One Kernel to Rule Them All: Universal Structure in the Bethe Ansatz for Stochastic Processes

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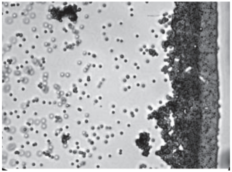
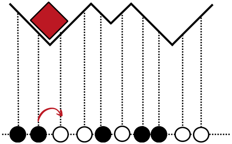
Interacting particle models and universality

Interacting particle models are a tool to study the universal scaling behaviour of a wide variety of large stochastic systems:





- random interface growth
- traffic flow
- polymers in random media
- crystal shapes
- surface deposition models

Why exactly solvable models?

The integrability allows to get exact solutions whose scaling limits reveal universal probabilistic and physical laws.

Outline

1. Problem
2. Method of solution
3. Some results

A continuous-time Markov process

on a (finite or countable) state space Ω, with configurations $\boldsymbol{\eta} \in \Omega$. Local transitions $\boldsymbol{\eta} \to \boldsymbol{\eta}'$ occur at rates $c(\boldsymbol{\eta}, \boldsymbol{\eta}')$, encoding the microscopic dynamics of the model.

The master equation

defines the evolution (given $P_0(\boldsymbol{\eta})$ for all $\boldsymbol{\eta} \in \Omega$)

$$\frac{d}{dt} P_t(\boldsymbol{\eta}) = \sum_{\boldsymbol{\eta}'} \Big(c(\boldsymbol{\eta}', \boldsymbol{\eta})\, P_t(\boldsymbol{\eta}') - c(\boldsymbol{\eta}, \boldsymbol{\eta}')\, P_t(\boldsymbol{\eta}) \Big).$$

Examples:

- Exclusion processes: left, right rates are 1 and q.
- Asym. avalanche process: left and right non-local with rates $R * r(q)$ and $L(1-q)r(q)$
- q-boson zero-range: only right jumps with rate $u(k) = \frac{1-q^k}{1-q}$

(A. M. Povolotsky 2004, T. Sasamoto and M. Wadati 1998)

Observable: an integrated current $Y(t)$

Fix $Y(0) = 0$; let $Y(t)$ count the total particle current in the model occurred up to time t.
$p_t(\boldsymbol{\eta}, Y)$ – the joint probability of $(\boldsymbol{\eta}(t), Y(t))$.
The moment generating function restricted to a final state $\boldsymbol{\eta}$

$$G_t(\boldsymbol{\eta}) = \sum_{Y} p_t(\boldsymbol{\eta}, Y)\, e^{\gamma Y}.$$

satisfies a deformed (tilted) master equation, obtained by weighting each jump event by e^{γ} (or by a jump-dependent factor) in the master equation:

$$\frac{d}{dt} G_t(\boldsymbol{\eta}) = \sum_{\boldsymbol{\eta}'} \Big(e^{\gamma\, q(\boldsymbol{\eta}', \boldsymbol{\eta})}\, c(\boldsymbol{\eta}', \boldsymbol{\eta})\, G_t(\boldsymbol{\eta}') - c(\boldsymbol{\eta}, \boldsymbol{\eta}')\, G_t(\boldsymbol{\eta}) \Big),$$

where $q(\boldsymbol{\eta}', \boldsymbol{\eta}) \in \mathbb{Z}$ is the increment of Y carried by the transition $\boldsymbol{\eta}' \to \boldsymbol{\eta}$.

The moment generating function,

$$\mathbb{E}\, e^{\gamma Y_t} = \sum_{\boldsymbol{\eta}\in\Omega} G_t(\boldsymbol{\eta}) = \sum_{\boldsymbol{\eta}\in\Omega} \left(e^{tM_\gamma} P_0\right)(\boldsymbol{\eta}).$$

Indeed, the solution to the tilted master equation is

$$\frac{d}{dt} G_t = M_\gamma G_t \quad \Longrightarrow \quad G_t = e^{tM_\gamma} P_0.$$

The exponential growth rate of the moment-generating function is given by **the largest eigenvalue of** M_γ (scaled cumulant generating function).

$$\lambda_1(\gamma) = \lim_{t\to\infty} \frac{\log \mathbb{E} e^{\gamma Y_t}}{t} = \sum_{n=1}^{\infty} c_n \frac{\gamma^n}{n!}.$$

(L.-H. Gwa and H. Spohn 1992, B. Derrida and J. L. Lebowitz 1998)
Its Legendre–Fenchel transform yields the large deviation rate function.

