# The third law of black hole mechanics

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HSR 2410.11956 (PRD)
A. McSharry & HSR 2507.06870 (PRD)
M. Gadioux, HSR & J. Santos 2512.10008 (PRD)
J. Crump, M. Gadioux, HSR & J. Santos, 2601.20955 (PRL)
A. McSharry, HSR & J. Santos to appear

# Black hole mechanics

In 1973, Bardeen, Carter and Hawking formulated the laws of black hole mechanics.

**Zeroth law.** The surface gravity $\kappa$ of a stationary (time-independent) black hole is constant across the event horizon.

**First law.** Linear perturbations of a stationary black hole obey $dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ$

**Second law.** The horizon area $A$ is a non-decreasing function of time.

The remarkable similarity to the corresponding laws of thermodynamics is explained by Hawking's discovery that black holes emit thermal radiation at temperature $T_H = \frac{\hbar\kappa}{2\pi}$ and so have entropy $S_{BH} = A/4\hbar$.

# Third Law v1

There are several inequivalent versions of the **third law of thermodynamics.**

"Planck version": *the entropy of a thermodynamic system should vanish at zero temperature.*

A black hole with zero temperature is called *extremal*. Classically, extremal black holes violate Planck's version of the third law.

There is a large quantum gravity effect near extremality that significantly reduces the entropy (Iliesu & Turiaci 2020) so Planck's version of the third law is respected.

This does not happen for *supersymmetric* black holes, for which the entropy is not corrected at zero temperature (Heydeman *et al* 2020): Planck's version of the third law is violated.

# Third Law v2

Nernst's "unattainability" statement of the **third law of thermodynamics**:

*It is impossible for any procedure, no matter how idealized, to reduce the temperature of a system to absolute zero in a finite number of operations.*

Bardeen-Carter-Hawking's (unproved) statement of the **third law of black hole mechanics**:

*It is impossible by any procedure, no matter how idealized, to reduce* $\kappa$ *to zero by a finite sequence of operations.*

This is the version of the third law that I will discuss in this talk. Note that it refers to *dynamical* black holes. Also it makes sense *classically* - one can hope to prove it as a theorem in GR, like the other laws of BH mechanics.

# Israel's proof (1986)

*It is impossible by any procedure, no matter how idealized, to reduce $\kappa$ to zero by a finite sequence of operations.*

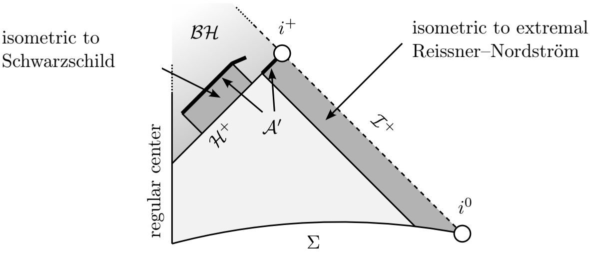
Israel's proposal: "finite sequence of operators" should mean "in finite advanced time", i.e., finite time for an observer near the horizon.

1970's: spherical gravitational collapse of a thin shell of charged matter onto a non-extremal black hole can produce an exactly extremal Reissner-Nordström black hole in finite time, in violation of third law.

In such examples, the apparent horizon jumps outwards discontinuously when crossed by the shell. Israel interpreted this as an artefact of the thin-shell model and presented a proof of the third law based on the assumption of a continuous apparent horizon.

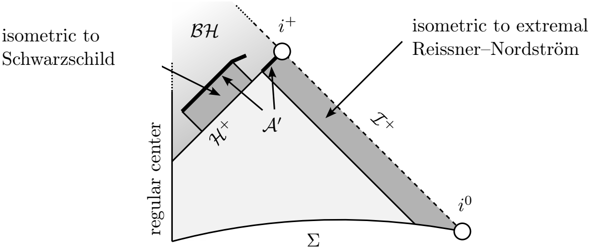
# Kehle-Unger third-law violating solutions (2022)

Einstein-Maxwell theory coupled to a massless charged scalar field. Spherically symmetric gravitational collapse of the scalar field can result in formation of an exactly extremal Reissner-Nordström black hole in finite advanced time, with an intermediate phase in which the spacetime is exactly Schwarzschild at the horizon:




