The diffusive correction to the generalized hydrodynamics equation is neither diffusive nor correct

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Introduction: Cold bosonic atom quench in 1D

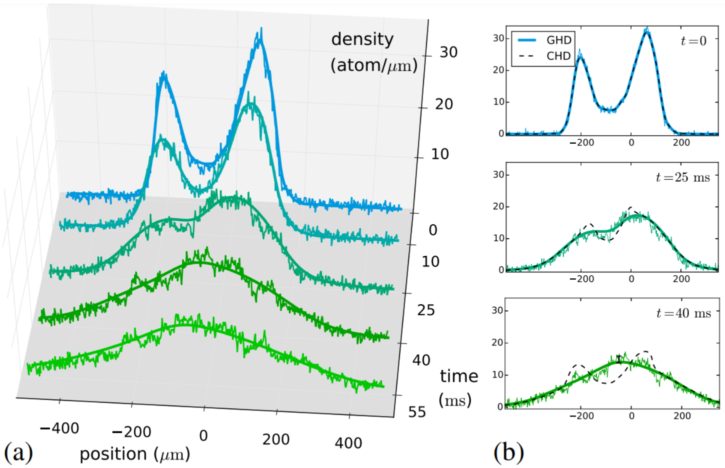


Figure: Cold bosonic atoms in 1D: Expansion after removal of double well potential (from (Schemmer, Bouchoule, Doyon, and Dubail 2019))

Introduction: Lieb-Liniger model

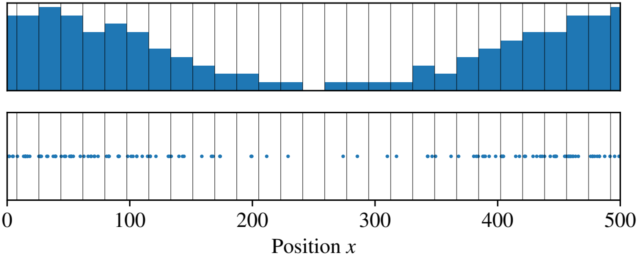
- Cold atoms in 1D can be described by Lieb-Liniger model:

$$\hat{H}_{\mathrm{LL}} = -\sum_i \partial_{x_i}^2 + 2c \sum_{i<j} \delta(x_i - x_j)$$

- Many-body physics: $N \approx 6300$ particles
- Separation of scales
  - Microscopic length scale $\sim 0.1\,\mu\mathrm{m}$
  - Trapping potential $\sim 100\,\mu\mathrm{m}$
- Microscopic analysis not possible
- Only able to measure space-averaged quantities, e.g. particle density
- Find simplified effective large-scale equations for averaged quantities

Hydrodynamics

- Divide space into cells of mesoscopic size $L_{\mathrm{micro}} \ll \ell \ll L$



- In non-integrable model: Thermal equilibrium in each cell

$$\hat{\rho} \sim e^{-\int \beta(\vec{x}/L,t)(\hat{H}-\mu(\vec{x}/L,t)\hat{N}-v(\vec{x}/L,t)\hat{P})}$$

- 3 conserved quantities $\rightarrow$ 3 continuity equations (Euler equations)

Generalized hydrodynamics (GHD)

- Integrable models have infinitely many (local) conserved quantities
- Local equilibrium in each cell: $\hat{\rho} = e^{-\sum_n \beta_n \hat{Q}_n}$
- TBA gives $q_n(t,x) = \langle \hat{q}_n \rangle_{\beta_m(t,x)}$ and $j_n(t,x) = \left\langle \hat{j}_n \right\rangle_{\beta_m(t,x)}$
- Infinitely many continuity equations

$$\partial_t q_n(t,x) + \partial_x j_n(t,x) = 0$$

- Generalized hydrodynamics (Castro-Alvaredo, Doyon, and Yoshimura 2016; Bertini, Collura, De Nardis, and Fagotti 2016)
- Mathematically appears in the Euler scaling limit $T \sim L \to \infty$