Problem: calculate $\lambda_1(\gamma)$ to study current large deviations.

ASEP (B. Derrida and K. Mallick 1997, S.Prolhac and K.Mallick, 2008)

$$c_1 = J = \frac{(1-q)p(N-p)}{N-1}.$$

$$c_2 = \Delta = \frac{2(1-q)N}{(N-1)\binom{N}{p}^2} \sum_{r=1}^{p} r^2 \frac{1+q^r}{1-q^r} \binom{N}{p+r}\binom{N}{p-r}$$

q-**boson ZRP**, $F(z) = \big(z(1-q);\, q\big)_\infty^{-1}, \qquad |q|<1.$

$$c_1 = J = \frac{N}{Z(N,p)} \oint \frac{F^N(z)}{z^{p+1}} \frac{dz}{2\pi i}.$$

$$c_2 = \Delta = pJ + \frac{2N^2}{Z(N,p)^2} \oint \frac{dy}{2\pi i} \frac{F^N(y)}{y^p} \oint_{|y|<|t|} \frac{dt}{2\pi i} \frac{F^N(t)}{t^p} \frac{\phi(y)}{t-y}$$

$$+ \frac{2N^2}{Z(N,p)^2} \sum_{i=1}^{\infty} \oint \frac{dt}{2\pi i} \frac{F^N(t)}{t^p} \oint_{|y|<|t|} \frac{dy}{2\pi i} \frac{F^N(y)}{y^p} \frac{q^{\pm i}\phi(yq^{\pm i}) + \phi(y)}{t - yq^{\pm i}}.$$

Here,

$$\phi(z) = \frac{J}{p}(\ln F(z))' - 1 = \frac{J}{p} \sum_{k=0}^{\infty} z^k \frac{(1-q)^{k+1}}{1-q^{k+1}} - 1.$$

(A. A. Trofimova and A. M. Povolotsky 2020)

Outline

Coordinate Bethe ansatz

Let particle position coordinates of a state $\boldsymbol{\eta}$ be $(1 \leq x_1, x_2, \ldots, x_p \leq N)$. The eigenvector ϕ of M_γ is presented as a linear combination of free waves

$$\phi(x_1, \ldots, x_p) := \sum_{\sigma \in S_p} A_\sigma z_{\sigma(1)}^{x_1} \ldots z_{\sigma(p)}^{x_p}, \quad z_1, \ldots, z_p \in \mathbb{C}^*$$

From $M_\gamma \phi = \lambda(\gamma)\phi$, one finds that $\lambda(\gamma)$ is a symmetric function of $z_1, \ldots, z_p$ iff z_1, ..., z_p satisfy Bethe Ansatz Equations.

$$BAE: \qquad \frac{a(z_i)^N}{b(z_i)^N} = (-1)^{p-1} \prod_{j=1}^{p} \frac{z_i - q z_j}{z_j - q z_i}, \qquad i = 1, \ldots, p.$$

$$\lambda(\gamma) = \mathit{symmetric\ of}\,(z_1, \ldots, z_p)$$

TQ-, *TP*-relation (A. Lazarescu and V. Pasquier 2014)