(image credit: Kehle and Unger)

# Kehle-Unger third-law violating solutions (2022)



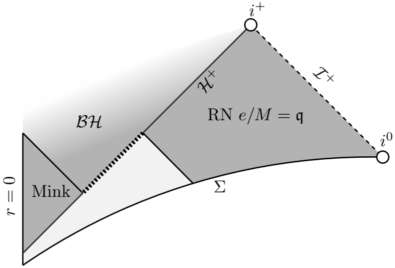

(image credit: Kehle and Unger)

The apparent horizon is discontinuous. This is why Israel's proof fails. The third law does not hold for the Einstein-Maxwell massless charged scalar theory.

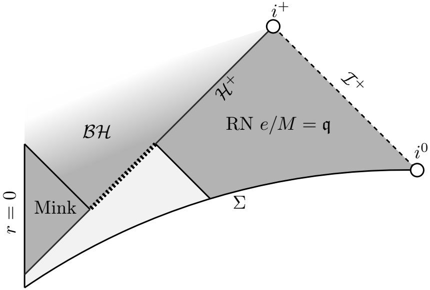
# Characteristic gluing (Aretakis, Czimek & Rodnianski 2023-25)

Einstein-Maxwell-charged scalar in spherical symmetry.

Kehle & Unger construct solutions by specifying data along pairs of characteristic (i.e. null) surfaces and solving forwards/backwards in time. Simplest example describes formation of RN, either subextremal or extremal:




(image credit: Kehle and Unger)




Double null coordinates $(U, V)$.

The free data is $\Phi(V)$ on the dotted line. Must be chosen to ensure matching of $Q$ and sufficient differentiability across the horizon.

$Q$ and $\partial_U^j \Phi$ satisfy transport eqs along dotted line so uniquely determined by either their Minkowski (above) or RN (below) values. Hence these will agree everywhere along the dotted line if they agree at the corners. For a $C^k$ solution this must be true for $j \leq k$ so have $2k + 1$ real equations.

Summary: to achieve $C^k$ gluing we need to choose $\Phi(V)$ such that $2k+1$ equations hold.

Kehle & Unger make an Ansatz for $\Phi(V)$ containing $2k+1$ real parameters $\alpha_i$

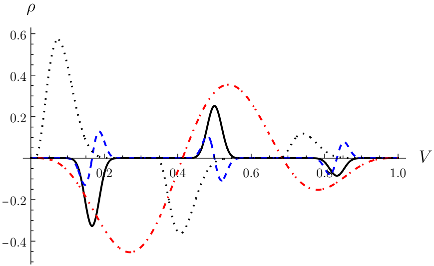
For large $eM$ (e = scalar charge), matching of $Q$ defines a sphere $S^{2k}$ in $\mathbb{R}^{2k+1}$. The remaining equations are odd under $\alpha_i \to -\alpha_i$ (arises from $\Phi \to -\Phi$ symmetry). The Borsuk-Ulam theorem then guarantees existence of a solution for any $k$.

The proof doesn't tell us anything about the form of the solution!

Construction of third-law violating solutions involves 2-stage gluing: Minkowki to Schwarzschild, then Schwarzschild to extremal RN.

# Numerics for RN (Gadioux, Santos & HSR 2025)

Profile solving $C^1$ gluing constraints for 4 different Ansätze:




Can also include a mass parameter $m$ for the scalar field: the largest $m/e$ for which we achieve $C^k$ gluing is about 0.2, 0.04, 0.01 for $k = 0, 1, 2$, (will see later that gluing is impossible for $m/e \geq 1$).

# Vacuum black holes

Kehle and Unger conjecture that it should be possible to form a black hole that is exactly *extremal Kerr* after a finite advanced time, starting from regular *vacuum* initial data (gravitational collapse of gravitational waves).

They proved (2023) that regular vacuum initial data can give a spacetime that is exactly a slowly-rotating ($|a| \ll M$) Kerr black hole after a finite advanced time.

Constructing third law violating solutions in 4d vacuum looks difficult: solutions will depend on all coordinates.

