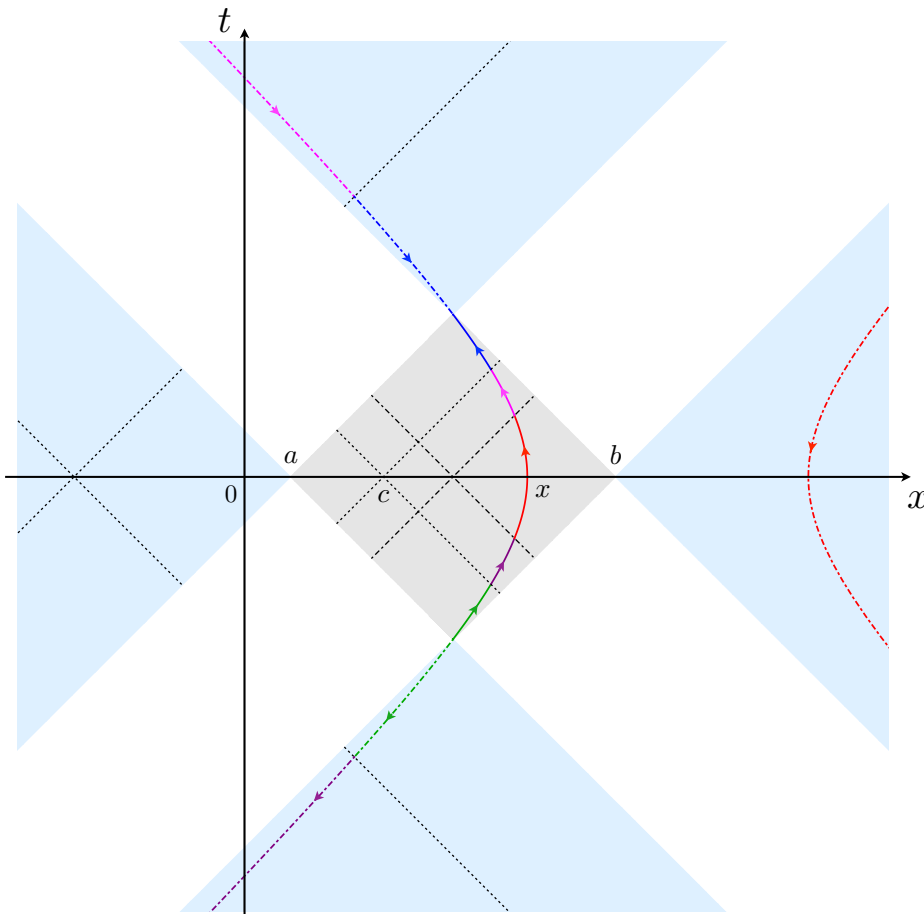


$T > 0$

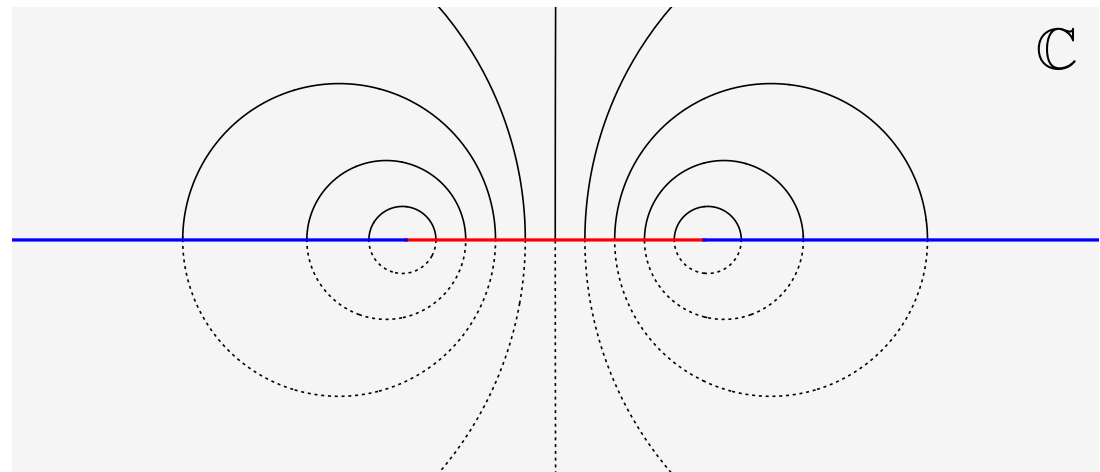


Geometric actions of the modular objects

- Bisognano-Wichmann theorem: σ_τ and J have geometric action
[Bisognano, Wichmann, (1975)] (Lorentz boost and reflection)
- CFT in Minkowski spacetime and in its ground state
 bipartite by a sphere (Hislop-Longo theorem) [Hislop, Longo, (1982)]



○ For an interval the geometric actions of σ_τ and J are known



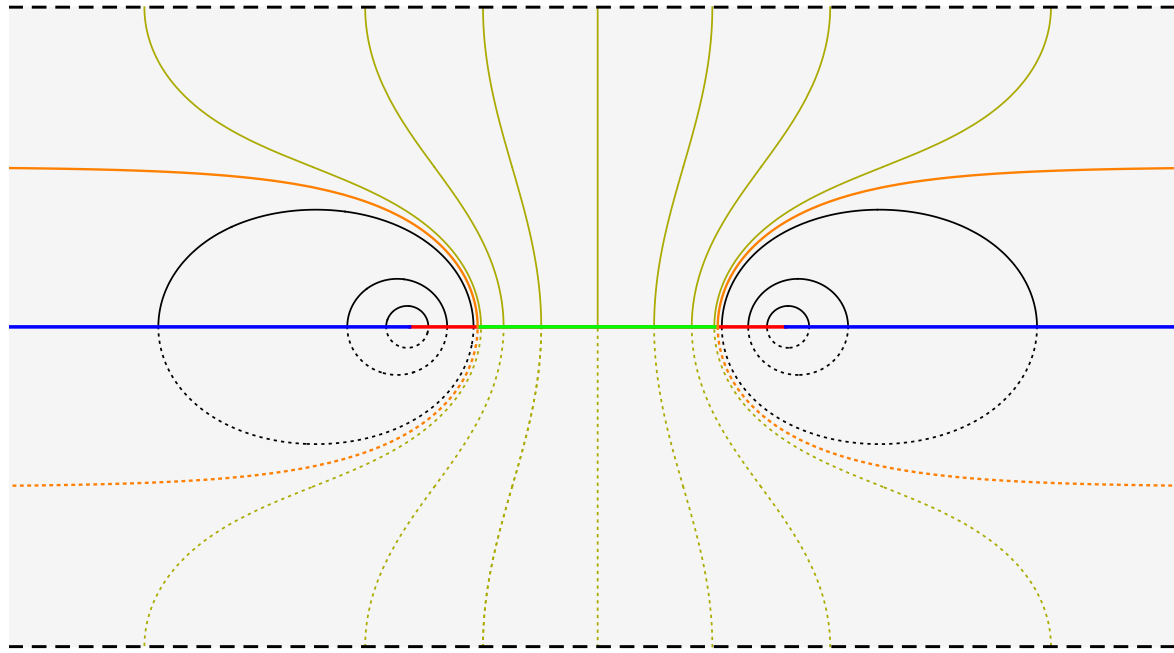
At $t = 0$

$$A = (-b, b)$$

$$x_{\text{inv}} = \frac{b^2}{x} \quad x \in A$$

Geometric actions of the modular objects

[Mintchev, E.T. (2022)]



→ At finite temperature, the modular trajectories in the Euclidean provide the interval $A_\beta \subsetneq A$ mapped on the other side of the cylinder (second copy of the spatial section)

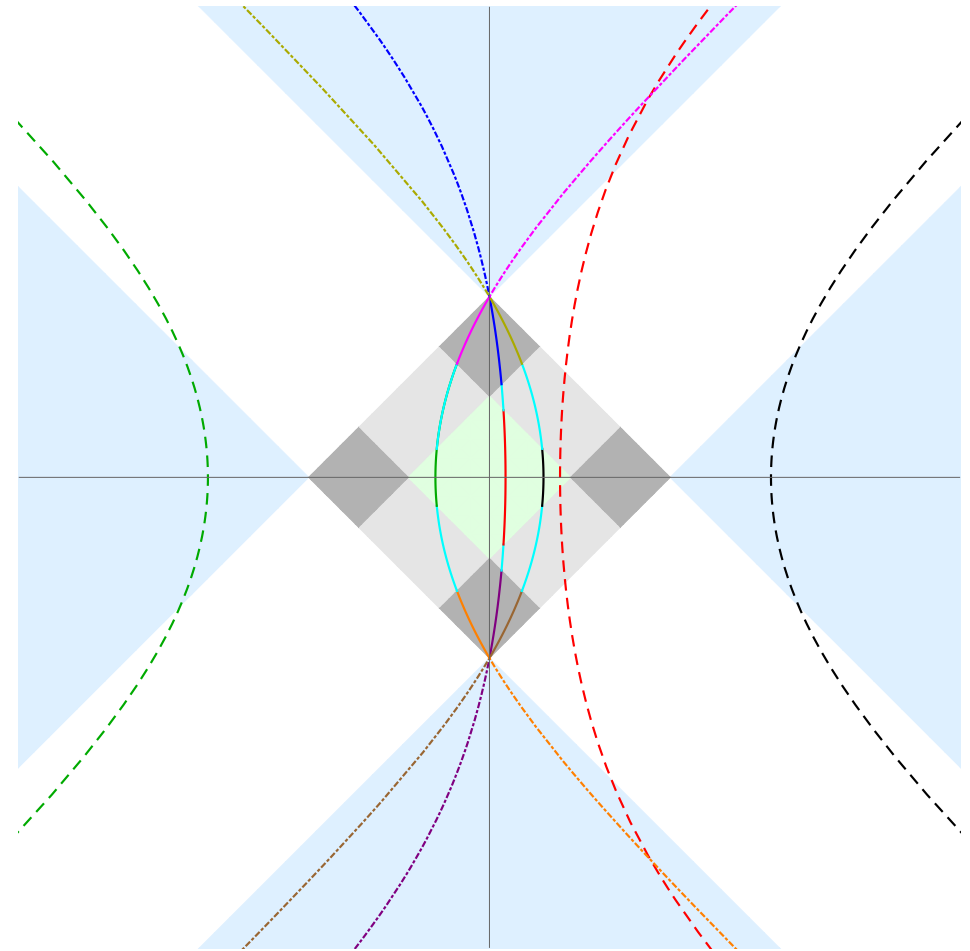
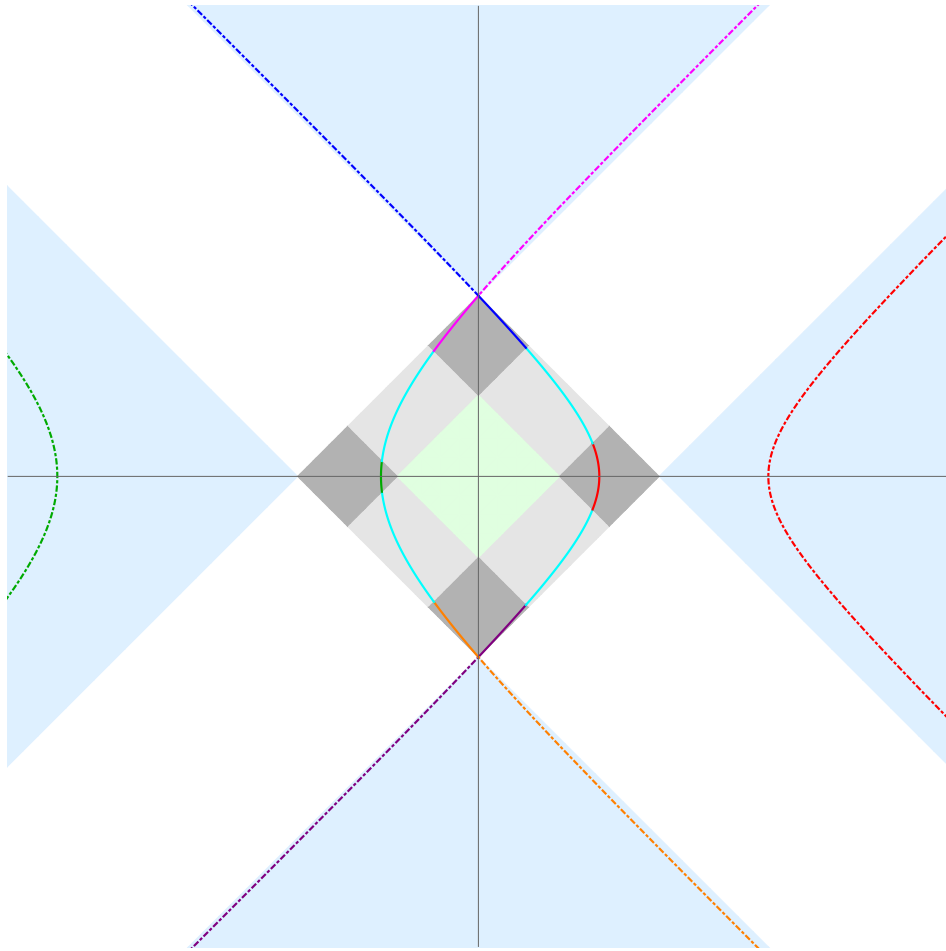
→ The contour integrated on A_β gives the thermal entropy for any size of the interval A

$$\int_{A_\beta} \mathcal{C}_A(x) dx = s_{\text{th}} \ell$$

$$s_{\text{th}} = \frac{\pi c}{3\beta}$$

Geometric actions of the modular objects

- Modular trajectories in Minkowski spacetime and their images under modular conjugation