Since the BAE are invariant under permutations of $z_1, \ldots, z_p$, they depend on the z_i only through symmetric combinations. The eigenvalue problem can be reformulated in terms of a Q-polynomial

$$Q(z) := \prod_{i=1}^{p} (z - z_i), \qquad \deg Q(x) = p.$$

This turns the BAE into a single functional identity for Q valid for *all* $z \in \mathbb{C}$, together with an a priori unknown polynomial $T(z)$ of degree N

$$T(z)\, Q(z) = a(z)^N\, Q(qz) + b(z)^N\, Q\left(\frac{z}{q}\right).$$

$$\lambda(\gamma) = \textit{via } Q(z) \textit{ at some } z. \tag{1}$$

Interacting particle models

Model	$\alpha(x)$	$\beta(x)$	$\lambda(\gamma)$
ASEP	$e^{\gamma}(1+z)$	$q^{\frac{p}{N}}(1+qz)$	$(1-q)\,\dfrac{d}{dz}\log\dfrac{Q(q/z)}{Q(1/z)}\Big\vert_{z=-1}$
AAP	$e^{\gamma}(1+z)$	$q^{\frac{p}{N}}(1+qz)$	(*)
q-boson ZRP	e^{γ}	$q^{\frac{p}{N}}(1-z)$	$\dfrac{1}{(p-1)!}\,\dfrac{d^{\,p-1}Q(z)}{dz^{\,p-1}}\Big\vert_{z=0}$

$$(*) = Rq^{-2}(1-q)\,\frac{Q'(z)}{Q(z)}\Big\vert_{x=-\frac{1}{q}} - L(1-q)\,\frac{Q'(z)}{Q(z)}\Big\vert_{x=-1} + p\left(Rq^{-1}+Lq^{-1}\right)$$

Second solution of the TQ-relation

Lemma 1: As a difference equation of the second order, TQ-relation

$$T(z)\,Q(z) = a(z)^N\,Q(qz) + b(z)^N\,Q\left(\frac{z}{q}\right),$$

with $a(z)^N, b(z)^N$ polynomials of degree at most N, and $T(x)$ be the associated polynomial of degree up to N generically has two distinct solutions

$$Q(z) = \prod_{i=1}^{p}(z - z_i)$$

$$P(z) := Q_2(z)$$

ASEP case: from TQ to PQ to TPQ

TQ-relation ($Q(z) = \prod_{i=1}^{p}(z - x_i)$, Bethe roots):

$$T(z)Q(z) = e^{-\gamma N}(1+z)^N Q(qz) + q^p(1+qz)^N Q\left(\tfrac{z}{q}\right).$$

Partial-fraction $R(z) := \dfrac{(1+qz)^N}{Q(z)Q(qz)} = \pi(z) + \dfrac{q_-(z)}{Q(z)} + \dfrac{q_+(z)}{Q(qz)}$,
$\deg \pi \le N - 2p$, $\deg q_\pm \le p$.

Pole cancellation on the $T(z)$ side forces

$$q_+\left(\tfrac{z}{q}\right) = -q^p e^{\gamma N} q_-(z).$$

Introduce $\rho(z)$, $\deg \rho \le N - 2p$, solving $\pi(z) = \rho(z) - q^p e^{\gamma N}\rho(qz)$, and define

$$P(z) := \rho(z)Q(z) + q_-(z), \qquad \deg P \le N - p.$$

Result — the PQ-relation = Wronskian relation

$$\boxed{(1+qz)^N = P(z)\,Q(qz) - q^p e^{\gamma N}\,P(qz)\,Q(z)}$$

$$\boxed{(1+qz)^N = P(z)\,Q(qz) - q^p e^{\gamma N}\,P(qz)\,Q(z)}$$

Define the logarithmic ratio

$$W(z) := -\log\left(\frac{P(z)\,Q(qz)}{e^{\gamma N} q^p\,P(qz)\,Q(z)}\right).$$

Dividing the PQ-relation by the second term

$$\frac{(1+qz)^N}{e^{\gamma N} q^p\,P(qz)Q(z)} = \frac{P(z)Q(qz)}{e^{\gamma N} q^p\,P(qz)Q(z)} - 1 = e^{-W(z)} - 1.$$

Define the convolution operator $X[f](z) := \oint_{C_1} \frac{d\tilde{z}}{2\pi i\,\tilde{z}}\,f(\tilde{z})\,K(z,\tilde{z})$,

with kernel

$$\boxed{K(z,\tilde{z}) := \sum_{k=1}^{\infty} \frac{q^k}{1-q^k}\left[\left(\frac{z}{\tilde{z}}\right)^k + \left(\frac{z}{\tilde{z}}\right)^{-k}\right]}$$

From PQ-relation

$$\boxed{(1+qz)^N = P(z)\,Q(qz) - q^p e^{\gamma N}\,P(qz)\,Q(z)},$$

we turn to the integral equation on $W(z)$.