# 5d vacuum gravity (Crump, Gadioux, Santos & HSR 2026)

In 5d vacuum gravity the analogue of Kerr is the Myers-Perry solutions, parameterized by $M$, $J_1$, $J_2$ where $J_i$ are angular momenta in two orthogonal 2-planes.

Generically the isometry group is $\mathbb{R} \times U(1) \times U(1)$ and the solution depends on two coordinates, just like Kerr.

However if $J_1 = \pm J_2$ then the isometry group is $\mathbb{R} \times SU(2) \times U(1)$ and the metric depends on only one coordinate:

$$-f(r)dt^2 + g(r)dr^2 + \frac{r^2}{4}\left[\sigma_1^2 + \sigma_2^2 + h(r)(\sigma_3 + \Omega(r)dt)^2\right]$$

where $\sigma_i$ are left-invariant 1-forms on $SU(2) \sim S^3$.

This belongs to much larger class of *dynamical* spacetimes with $SU(2)$ isometry group (with $S^3$ orbits), depending on only 2 coordinates (e.g. Bizon, Chmaj & Schmidt 2005).

We'll seek third-law violating solutions within the class of $SU(2)$-symmetric spacetimes.

These can describe a Schwarzschild black hole evolving to an extremal MP black hole. Can also construct solutions describing gravitational collapse to form extremal MP.

We start from an Ansatz for a subset of $SU(2)$-symmetric spacetimes for which the equations of motion are qualitatively similar to those of Kehle & Unger for the Einstein-Maxwell-charged scalar system:

Charged scalar $\Phi \leftrightarrow$ some $SU(2)$-symmetric deformation of $S^3$

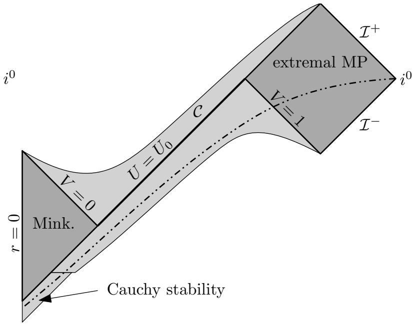
Maxwell 1-form $\leftrightarrow$ twist 1-form describing rotation of $S^3$

Quasilocal charge $Q \leftrightarrow$ quasilocal angular momentum $J$

We try to construct solutions using characteristic gluing: no large parameter analogous to $eM$ so proceed numerically.

# Characteristic gluing

Aim to "glue" Minkowski or Schwarzschild to extremal MP along a common null hypersurface $\mathcal{C}$.

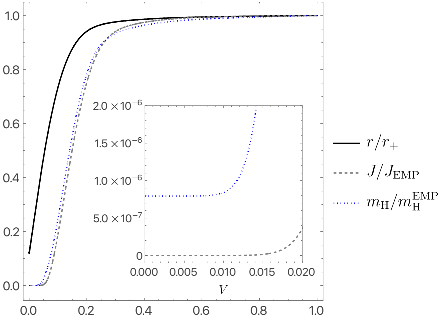



Construct solutions by specifying data along pairs of characteristic (i.e. null) surfaces and solving forwards/backwards in time.

# Results

We have solved the gluing problem numerically to demonstrate the existence of two types of solution:

(i) Schwarzschild to extremal MP gluing: a non-extremal BH that absorbs gravitational waves to become an extremal rotating BH in finite time. Hence *third law is violated in 5d vacuum gravity.*




(ii) Minkowski to extremal MP gluing: formation of extremal rotating BH in gravitational collapse of gravitational waves.

# Third law for supersymmetric black holes

The third-law violating solutions of Kehle and Unger involve a massless charged scalar field or massless Vlasov matter: matter with large charge to mass ratio.

What happens if the charge to mass ratio of matter is bounded?

# Local mass-charge inequality

We'll consider Einstein-Maxwell theory coupled to matter satisfying the local mass-charge inequality of Gibbons & Hull (1981):

$$T_{00}^{(m)} \geq \sqrt{T_{0i}^{(m)} T_{0i}^{(m)} + J_0^2 + \tilde{J}_0^2}$$

where indices $(0, i)$ refer to an arbitrary orthonormal frame, $T_{ab}^{(m)}$ is the energy-momentum tensor of matter (excluding the Maxwell field) and $J_a$, $\tilde{J}_a$ are the electric and magnetic currents of matter.