Modular conjugations in CFT
&
Holographic bit threads

Holographic Entanglement Entropy

- Constant time slice in AdS_{d+2}
Hypersurfaces γ_A s.t. $\partial\gamma_A = \partial A$
Find the minimal area surface $\hat{\gamma}_A$

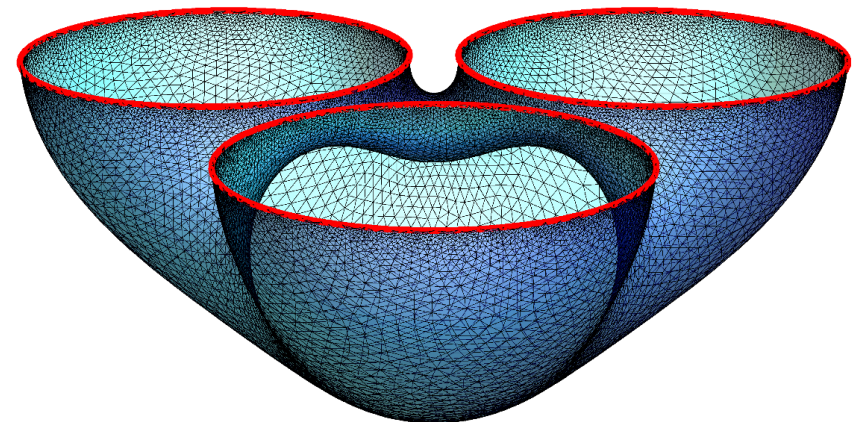
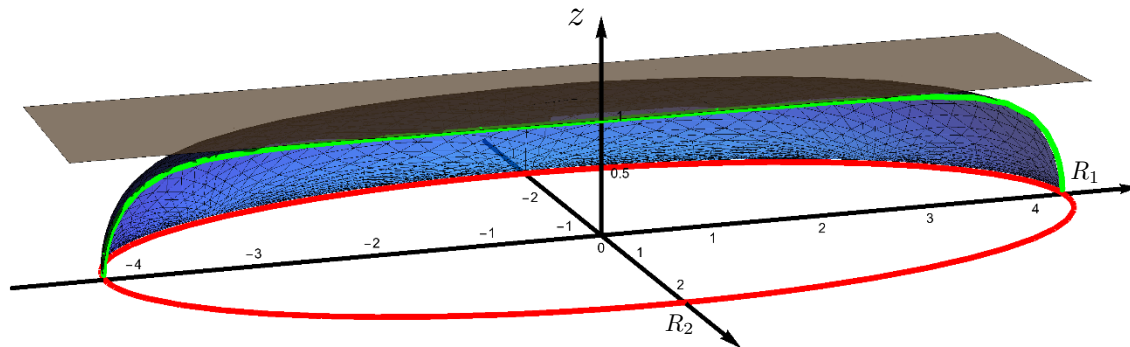
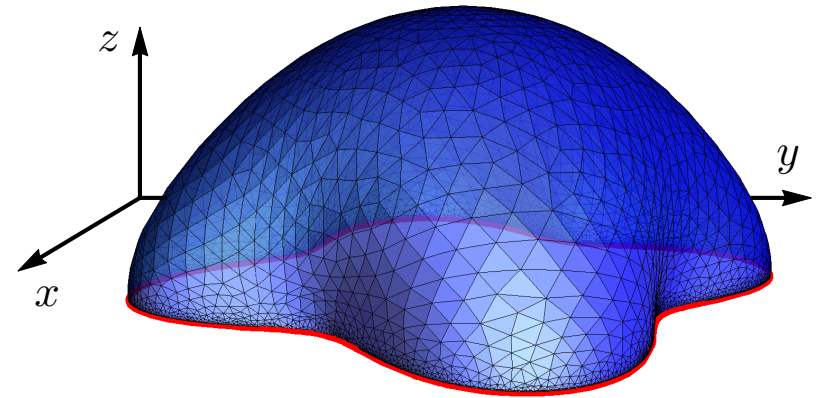
[Ryu, Takayanagi, (2006)]

$$S_A = \frac{\text{Area}(\hat{\gamma}_\varepsilon)}{4G_N^{(d+2)}}$$

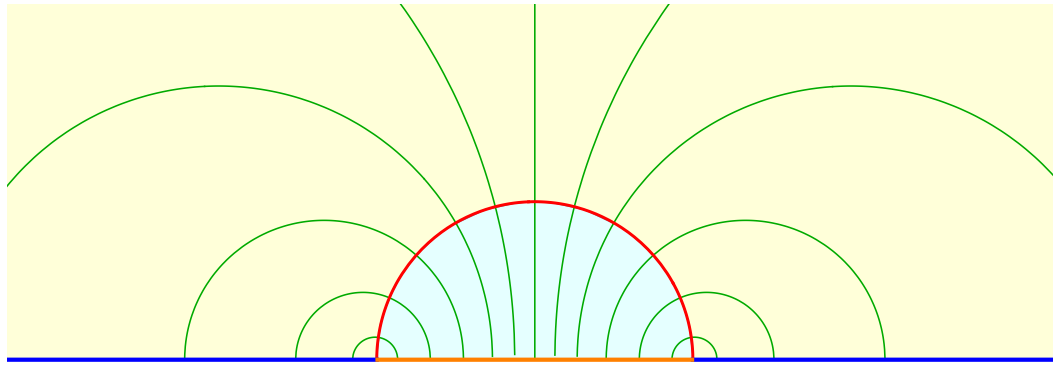
- Various non trivial checks. E.g. strong subadditivity [Headrick, Takayanagi, (2007)]

- E.g.: Asymptotically AdS_4 geometries

$$\mathcal{A}[\hat{\gamma}_\varepsilon] = \frac{P_A}{\varepsilon} - F_A + o(1)$$



Holographic bit threads



→ while $\mathbf{V}$ is fully determined on the RT hypersurface (where $|\mathbf{V}| = 1$) it is highly degenerate on A

→ Gravitational dual of the contour function $\mathcal{C}_A(\mathbf{x})$ for S_A
[E.T. (2018)]
 [Kudler-Flam, MacCormack, Ryu (2019)]

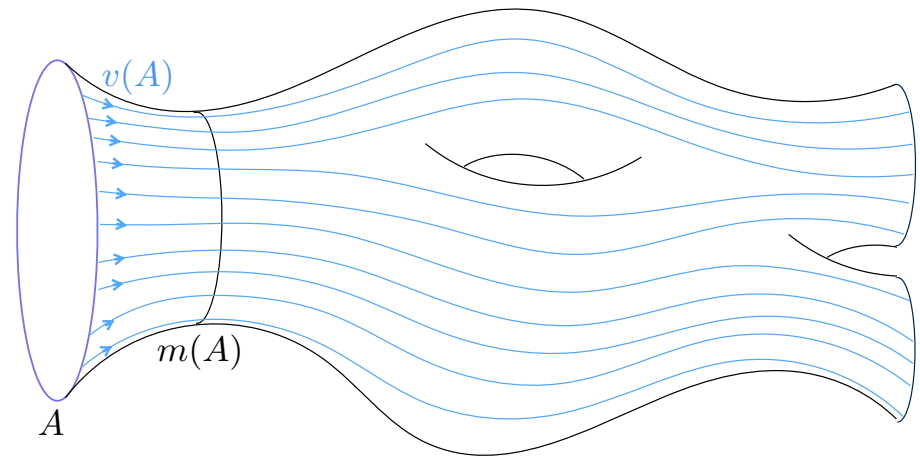
$$S_A = \int_A \mathcal{C}_A(\mathbf{x}) d^d \mathbf{x}$$

■ Some explicit constructions have been explored [Agon, de Boer, Pedraza (2018)]
 (e.g. *geodesic* bit threads, maximally packed bit threads, etc.)

Bit threads formulation of the RT proposal [Freedman, Headrick (2016)]

$$\mathcal{V} \equiv \left\{ \mathbf{V} \mid \nabla_\mu V^\mu = 0 \text{ and } |\mathbf{V}| \leq 1 \right\}$$

$$S_A = \frac{1}{4G_N} \max_{\mathbf{V} \in \mathcal{V}} \int_A \mathbf{V} \cdot \mathbf{n} \sqrt{h} d^d s$$



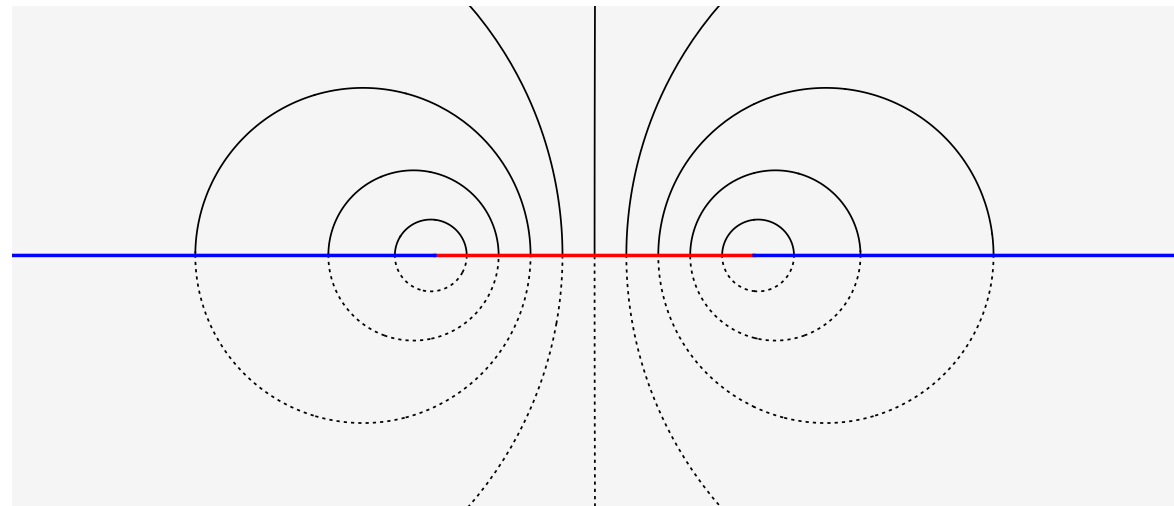
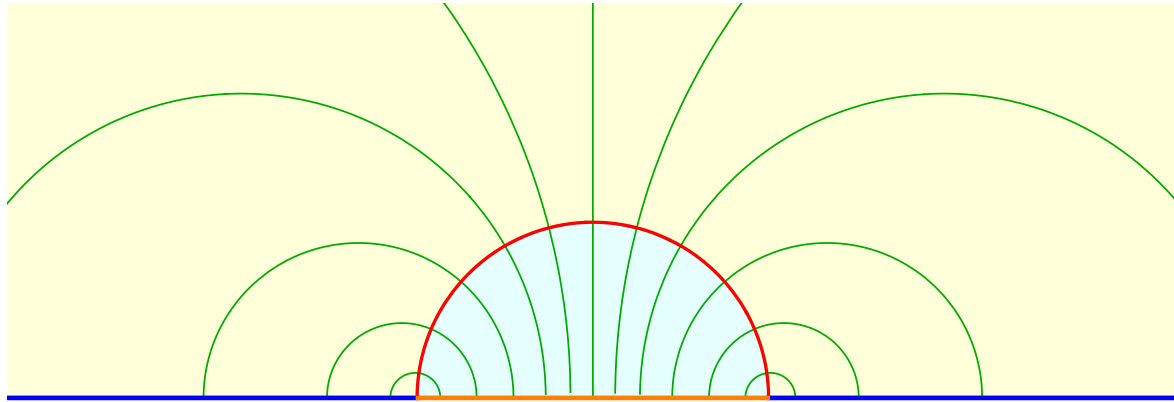
Geodesic bit threads & Modular conjugation: $AdS(3)$

- Consider the case where the threads are *geodesics*
[Agon, de Boer, Pedraza (2018)]
- The geodesic bit threads provide the map from A to B characterising the geometric action of J
[Mintchev, E.T. (2022)]

➔ For $t = 0$ and $A = (-b, b)$

$x \in A$

$$x_{\text{inv}} = \frac{b^2}{x}$$



Geodesic bit threads & Modular conjugation: BTZ brane

[Mintchev, E.T. (2022)]

At finite temperature $T = 1/\beta$

$$ds^2 = \frac{L_{\text{AdS}}^2}{z^2} \left(\frac{dz^2}{1 - (z/z_h)^2} + dx^2 \right)$$

Geodesic bit threads implement the geometric action of J obtained in 2D CFT at finite T