Diffusive correction

- Euler GHD **conserves entropy**
  $\rightarrow$ No “thermalization” to GGE!
- Need to understand corrections to Euler GHD
- Gradient expansion: $j_n = j_n(\{q_m\}, \{\partial_x q_m\}, \{\partial_x^2 q_m\}, \cdots)$ and $\partial_x^a q_m = \mathcal{O}(1/L^a)$
- First order correction: diffusive (“Navier-Stokes”) correction (Nardis, Bernard, and Doyon 2019)

$$\partial_t q_n + \partial_x j_n[q_m] = \tfrac{1}{2}\partial_x \left[\sum_k D_{nk}[q_m]\partial_x q_k \right] + \mathcal{O}\left(1/L^2\right)$$

- **Diffusion matrix** $D_{nk}[q_m]$ explicitly known (Kubo formula)
- Intuitive explanation: diffusion due to random walk of quasi-particles

Diffusive correction

- Diffusive correction increases entropy
  $\rightarrow$ Explains "thermalization" to a GGE
- Applying same logic to non-integrable fluids $\rightarrow$ Navier-Stokes equation
  - Extremely well established equation (experimentally, numerically)
- However...

The correct diffusive correction

- Navier-Stokes correction

$$\partial_t q_n + \partial_x j_n[q_m] = \tfrac{1}{2}\partial_x \left[\sum_k D_{nk}[q_m]\partial_x q_k\right] + \mathcal{O}\left(1/L^2\right)$$

- In integrable models the Navier-Stokes correction **is wrong**!

Correct diffusive equation

The correct expansion up to order $1/L$ is given by

$$\begin{aligned}\partial_t \left\langle \hat{q}_n(x)\right\rangle + \partial_x j_n[\langle \hat{q}_m(x)\rangle] &= \partial_x(\ldots \left\langle \hat{q}_n(x)\hat{q}_m(x)\right\rangle^{\mathrm{c}}) + \mathcal{O}\left(1/L^2\right)\\ \partial_t \left\langle \hat{q}_n(x)\hat{q}_m(y)\right\rangle^{\mathrm{c}} &= \qquad\qquad \ldots \qquad\qquad + \mathcal{O}\left(1/L^2\right)\end{aligned}$$

- **Conserves entropy** $\rightarrow$ no “thermalization” towards GGE
- Physical meaning: importance of long range correlations

Long range correlations

- At initial times $\hat{\rho} = e^{-\sum_n \int \mathrm{d}x \beta_n(x/L)\hat{q}_n(x)}$
- From TBA $\langle \hat{q}_n(x)\hat{q}_m(y)\rangle^{\mathrm{c}} \sim \mathcal{O}(1/L)\delta(x-y)$
- However, in time long range correlations develop (and remain)

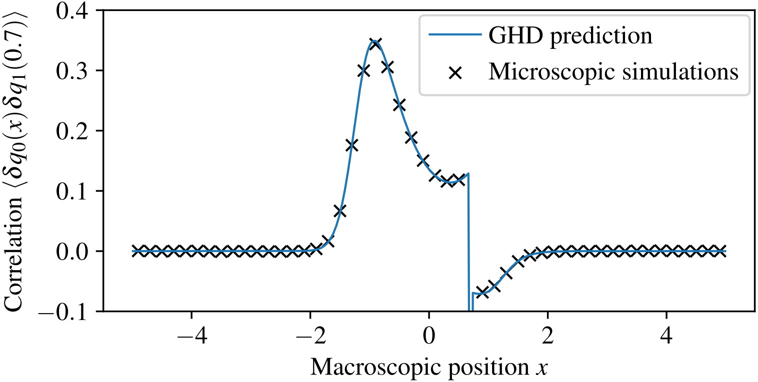


Figure: Long range correlations in hard rods between x and $y = 0.7$

Long range correlations

- Long range correlations develop from non-linear transport of initial correlations (Doyon, Perfetto, Sasamoto, and Yoshimura 2023)
- They represent a **failure of local "thermalization"** towards GGE

$$\hat{\rho}(t) \neq e^{-\sum_n \int \mathrm{d}x \beta_n(x/L)\hat{q}_n(x)}$$

- Fundamental assumption behind GHD broken (with error $\mathcal{O}(1/L)$)!
  - Does not affect Euler hydrodynamics, but next order correction
- Why are we so sure about this?
  - Because can **explicitly compute it** in special model $\rightarrow$ Hard rods

Hard rods

- Hard rods: **classical** hard spheres in 1D

$$H = \sum_i \frac{p_i^2}{2} + \sum_{i \neq j} V(x_i - x_j) \qquad V(x) = \begin{cases} \infty & |x| < a \\ 0 & |x| > a \end{cases}$$