Key computation: $X[W](z) = -\log \dfrac{P(qz)}{P(0)}\dfrac{Q(z)}{z^p}$ (residues at the roots of P, Q).

Result — the integral equation, with $B := \dfrac{1}{e^{\gamma N}P(0)}$:

$$\boxed{e^{-W(z)} - 1 = B\, e^{X[W](z)}\, \frac{(1+qz)^N}{(qz)^p}}$$

Finally, the eigenvalue is

$$\lambda(\gamma) = -(1-q)\oint_C \frac{dz}{2\pi i\,(1+z)^2}\, W(z).$$

q-boson ZRP case: the PQ-relation

TQ-relation ($Q(z) = \prod_{i=1}^{p}(z - x_i)$, Bethe roots):

$$T(z)Q(z) = e^{\gamma N}\, Q(qz) + q^p(1-z)^N\, Q\left(\tfrac{z}{q}\right).$$

Partial-fraction $R(z) := \dfrac{1}{Q(z)Q(qz)} = \dfrac{q_-(z)}{Q(z)} + \dfrac{q_+(z)}{Q(qz)}$, $\deg q_\pm \le p$.

Pole cancellation forces

$$q_+\left(\tfrac{z}{q}\right) = -e^{-\gamma N}\, q_-(z).$$

Introduce $\rho(z)$, $\deg \rho \le N-1$, solving
$\pi(z) = \rho(z) - e^{-\gamma N} q^p (1-z)^N \rho(qz)$, and define

$$P(z) := \rho(z)Q(z) + q_-(z).$$

Result — the PQ-relation:

$$\boxed{q^p(1-z)^N = P(z)\,Q(qz) - q^p(1-z)^N\, e^{-\gamma N}\, P(qz)\,Q(z)}$$

Outline

Result for asymmetric avalanche process AAP

$$\lambda(\gamma) = \sum_{k=1}^{p} \left(R \frac{1+z_k}{1+qz_k} + L \frac{1+qz_k}{1+z_k} - p \right),$$

where for γ near zero we have $z_i \to 0$.

Let *pos* denote the projection onto non-negative Fourier modes of a Laurent series on the unit circle

$$\begin{aligned} \lambda(\gamma) &= -q^{-2}(1-q) \oint_{C_q} \frac{RposX[W](z) - LqposX[W](qz)}{(z+q^{-1})^2} \frac{dz}{2\pi i} \\ &= -q^{-2}(1-q) \sum_{k=1}^{\infty} D_k \frac{B^k}{k}, \end{aligned} \tag{2}$$

$$W(0) = \gamma N \qquad \longrightarrow \qquad \gamma N = \oint \frac{dz}{2\pi i z} W(z) = \sum_{k=1}^{\infty} C_k \frac{B^k}{k}.$$

Result for asymmetric avalanche process AAP

$$c_1 = -q^{-2}(1-q)LD_1C_1$$
$$c_2 = q^{-2}(1-q)L^2\frac{(1-q)L^2D_1C_2 - D_2C_1}{2C_1^3}$$

and so on.

Conclusion

- For the study of large deviations of additive observables in interacting particle models, we can separate out those that are **Bethe integrable**.
- The Bethe equations contain all the information, and we access them through the functional **TQ**, **TP**, and **PQ** relations. Physically different processes turn out to share the **same structure** of PQ-relation, which can be rewritten as an integral equation whose **kernel coincides** across models.
- **Open problem:** understand the origin of this coincidence, and separate the **universal** from the **model-dependent** data.

Thank you!