Geodesic bit threads provide A_β and their flow through A_β gives the thermal entropy of A

$$\int_{A_\beta} \mathcal{C}_A(x) dx = s_{\text{th}} \ell$$

$$s_{\text{th}} = \frac{\pi c_{\text{BH}}}{3\beta}$$

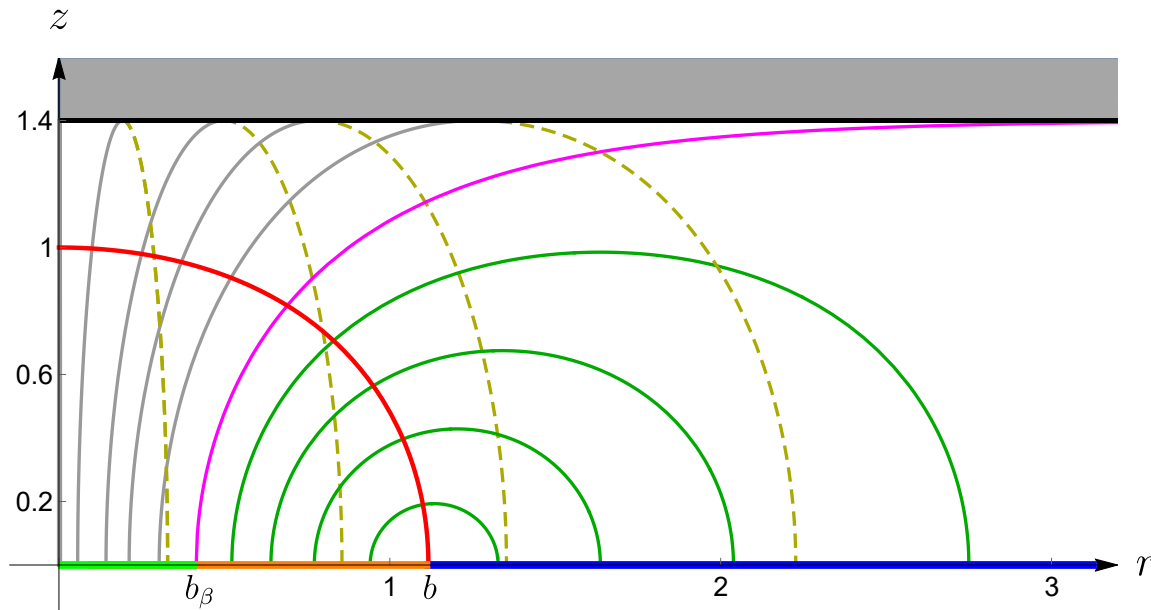
Geodesic bit threads & Modular conjugation: BTZ brane



- Geodesic bit threads map A_β onto the horizon
- Geodesic bit threads map A_β onto the second copy of the CFT by employing auxiliary geodesics

Sphere and Schwarzschild $AdS(d+2)$ black brane

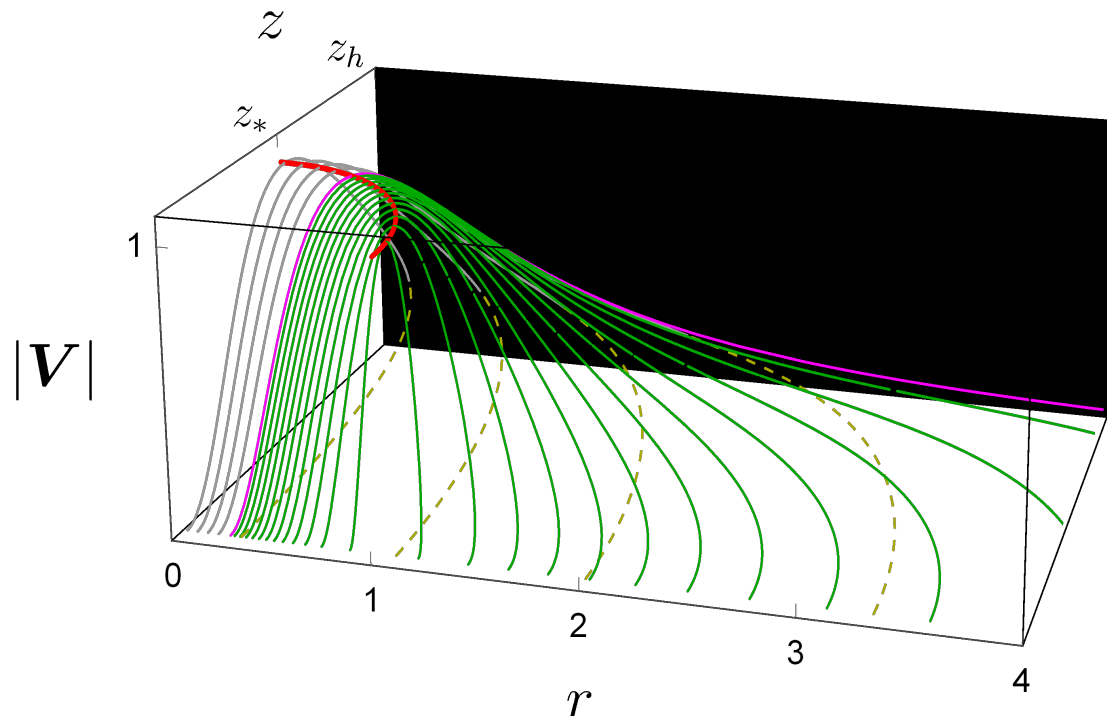
[Caggioli, Gentile, Seminara, E.T. (2024)]



Sphere $A \subset \mathbb{R}^d$ with radius b and

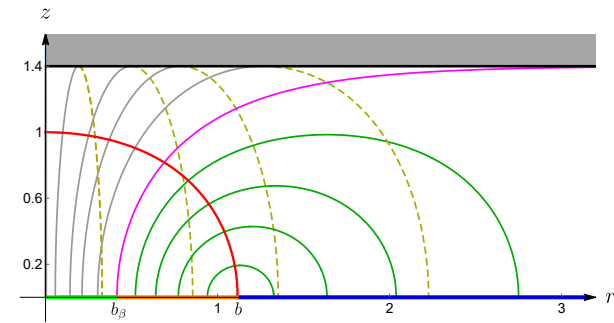
$$ds^2 = \frac{L_{\text{AdS}}^2}{z^2} \left(\frac{dz^2}{1 - (z/z_h)^{d+1}} + d\mathbf{x}^2 \right)$$

→ Numerical analysis
(RT hypersurface unknown analytically)



→ Geodesic bit threads
give a candidate for the
geometric action of J
(not known in CFT_d)

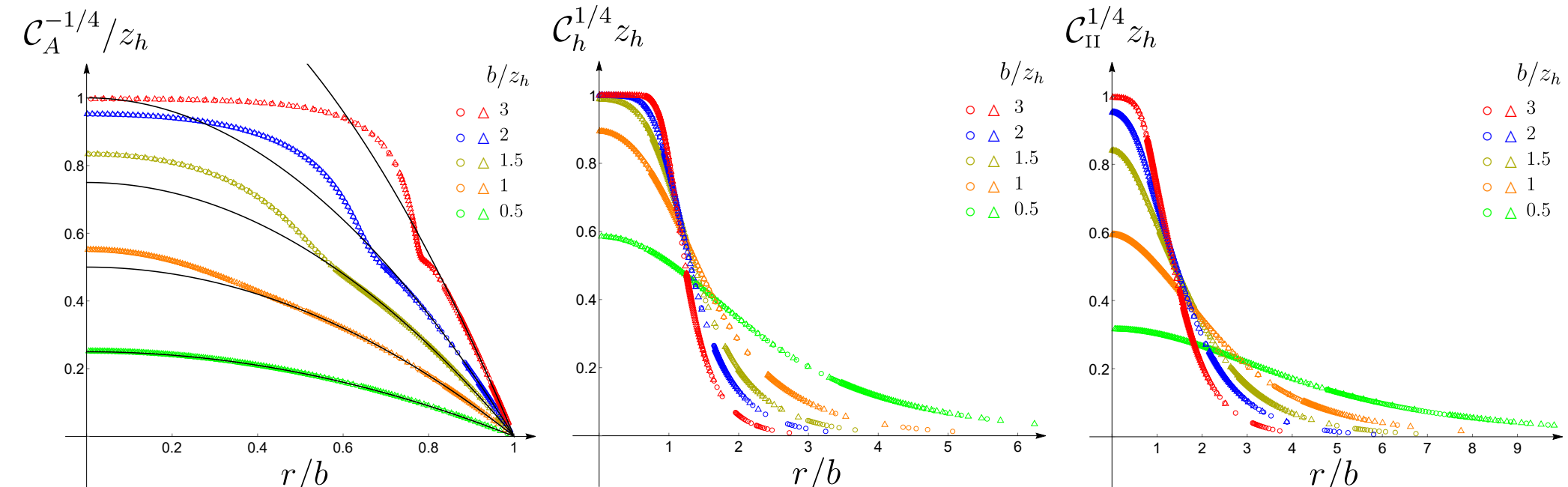
Holographic thermal entropy from geodesic bit threads



The holographic thermal entropy of A is obtained from the flux of the geodesic bit threads either through A_β or through the horizon, or of the auxiliary geodesics through the (second copy of the) entire boundary

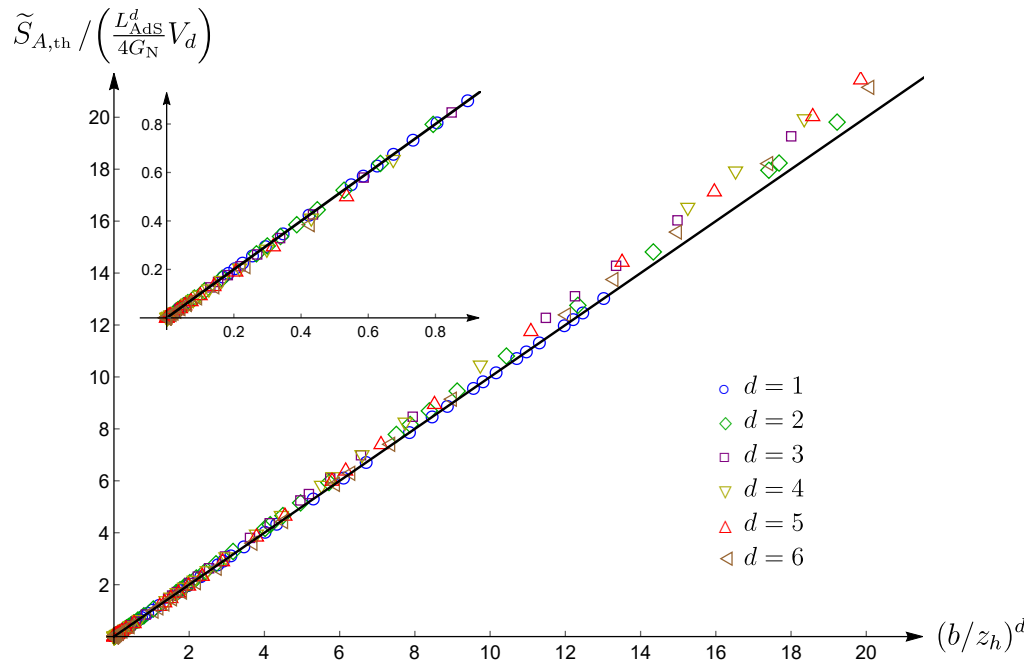
$$S_{A,\text{th}} = s_{\text{th}} \text{Vol}(A) \quad s_{\text{th}} = \frac{1}{4G_{\text{N}}} \frac{L_{\text{AdS}}^d}{z_h^d} \quad \text{Vol}(A) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} b^d$$

$$\tilde{S}_{A,\text{th}} = \frac{1}{4G_{\text{N}}} \int_{\tilde{\gamma}_{A,\beta}} \sqrt{h} \, d^d \mathbf{r} = \int_{A_\beta} \mathcal{C}_A(\mathbf{r}) \, d^d \mathbf{r} = \int_{\mathbb{R}^d} \mathcal{C}_h(\mathbf{r}) \, d^d \mathbf{r} = \int_{\mathbb{R}^d} \mathcal{C}_{\text{II}}(\mathbf{r}) \, d^d \mathbf{r}$$