- Conserved quantities: $Q_n = \sum_i p_i^n$

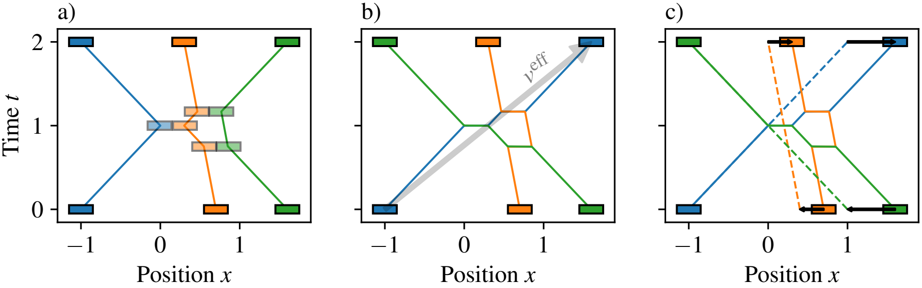


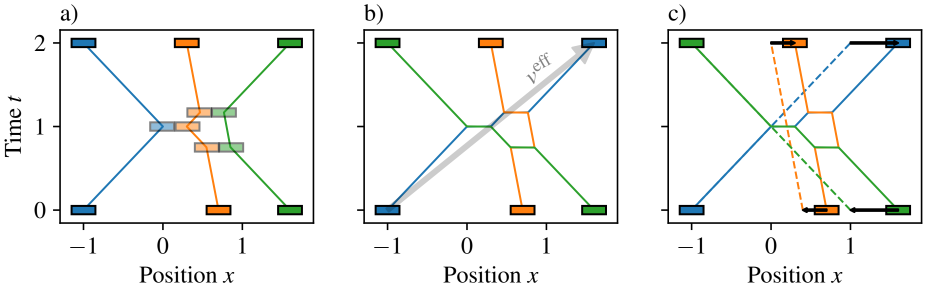
Figure: Dynamics of hard rods

Explicit solution formula for hard rods

- Hard rods evolution can be solved explicitly

$$x_i(t) = \hat{x}_i + p_i t + a \sum_{j \neq i} \theta(\hat{x}_i + p_i t - \hat{x}_j - p_j t)$$

$$\hat{x}_i = x_i - a \sum_{j \neq i} \theta(x_i - x_j)$$



Why do long range correlations appear?

- Hard rods formula is non-linear in $\hat{q}_n(x)$
- This means $\langle f(\hat{q}_n)\rangle \neq f(\langle \hat{q}_n\rangle)$, instead

$$\langle f(\hat{q}_n)\rangle = f(\langle \hat{q}_n\rangle) + \tfrac{1}{2} f''(\langle \hat{q}_n\rangle)\, \langle \hat{q}_n \hat{q}_n\rangle^{\mathrm{c}} + \mathcal{O}\big(1/L^2\big)$$

- This is the mathematical reason. But what does it mean physically?

Physical interpretation of diffusive correction

- Idea: use local microcanonical state instead of local GGE (Hübner 2026)
  - $q_n(t, x)$ is deterministic $\Rightarrow$ Long-range correlations exactly vanish
- Euler GHD equation works **without correction**
- When using local GGE $\Rightarrow$ average over microcanonical result
  - Inequivalence of ensembles on $\mathcal{O}(1/L)$
  - Introduces $\mathcal{O}(1/L)$ shift $\Rightarrow$ additional contribution
- Furthermore: Euler GHD also works for "unphysical states"
  - Initial states that are locally not in equilibrium

Conclusion

- The naive Navier-Stokes diffusive correction in GHD is not correct
- Using local GGE states $\Rightarrow$ complicated correction depending on long-range correlations
- Using local microcanonical state $\Rightarrow$ no correction
  - Diffusion constant exactly vanishes in integrable systems
  - Integrable systems are not chaotic
  - Also no hydrodynamic noise (Doyon 2026)
- Also relevant for 1D non-integrable systems with diffusion
  - Kubo formula does not give “correct” diffusion constant

$$(\text{Kubo formula}) = (\text{“correct” diffusion}) + (\text{long range correlations})$$

Thank you

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