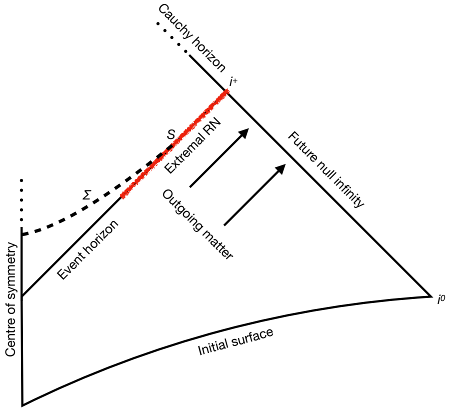
This is a strengthened version of the dominant energy condition.

For a scalar field of mass $m$ and charge $q$ it is equivalent to $m \geq |q|$.

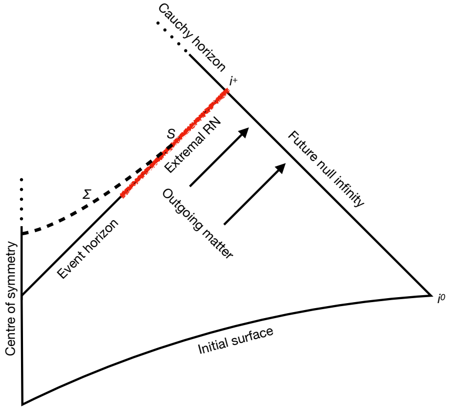
(Gibbons & Hull used this to prove a strengthened version of the positive mass theorem: $M \geq \sqrt{Q^2 + P^2}$.)

# Compact interior

We are considering the possibility of forming an extremal RN black hole in gravitational collapse. In such a situation, the black hole would have *compact interior*, i.e., a horizon cross-section $S$ would have $S = \partial\Sigma$ for a compact spacelike surface $\Sigma$. (The maximal analytic extension of extremal RN does *not* have compact interior because of the singularity at $r = 0$.)



**Theorem** (HSR 2024). If matter satisfies the local mass-charge inequality and $S$ has the same metric, Maxwell field and extrinsic curvature as a horizon cross-section of extremal RN then one *cannot* write $S = \partial\Sigma$ with $\Sigma$ a compact spacelike surface, i.e., $S$ does not have compact interior.




Proof uses spinors, supersymmetry and quasilocal energy.

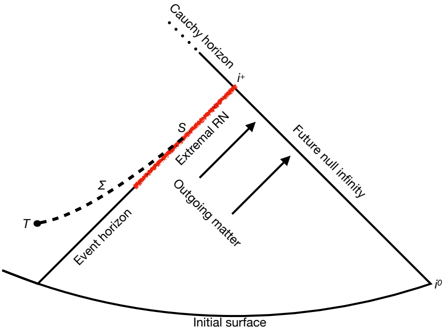
If matter satisfies the local mass-charge inequality then an extremal RN black hole cannot form in gravitational collapse in finite time.

Hence a non-extremal black hole cannot become extremal *if the initial black hole was formed from collapse*.

What if the initial black hole was not formed in collapse e.g. if we start with a a *two-sided* non-extremal black hole? Could we make it extremal by throwing in charged matter?

Israel: "non-extremal" means "there exists a trapped surface"

**Theorem** (McSharry-HSR 2025) Let $\Sigma$ be a compact spacelike surface with $\partial\Sigma = S \cup T$ where $T$ is a trapped surface. Assume that matter satisfies the local mass-charge inequality on $\Sigma$. Then $S$ cannot be a surface with the same metric, Maxwell field and extrinsic curvature as a horizon cross-section of extremal RN . In other words, this is impossible:



# Idea of proof

If local mass-charge inequality holds on a complete asymptotically flat hypersurface $\Sigma$ then a BPS inequality holds (Gibbons & Hull 1981)

$$M \geq \sqrt{Q^2 + P^2}$$

where $M, Q, P$ defined *at infinity*, equality iff there exists "supercovariantly constant" spinor on $\Sigma$.

Define *quasilocal* $M, Q, P$ associated with a finite 2-surface $S$ in spacetime.