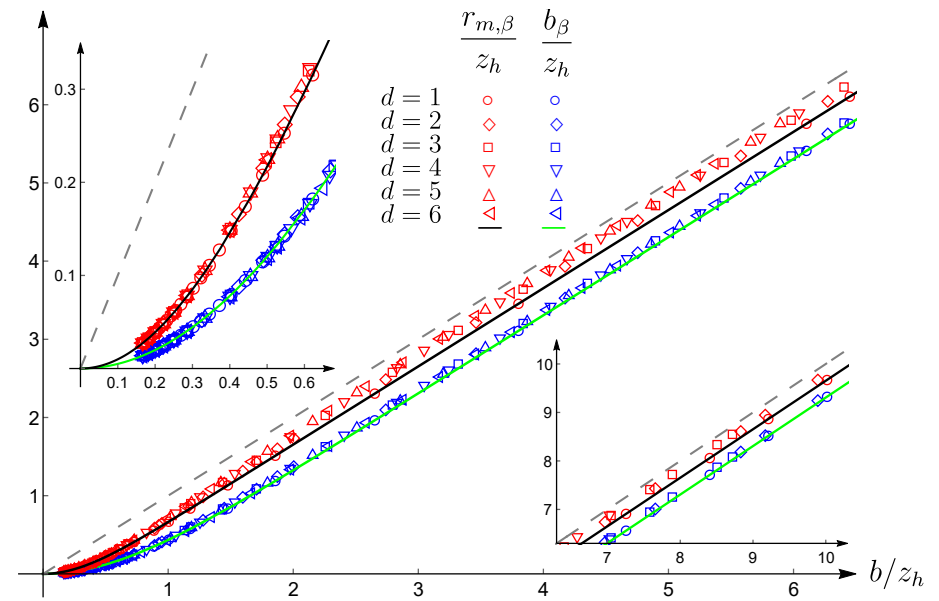


Holographic thermal entropy from geodesic bit threads

Holographic thermal entropy

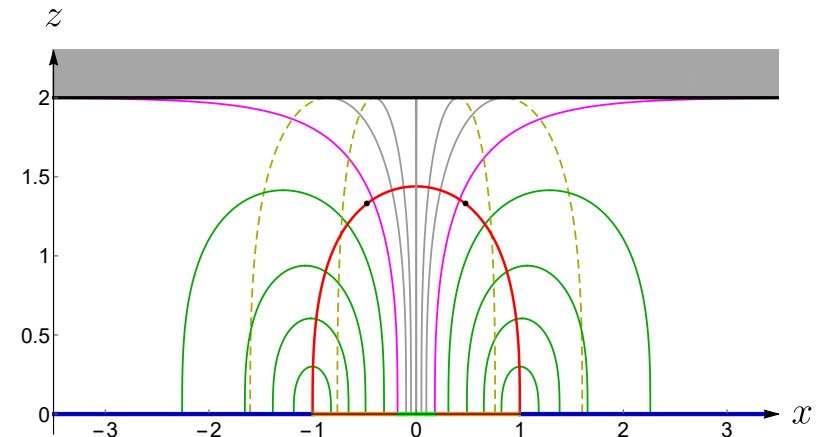


Size of A_β

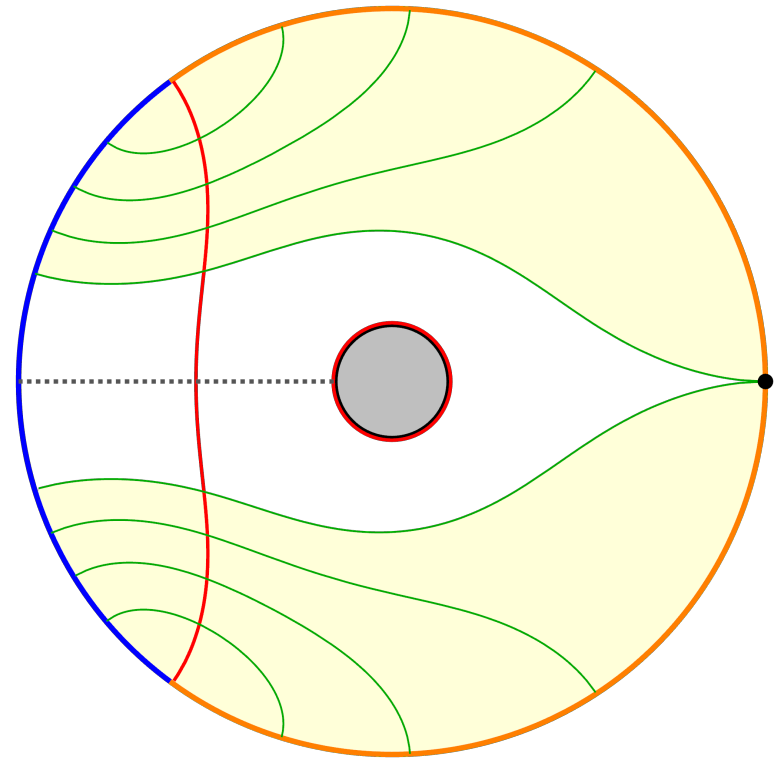
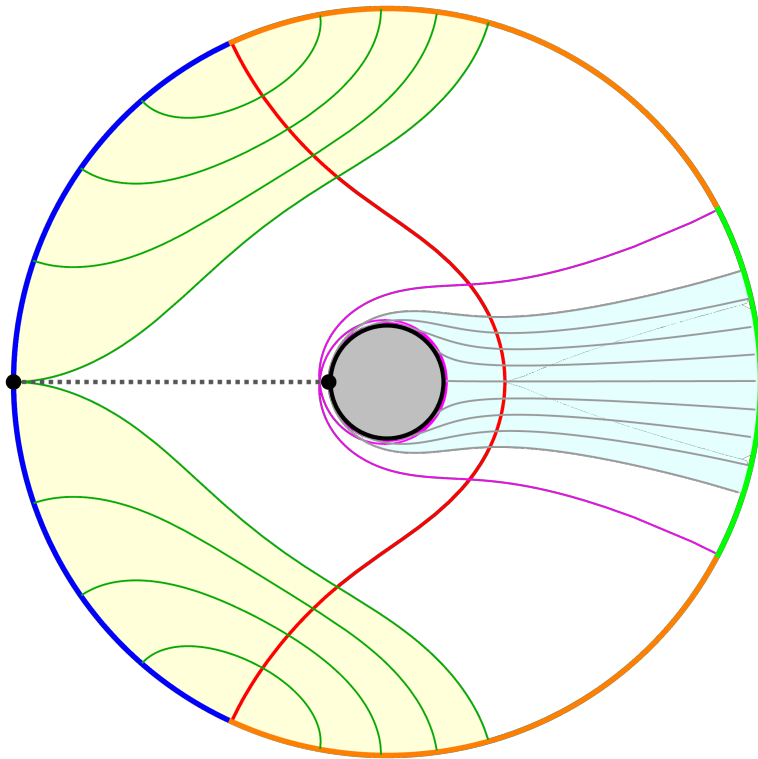


Other constructions of bit threads have been explored and the relation $\tilde{S}_{A,\text{th}} = S_{A,\text{th}}$ has been observed only for the geodesic bit threads

Strip: For geodesic bit threads and another type of bit thread construction we found $\tilde{S}_{A,\text{th}} \neq S_{A,\text{th}}$



Finite temperature and finite volume: BTZ black hole



- In the BTZ black hole the geodesics do not provide well defined bit threads
- In order to recover the thermal entropy of the arc A of width $\phi_A < \phi_c$ geodesics with all windings numbers around the horizon must be considered

Entanglement Hamiltonians
in the harmonic chain

Harmonic chain in the ground state: EH

□ Harmonic chain (periodic b.c.)

$$\hat{H} = \sum_{i=0}^{L-1} \left(\frac{1}{2m} \hat{p}_i^2 + \frac{m\omega^2}{2} \hat{q}_i^2 + \frac{\kappa}{2} (\hat{q}_{i+1} - \hat{q}_i)^2 \right) \quad \hat{\mathbf{r}} = \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}$$

Consider a Gaussian state (e.g. the ground state)

$$\hat{K}_A = \frac{1}{2} \hat{\mathbf{r}}^t H_A \hat{\mathbf{r}}$$

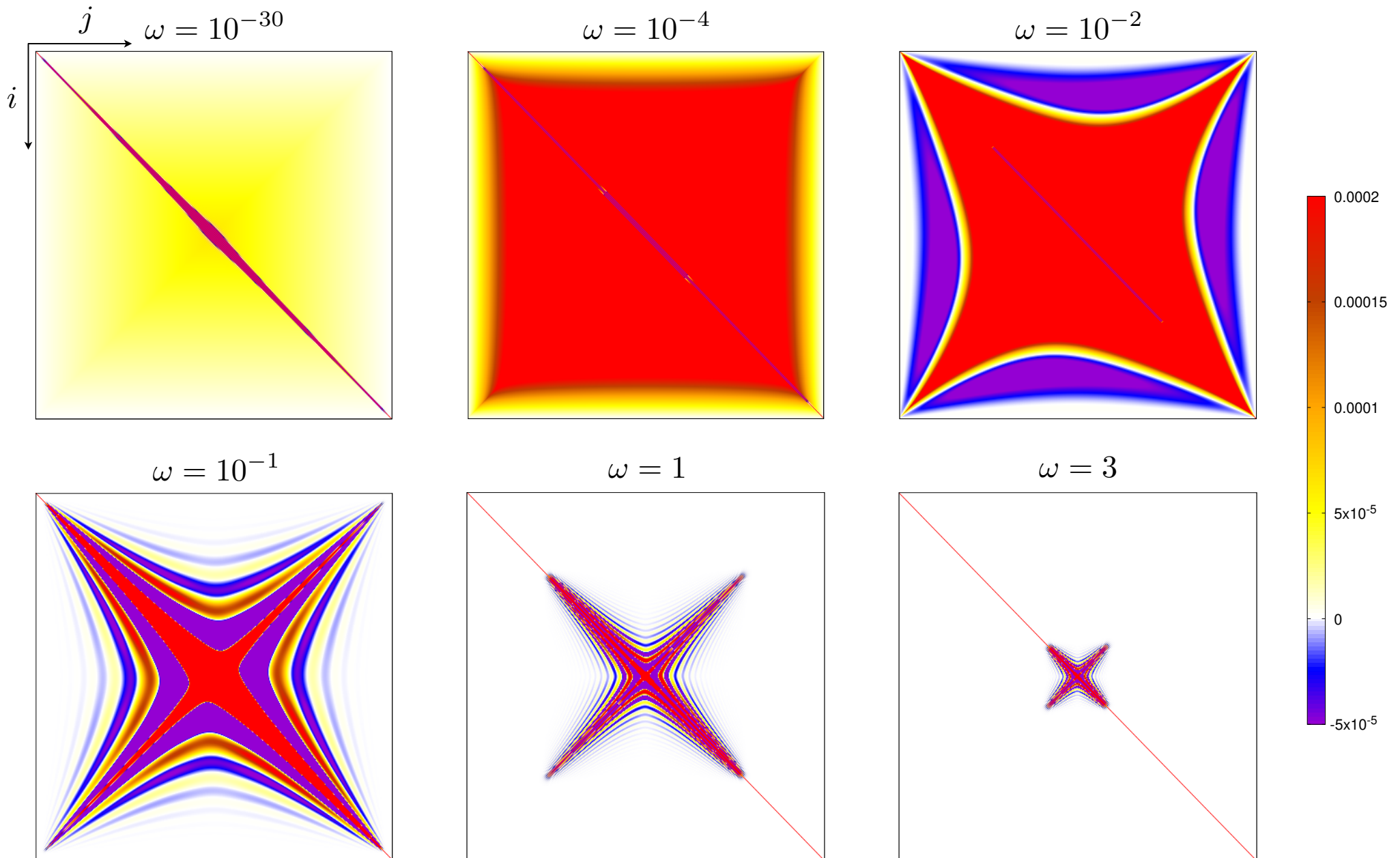
Gaussianity and Wick theorem allow to write the EH matrix H_A in terms of the reduced covariance matrix $\gamma_A \equiv \text{Re}\langle \hat{\mathbf{r}} \hat{\mathbf{r}}^t \rangle|_A = Q_A \oplus P_A$
[Peschel, (2003)] [Casini, Huerta, (2009)]

$$H_A = V \oplus T = \left(h(\sqrt{P_A Q_A}) \oplus h(\sqrt{Q_A P_A}) \right) (P_A + Q_A)$$

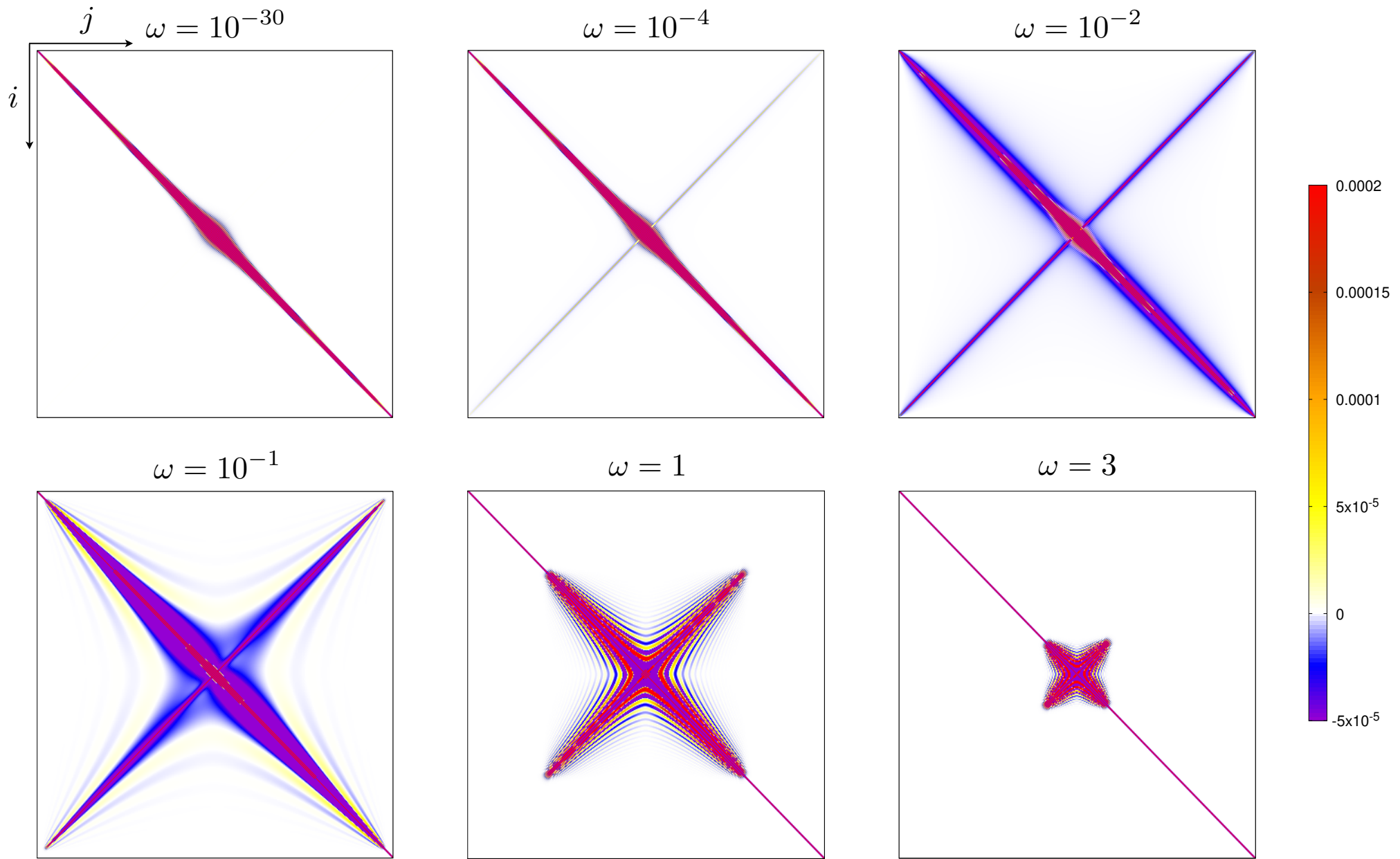
$$h(y) \equiv \frac{1}{y} \log \left(\frac{y + 1/2}{y - 1/2} \right)$$

EH of two a single block varying mass: Matrix T

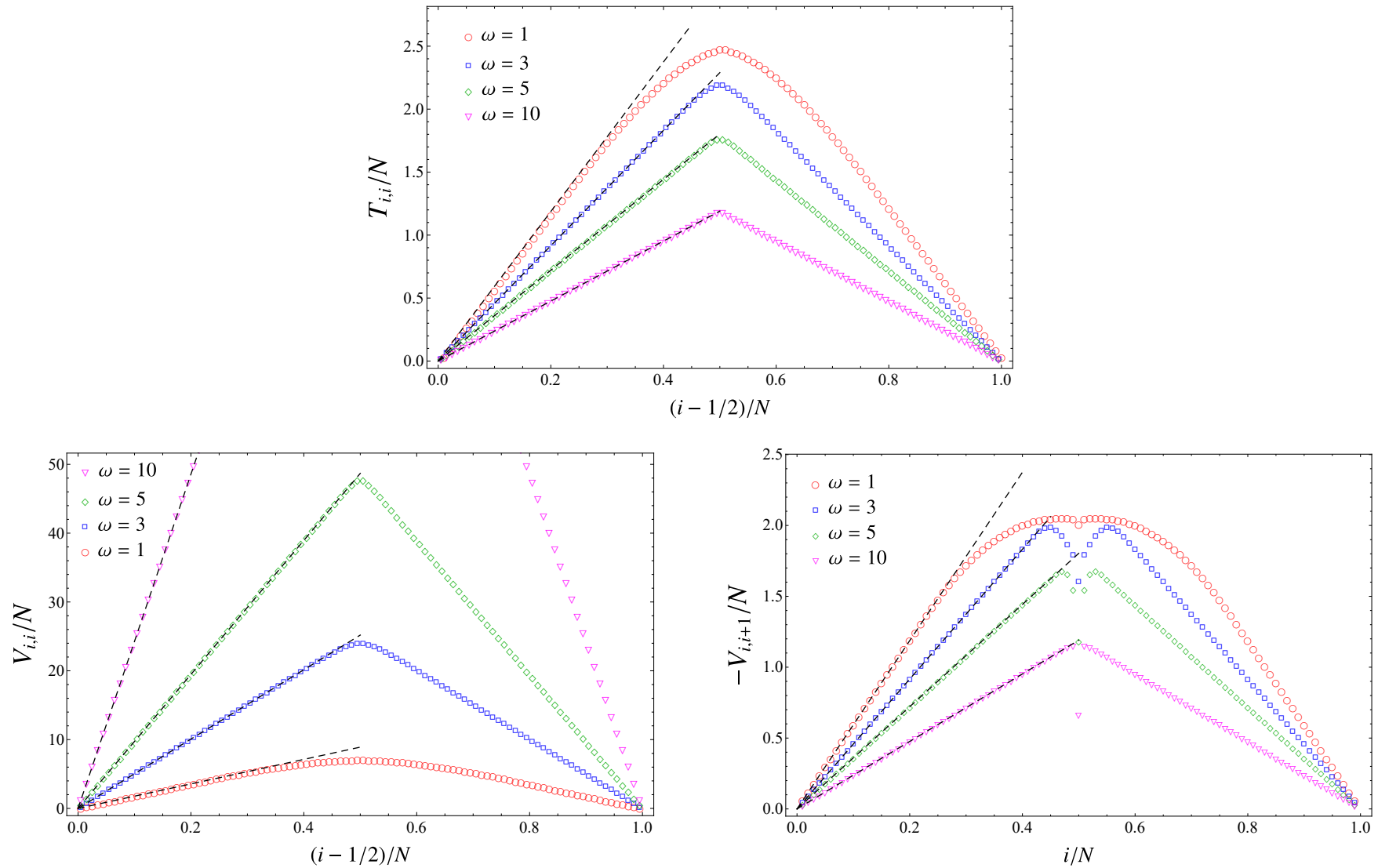
[Eisler, Di Giulio, E.T., Peschel (2020)]



EH of two a single block varying mass: Matrix V



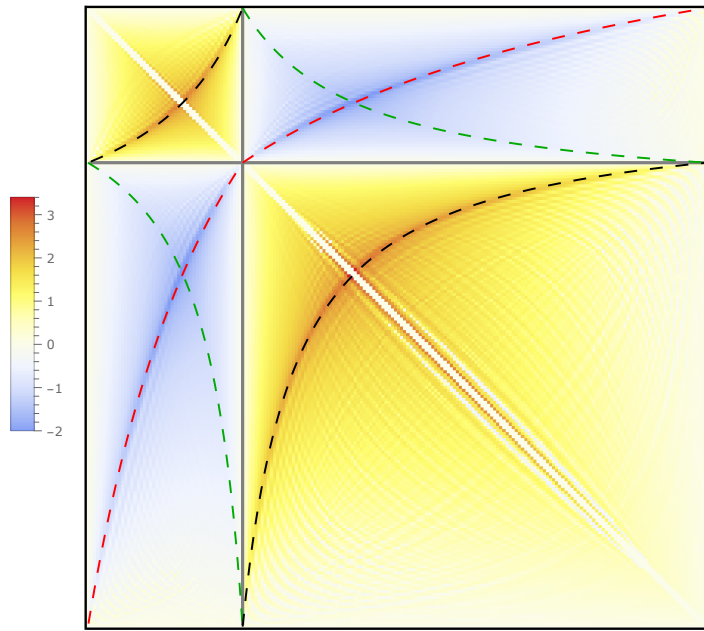
EH of two a single block varying mass: Three diagonals



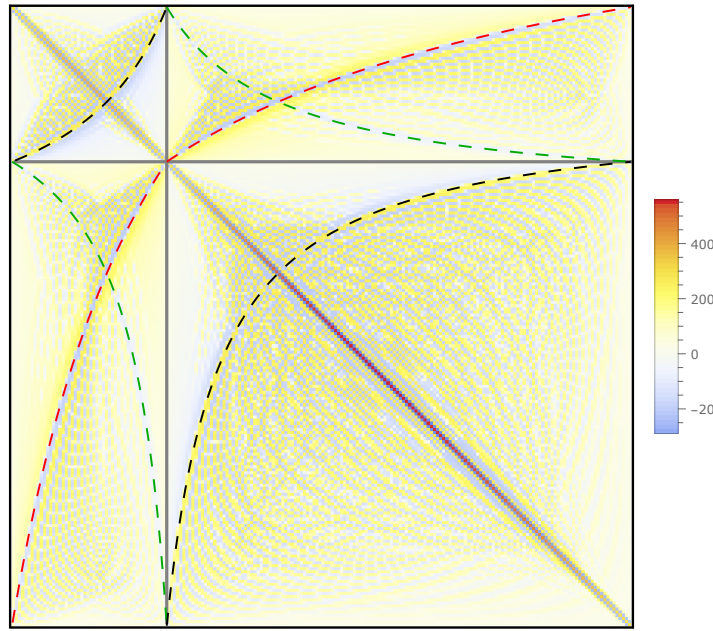
- Three diagonal approximation: only $T_{i,i}$, $V_{i,i}$ and $V_{i,i+1}$ are non vanishing
- Here the slopes are known from the half-chain case (corner transfer matrix)

EH of two disjoint unequal blocks: Massless regime

[Gentile, Rotaru, E.T. (2025)]



matrix T



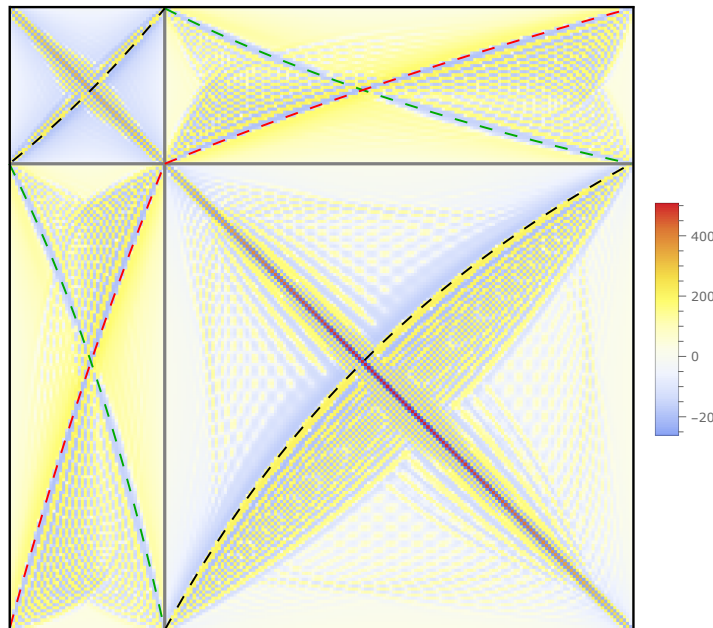
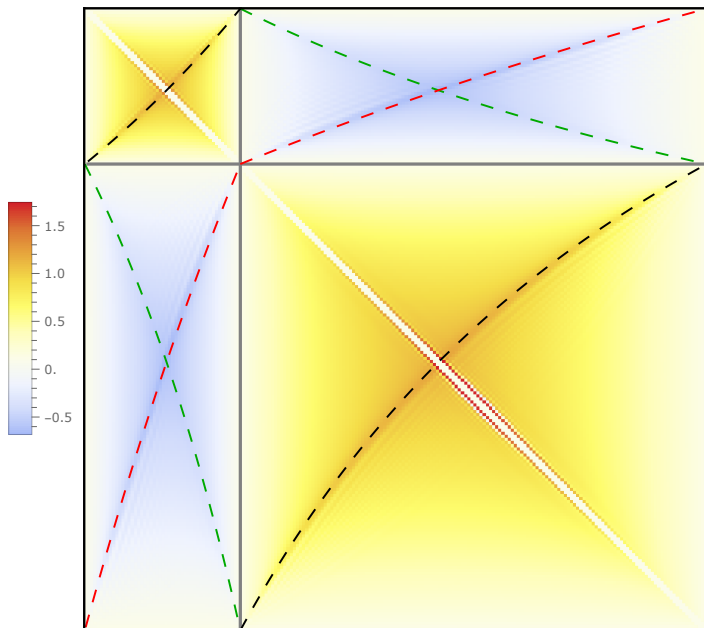
matrix V