Obtain a *quasilocal* BPS bound for compact surface $\Sigma$ bounded by $S$, possibly inner boundary at trapped $T$. Inequality is strict if $T$ non-empty.

If geometry and Maxwell field on $S$ coincides with those on extremal RN horizon then quasilocal BPS bound is saturated. Immediate contradiction if $T$ non-empty.

If $T$ empty then saturation of BPS bound implies existence of supercovariantly constant spinor on $\Sigma$. Can show this implies $Q = P = 0$.

# Anti-de Sitter black holes

$d = 4$ Einstein-Maxwell theory with negative cosmological constant and charged matter.

A solution is *supersymmetric* if it admits a globally defined spinor that is "supercovariantly constant".

Extremal RN-AdS is not supersymmetric but there exists a 1-parameter family of supersymmetric ($\Rightarrow$ extremal) Kerr-Newman-AdS black holes (Kostalecky-Perry 1995).

McSharry-HSR 2025: we proved that the third law holds for these black holes if matter satisfies local mass-charge inequality.

# BTZ black hole (McSharry, HSR & Santos, to appear)

3d gravity with negative cosmological constant: BTZ black hole.

Extremal BTZ is supersymmetric (Coussaert & Henneaux 1994).

**Theorem (third law for BTZ)** Let $\Sigma$ be a compact spacelike surface with annulus topology and $\partial\Sigma = S \cup T$ where $T$ is a trapped surface. Assume that matter satisfies the dominant energy condition on $\Sigma$. Then $S$ cannot be a surface with the same metric and extrinsic curvature as a horizon cross-section of extremal BTZ.

In other words, a non-extremal BH cannot evolve to an extremal BTZ BH in finite time.

Proof involves defining quasi-local mass and angular momentum in 3d gravity and proving a quasi-local BPS bound.

(DEC can probably be relaxed to allow for tachyonic scalars that respect the BF bound.)

Unlike in 4d, we *cannot* rule out the possibility of forming an extremal BTZ black hole in finite time in gravitational collapse.

Reason: in this case $\Sigma$ would have disc topology with boundary $S$. Spin structure regular on $\Sigma$ gives antiperiodic spinors around $S$. This excludes exploiting the supersymmetry of extremal BTZ, which is associated with periodic spinors around $S$. ("Gravitational collapse geometries belong to NS sector".)

Using gluing we can indeed construct solutions describing gravitational collapse of a massless scalar to form an extremal BTZ in finite time in gravitational collapse.

Our third law implies that such solutions cannot possess trapped surfaces.

# Summary

Third law (v2) is violated by extremal RN in theories with large charge to mass ratio and by extremal MP in 5d vacuum gravity.

Third law (v2) is not violated by supersymmetric BHs if matter satisfies a suitable energy condition.

This is the opposite of what happens for third law (v1)!

# To do

So far the explicit form of third-law violating solutions has been determined only on the horizon (by gluing). For Kehle-Unger solutions or our 5d vacuum solutions it should be straightforward to determine (numerically) the solution off the horizon.

4d vacuum gluing: much harder!

# Discussion: critical behaviour

Consider the moduli space of all solutions arising from some class of initial data. Some solutions collapse to form black holes, others disperse.

*Critical* solutions lie on the codimension-1 boundary between these two behaviours.

Choptuik (1993): massless scalar field, spherical symmetry. Nakedly singular critical solutions.

Kehle-Unger (2024): massless charged Vlasov matter, spherical symmetry. Critical solutions describing gravitational collapse to form an extremal BH. (East (2025): massive charged Vlasov, also has unstable stars as critical solutions.)

A critical solution cannot possess a trapped surface. Likely that most third law-violating solutions do have trapped surfaces, so are not critical. However, our solutions describing formation of extremal BTZ in collapse might be critical.

In general, expect extremal BH critical solutions to approach extremality only *asymptotically* (so don't violate third law).

Such solutions have been constructed for Einstein-Maxwell-scalar field in spherical symmetry (Murata, HSR & Tanahashi 2012, Angelopoulos, Kehle & Unger 2026, Gelles & Pretorius 2026)

Can we construct similar solutions for 5d vacuum gravity?

Does supersymmetry let us prove anything about such solutions? For example, do they require "more fine-tuning", i.e., is co-dimension of critical surface greater than 1?