$$\delta \equiv \frac{D}{L_A}$$

$$\delta = \frac{1}{4}$$

$$L_1 = 50 \quad L_2 = 3L_1$$

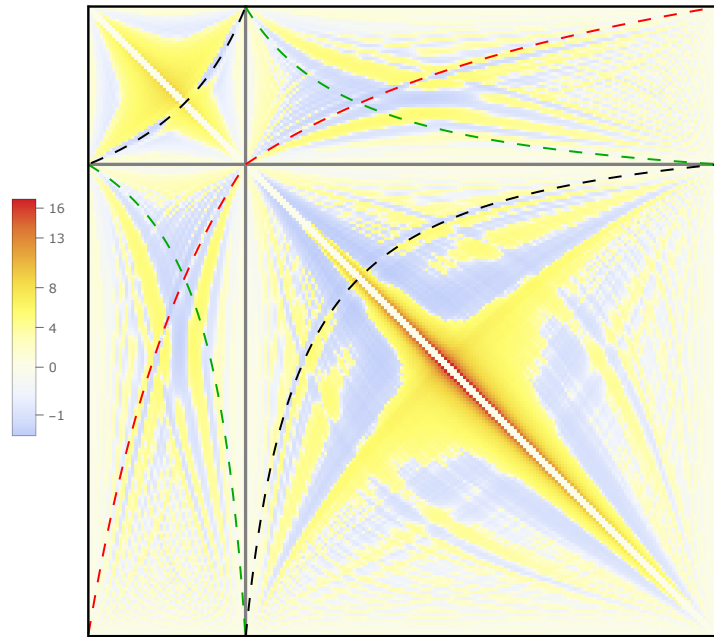
$$\omega = 10^{-500}$$



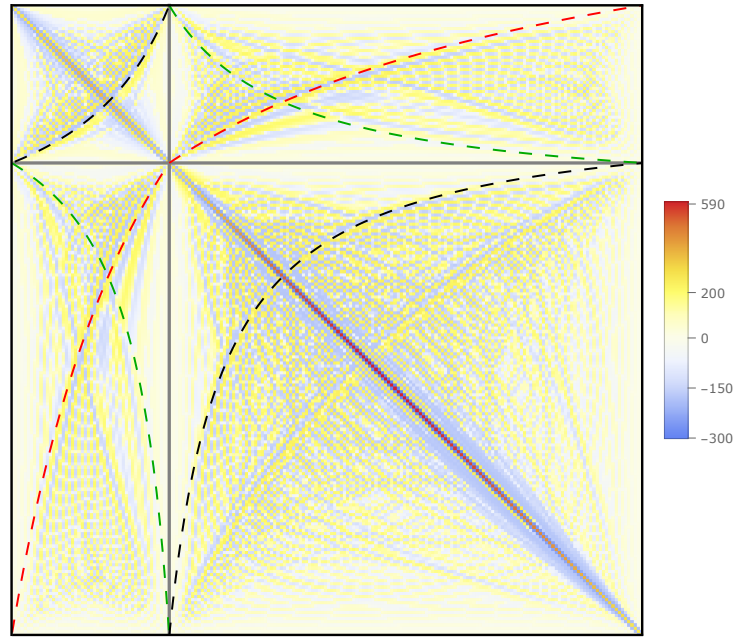
$$\delta = 1$$

EH of two disjoint blocks varying mass

matrix T

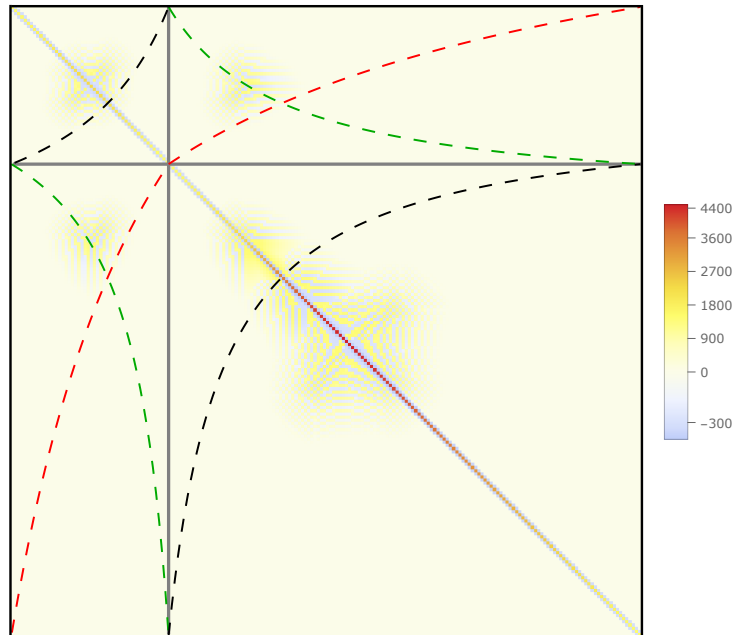
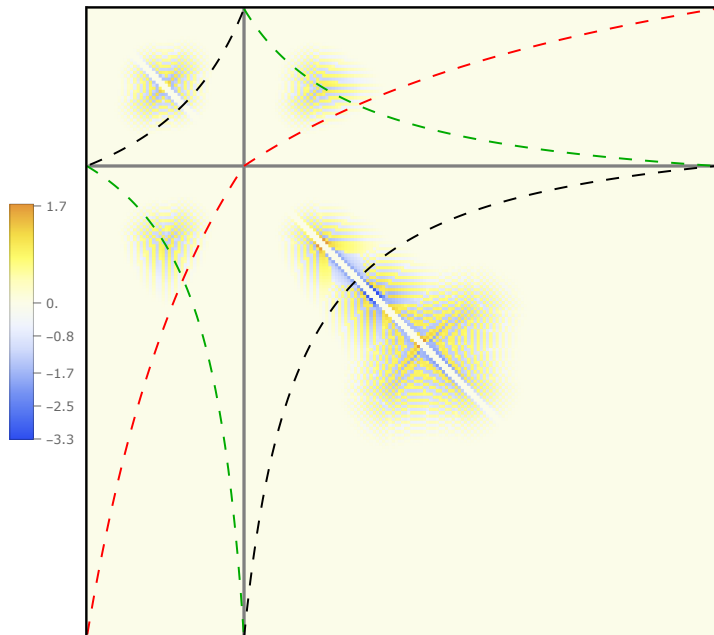


matrix V



$$\omega = 0.1$$

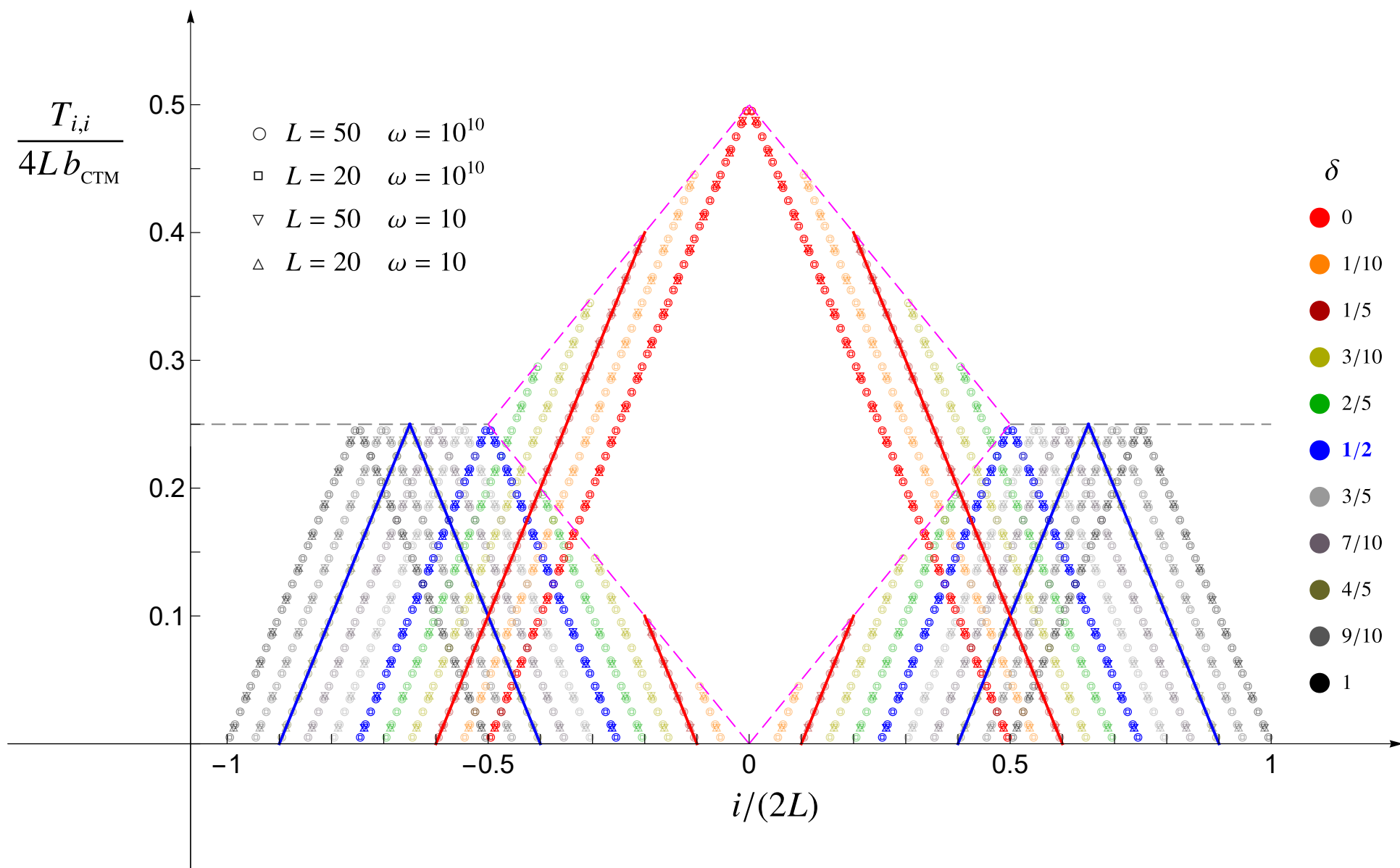
$$\delta = \frac{1}{4}$$



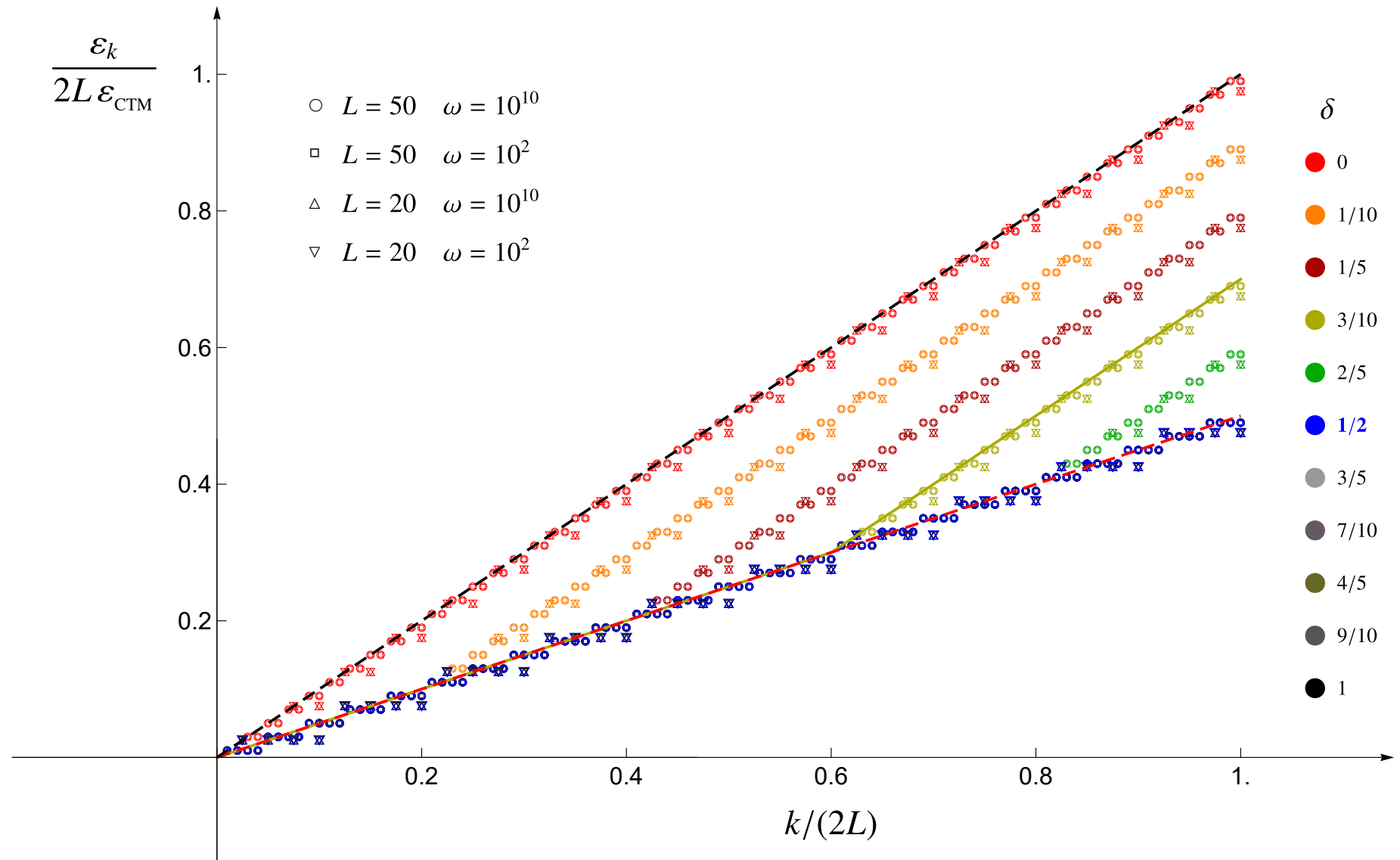
$$\omega = 3$$

EH of two disjoint equal blocks at large mass

→ At large ω , the three diagonals approximation holds



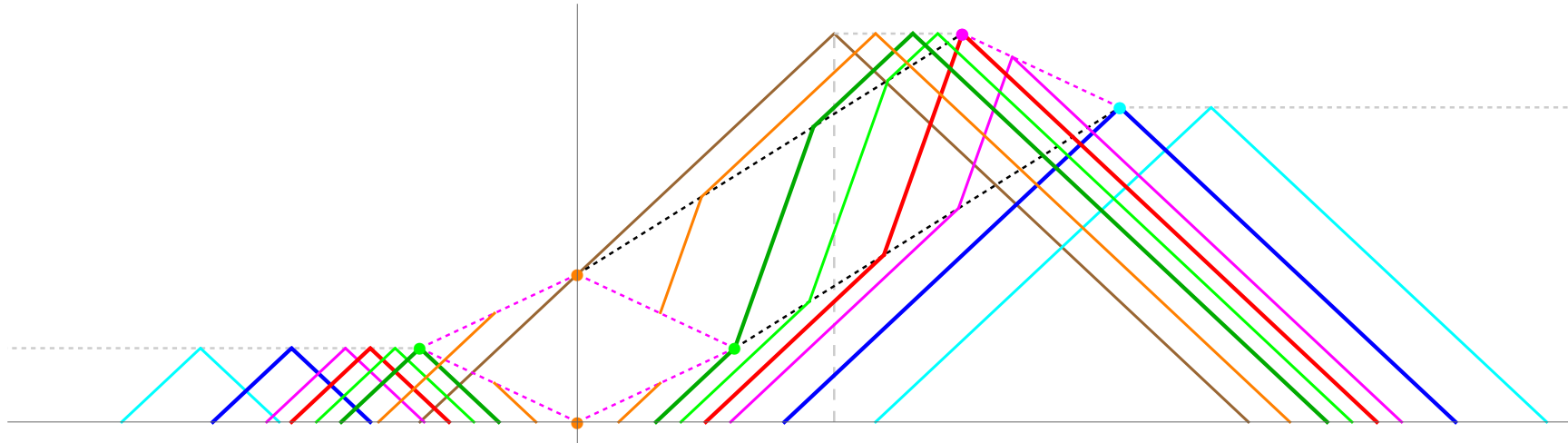
Large mass: Single-particle entanglement spectrum



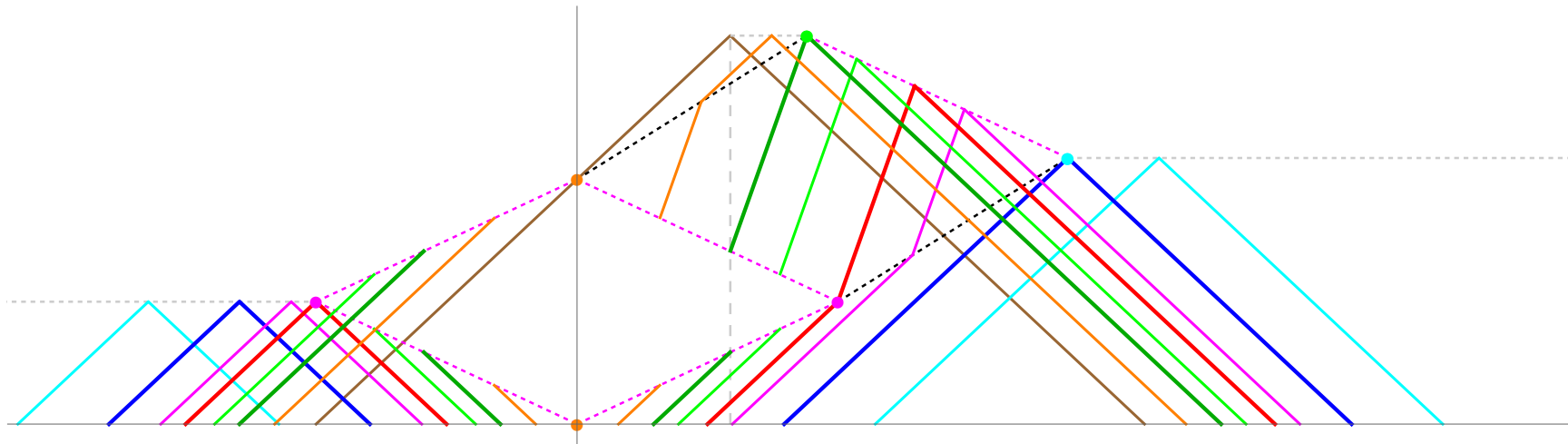
EH of two disjoint unequal blocks at large mass

➡ For $L_1 < L_2$, two different sequences of profiles are observed

$$\rho \equiv \frac{L_1}{L_A}$$



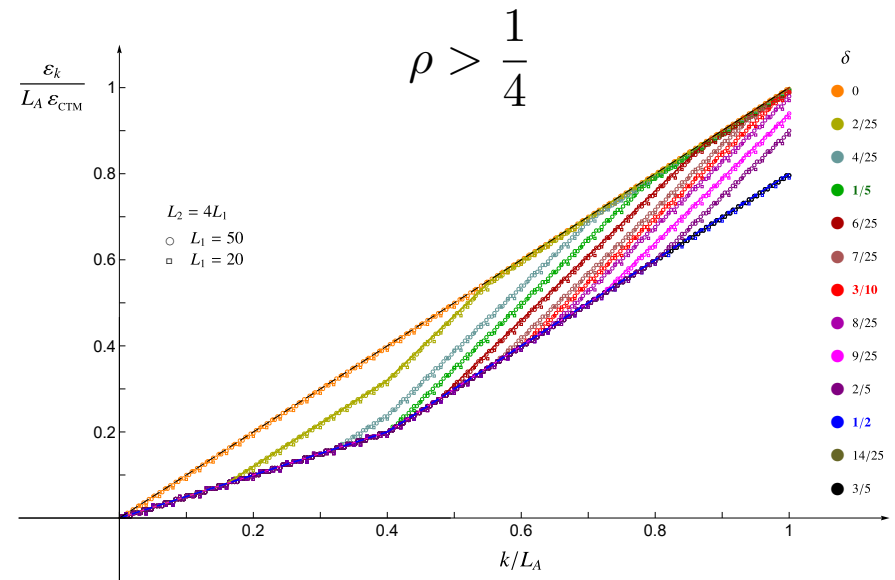
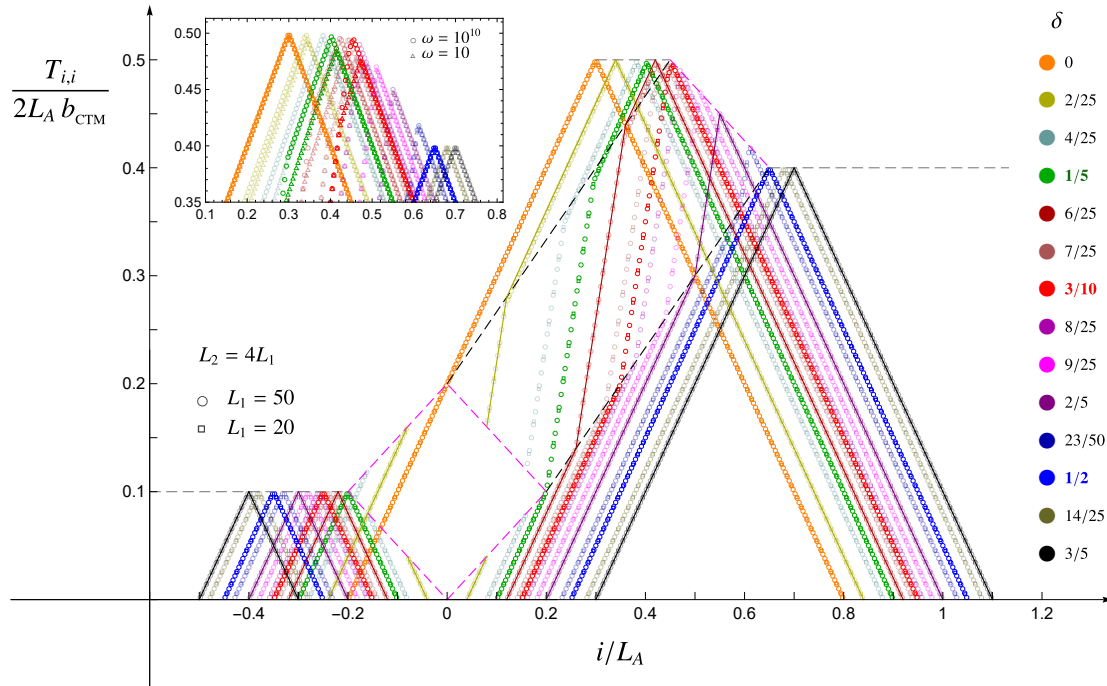
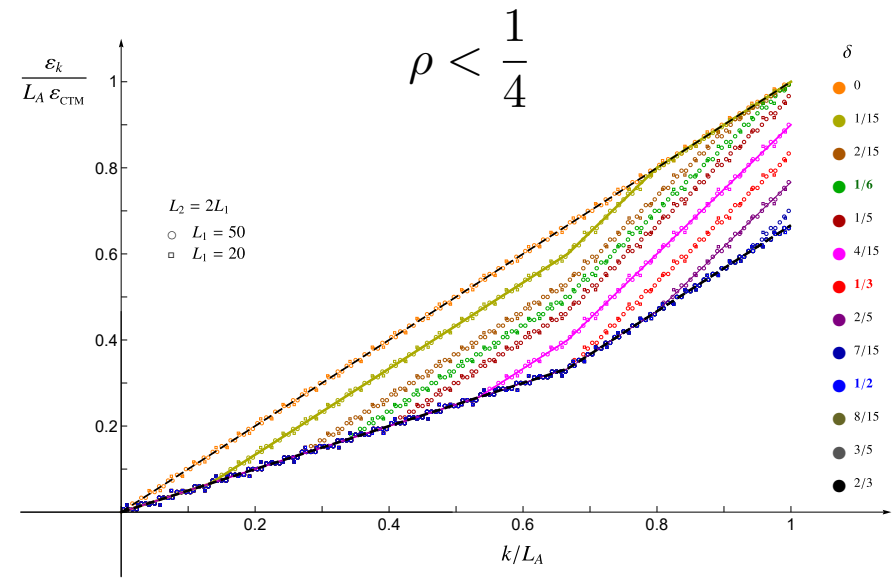
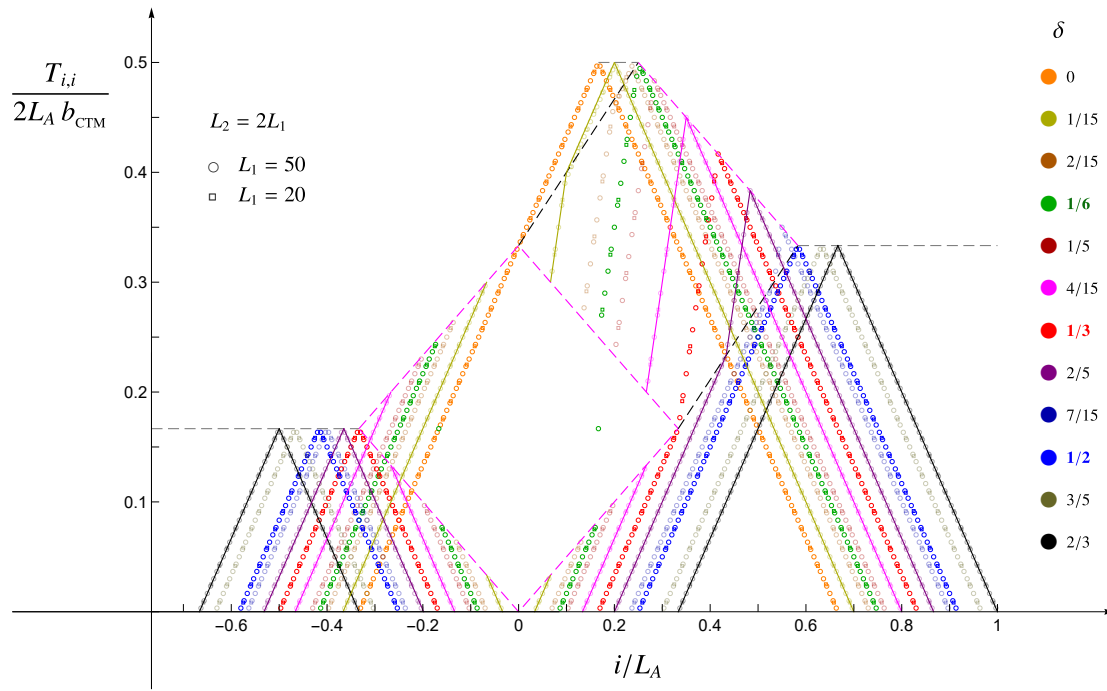
$$\rho < \frac{1}{4}$$



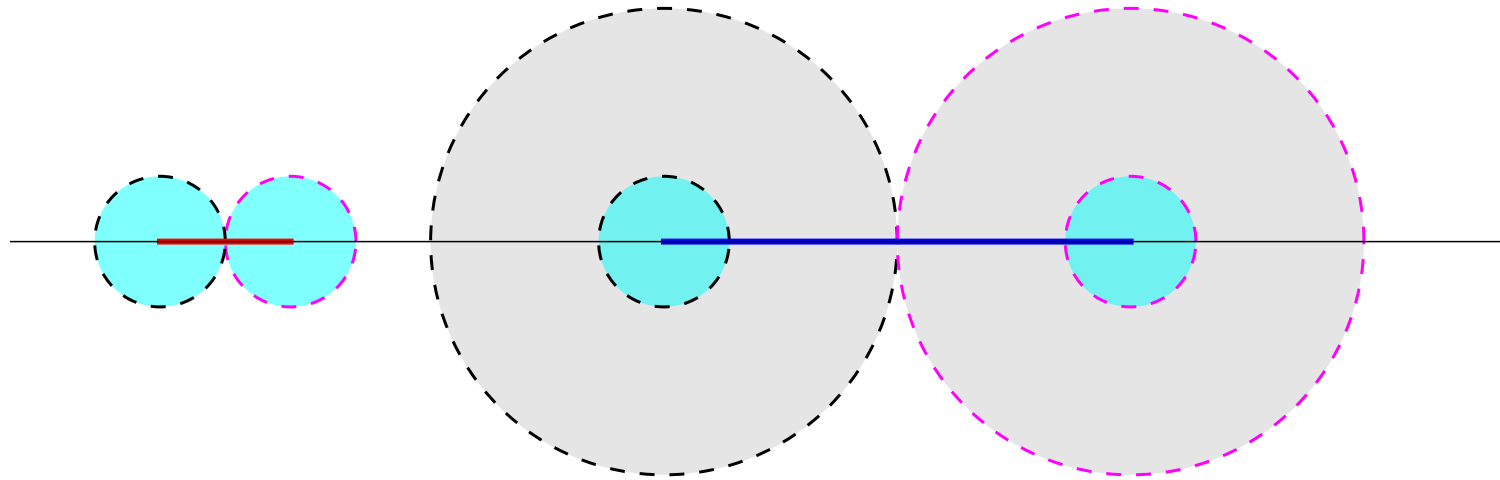
$$\rho > \frac{1}{4}$$

Critical distances $D_c \in \left\{ \frac{L_1 + L_2}{2}, \frac{L_2 - L_1}{2}, L_1 \right\}$

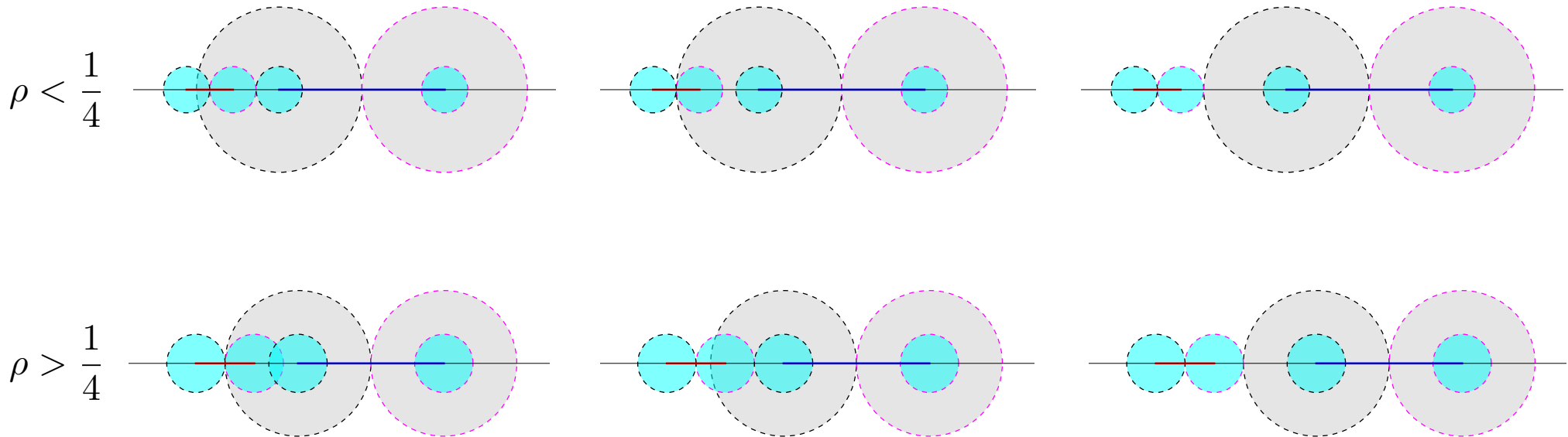
EH of two disjoint unequal blocks at large mass



EH of two disjoint unequal blocks at large mass



➡ A heuristic geometric picture provides the critical distances



Conclusions

- Geometric actions of modular objects in CFT_2 for an interval either in the ground state or in the thermal state
 - ➔ The modular conjugation and the contour function provide the thermal entropy of the interval
 - In AdS/CFT, the geodesic bit threads implement the geometric action of the modular conjugation
 - ➔ Holographic thermal entropy of spherical region
 - Entanglement Hamiltonian of two disjoint blocks at large mass displays transitions in the separation distance
-
- Some future directions:
 - Other spatial configurations or other states
 - Other AAdS gravitational backgrounds
 - Out of equilibrium



Two-weeks workshop at IGAP (SISSA, Trieste) in **November 2026**

Von Neumann Algebras, Quantum Information and Gravity

Organisers: Roberto Longo, Stefan Hollands, Horacio Casini, Javier Magan, Erik Tonni

THANK YOU

BACK UP SLIDES

Entanglement contours in the lattice: further properties

[Chen, Vidal, (2014)]

■ Besides $S_A^{(n)} = \sum_{i \in A} s_A^{(n)}(i)$ and $s_A^{(n)}(i) \geq 0$, other properties can be imposed on the contour function.

(a) Spatial symmetry. If ρ_A is invariant under a transformation relating the sites i and j in the subsystem A , then $s_A^{(n)}(i) = s_A^{(n)}(j)$.

The contour $s_A^{(n)}(G)$ of a subregion $G \subseteq A$ is $s_A^{(n)}(G) \equiv \sum_{i \in G} s_A^{(n)}(i)$.

(b) Invariance under local unitary transformations. Given a system in the state ρ and a unitary transformation U_G acting non trivially only on $G \subseteq A$, denoting by ρ' the state of the system after such transformation, the same contour $s_A^{(n)}(G)$ should be found for ρ and ρ' .

(c) A bound. Given a system in the pure state $|\Psi\rangle$ and $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, let us assume that the further decompositions $\mathcal{H}_A = \mathcal{H}_{\Omega_A} \otimes \mathcal{H}_{\bar{\Omega}_A}$ and $\mathcal{H}_B = \mathcal{H}_{\Omega_B} \otimes \mathcal{H}_{\bar{\Omega}_B}$ lead to the factorisation of the state $|\Psi\rangle = |\Psi_{\Omega_A \Omega_B}\rangle \otimes |\Psi_{\bar{\Omega}_A \bar{\Omega}_B}\rangle$. Considering a subregion $G \subseteq A$ such that $\bigotimes_{i \in G} \mathcal{H}_i \subseteq \mathcal{H}_{\Omega_A}$, we must have $s_A^{(n)}(G) \leq S^{(n)}(\Omega_A)$, where $S^{(n)}(\Omega_A)$ are the entanglement entropies obtained by tracing over $\mathcal{H}_{\bar{\Omega}_A} \otimes \mathcal{H}_B$.

■ A contour function $s_A^{(n)}(i)$ constructed for free fermions

Entanglement entropies in harmonic lattices

- E.g.: Harmonic chain

$$H = \sum_{i=0}^{L-1} \left(\frac{1}{2m} \hat{p}_i^2 + \frac{m\omega^2}{2} \hat{q}_i^2 + \frac{\kappa}{2} (\hat{q}_{i+1} - \hat{q}_i)^2 \right)$$

Considering $\hat{\mathbf{r}} \equiv (\hat{q}_1, \dots, \hat{q}_L, \hat{p}_1, \dots, \hat{p}_L)^t$, the canonical commutation relations are preserved by symplectic transformations $\hat{\mathbf{r}} \rightarrow \hat{\mathbf{r}}' = M\hat{\mathbf{r}}$, where $M \in \text{Sp}(L)$

- ρ_A is Gaussian and fully described by the $2\ell \times 2\ell$ reduced covariance matrix $\gamma_A \equiv \text{Re}\langle \mathbf{r} \mathbf{r}^t \rangle|_A$, which is real and symmetric.

- The Williamson's theorem guarantees that a symplectic matrix $W \in \text{Sp}(\ell)$ exists such that

$$\gamma_A = W^t (D \oplus D) W$$

where $D = \text{diag}(\sigma_1, \dots, \sigma_\ell)$ (with $\sigma_k > 0$) gives the symplectic spectrum.

- The symplectic spectrum of γ_A provides the entanglement entropies

$$S_A^{(n)} = \sum_{k=1}^{\ell} s_n(\sigma_k)$$

$$s_n(y) \equiv \frac{1}{n-1} \log [(y + 1/2)^n - (y - 1/2)^n]$$

A proposal for the contour function in harmonic lattices

- Given a set $p_k(i)$ of probabilities ($\sum_{i=1}^{\ell} p_k(i) = 1$ with $p_k(i) \geq 0$)

$$s_A^{(n)}(i) = \sum_{k=1}^{\ell} p_k(i) s_n(\sigma_k)$$

[Chen, Vidal, (2014)]
[Botero, Reznik, (2004)]

We call mode participation function the function $p_k(i)$.

- Any orthogonal matrix leads to $s_A^{(n)}(i) \geq 0$ such that $S_A^{(n)} = \sum_{i \in A} s_A^{(n)}(i)$
- We employed the Euler decomposition of the symplectic matrix W such that $\gamma_A = W^t(D \oplus D)W$, i.e. [Coser, De Nobili, E.T. (2017)]

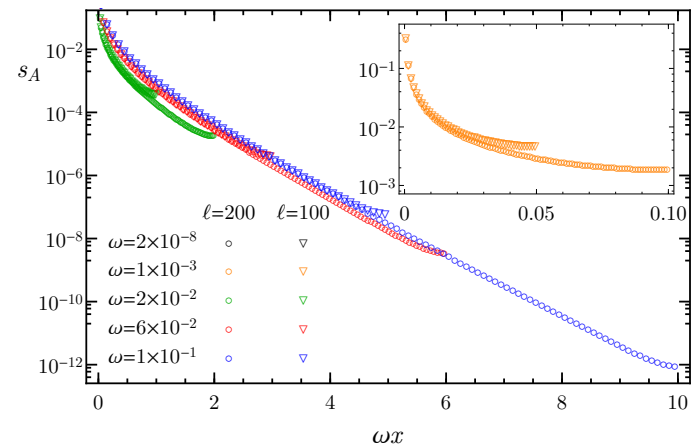
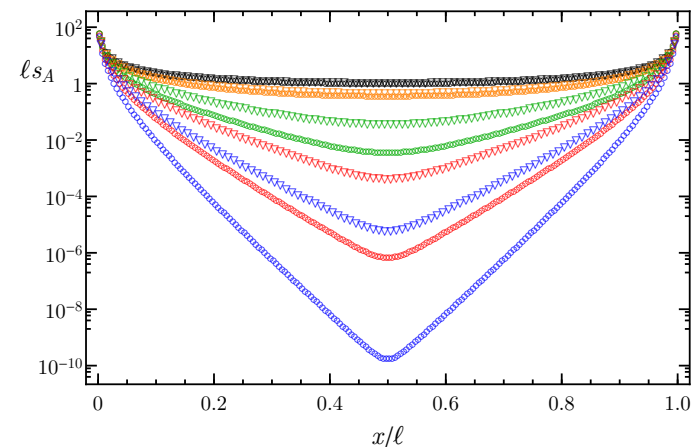
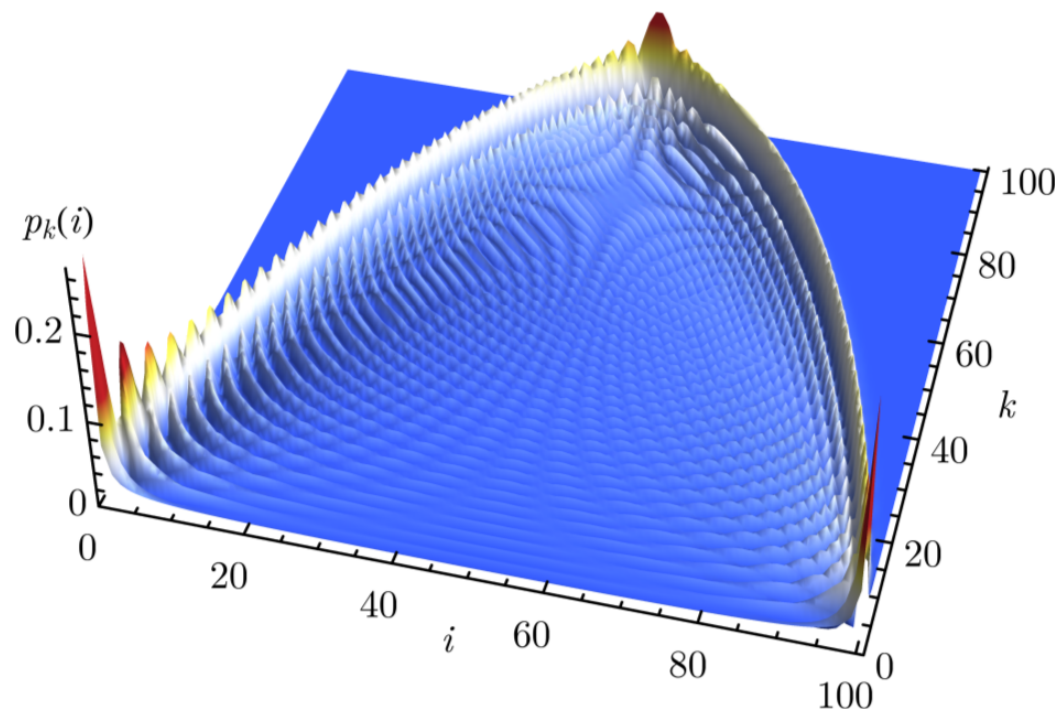
$$\boxed{W = K_L E K_R} \quad \left\{ \begin{array}{l} E = e^{\chi} \oplus e^{-\chi} \quad \chi \equiv \text{diag}(\chi_1, \dots, \chi_{\ell}) \geq 0 \\ K_L \text{ and } K_R \text{ are symplectic and orthogonal} \end{array} \right.$$

➔ Construct the contour function from the orthogonal matrix $K_L K_R$.

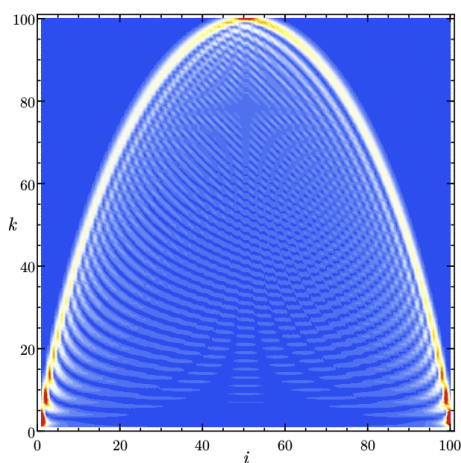
- This contour function satisfies the properties (a), (b) and (c) for the transformations implemented by symplectic and orthogonal matrices.

Mode participation function & contour function

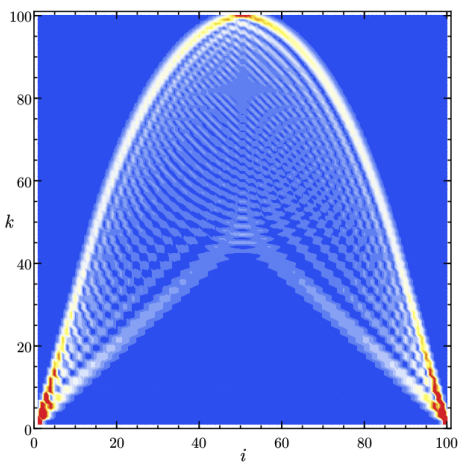
Massive infinite harmonic chain



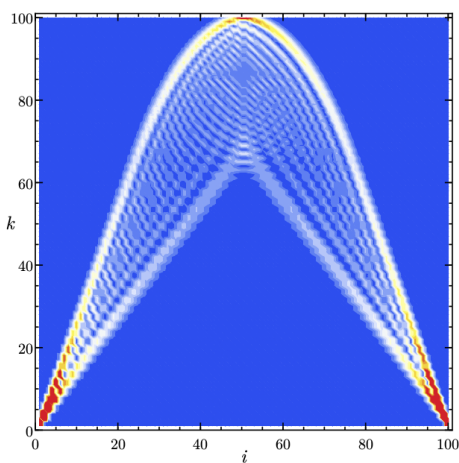
$\omega = 10^{-8}$



$\omega = 0.5$



$\omega = 1$



$\omega = 4$

