Exact asymptotic analysis of the correlation functions of the Bose gas beyond thermal equilibrium

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Dynamical correlation functions

Among the most important quantities to study in quantum many-body systems are dynamical two-point correlation functions:

- measure how physical properties at two points in space and time are related;
- reveal information about system's response to external perturbations, describe the transport properties, etc.;
- microscopic quantum mechanical description $\rightarrow$ experimental observables;
- in long-time, large-distance limit they reveal generic behaviour.

They are usually very hard to compute even for integrable models.

The Lieb–Liniger model

The Hamiltonian of the Lieb–Liniger model is

$$H = \int_0^L \left[\partial_y \Psi^\dagger(y) \partial_y \Psi(y) + c\, \Psi^\dagger(y)\Psi^\dagger(y)\Psi(y)\Psi(y)\right] \mathrm{d}y\,.$$

$\Psi(y)$ and $\Psi^\dagger(y)$ are canonical Bose fields,

$$\left[\Psi(x), \Psi^\dagger(y)\right] = \delta(x-y), \quad \left[\Psi(x), \Psi(y)\right] = \left[\Psi^\dagger(x), \Psi^\dagger(y)\right] = 0.$$

L is the length of the system, $c > 0$ is the coupling constant, and periodic boundary conditions are implied.
The number of particles and momentum are conserved, and the model is equivalent to N non-relativistic bosons with pairwise δ-function interaction.
When $c \to +\infty$ the model is called the impenetrable Bose gas or Tonks–Girardeau gas.

Dynamical field-field correlation function

The dynamical field-field correlation function:

$$g_N(x,t) = \langle \mathbf{k}_N | \Psi(x,t)\Psi^\dagger(0,0) | \mathbf{k}_N \rangle \,.$$

$|\mathbf{k}_N\rangle$ is an N-particle Bethe state with momentum and energy

$$P(\mathbf{k}_N) = \sum_{j=1}^{N} p(k_j), \qquad E(\mathbf{k}_N) = \sum_{j=1}^{N} \varepsilon(k_j).$$

Quasi-momenta $k_1, \ldots, k_N \in \mathbb{R}$ are the Bethe roots. We call $|\mathbf{k}_N\rangle$ the reference state.

We study $g_N(x,t)$ at free fermion point ($c \to +\infty$) and in the thermodynamic limit ($N, L \to \infty$ at fixed $D = N/L$),

$$g(x,t) = \lim_{\substack{N,L\to\infty \\ D=N/L}} \lim_{c\to+\infty} g_N(x,t).$$

We are interested in the long-time, large-distance asymptotics of $g(x,t)$, when $x, t \to +\infty$ for a fixed distance-to-time ratio x/t.

The form factor series: the free fermion point

The dynamical field-field correlation function:

$$g_N(x,t) = \langle \mathbf{k}_N | \Psi(x,t) \Psi^\dagger(0,0) | \mathbf{k}_N \rangle \, .$$

The form factor series:

$$g(x,t) = \sum_{\mathbf{q}_{N+1}} \left| \langle \mathbf{q}_{N+1} | \Psi^\dagger(0,0) | \mathbf{k}_N \rangle \right|^2 \mathrm{e}^{\mathrm{i}t[E(\mathbf{k}_N) - E(\mathbf{q}_{N+1})] - \mathrm{i}x[P(\mathbf{k}_N) - P(\mathbf{q}_{N+1})]} .$$

At free fermion point ($c \to +\infty$) the Bethe roots satisfy

$$1 + \mathrm{e}^{-\mathrm{i}k_j L} (-1)^N = 0, \qquad j = 1, \ldots, N$$

the one-particle momentum and energy take the form $p(k) = k$, $\varepsilon(k) = k^2$, and the form factor simplifies

$$\lim_{c \to +\infty} \left| \langle \mathbf{q}_{N+1} | \Psi^\dagger(0,0) | \mathbf{k}_N \rangle \right|^2 = \frac{2^{2N}}{L^{2N+1}} D\big(\mathbf{q}_{N+1}, \mathbf{k}_N\big)^2,$$

where $D(\mathbf{q}_{N+1}, \mathbf{k}_N)$ is a generalized Cauchy determinant.

The form factor series: the thermodynamic limit

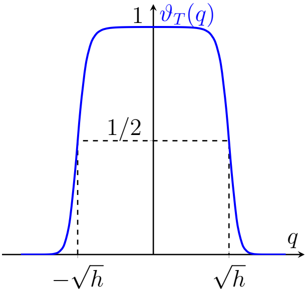
A sequence of sets of Bethe roots $\mathbf{k}_N$ satisfies the condensation property, if for any continuous function f there exists a smooth function $\vartheta : \mathbb{R} \to [0,1]$ such that

$$\lim_{\substack{N,L\to+\infty \\ D=N/L}} \frac{1}{L}\sum_{j=1}^{N} f(k_j) = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi}\vartheta(k)f(k)$$

The function $\vartheta(k)$ is the probability of the single-particle state with quasi-momentum k to be occupied and is called the filling fraction.

In thermal equilibrium with temperature T and chemical potential h:

$$\vartheta_T(q) = \Big[1+\exp\Big(\frac{q^2-h}{T}\Big)\Big]^{-1}.$$



In a generalized Gibbs ensemble:

$$\vartheta_{\mathsf{GGE}}(q) = \Big[1+\exp\Big(\frac{1}{T}\sum_{k=0}^{n}\beta_k q^k\Big)\Big]^{-1}$$

In what follows we treat ϑ as a functional parameter!

Fredholm determinant representation

$g(x,t)$ can be expressed as a Fredholm determinant (minor)

$$g(x,t) = A(x,t)\ \det_{\mathbb{R}}\left(\mathrm{id} + \mathrm{V}\right),$$

of an integrable integral operator V with kernel

$$V(p,q) = \frac{\vartheta(q)\mathrm{e}^{\mathrm{i}\varphi(p)/2}\mathrm{e}^{\mathrm{i}\varphi(q)/2}}{\pi\mathrm{i}(p-q)}\Big[\frac{\mathrm{erf}(\sqrt{-\mathrm{i}t}(p-q_0))}{\mathrm{e}^{\mathrm{i}\varphi(p)}} - \frac{\mathrm{erf}(\sqrt{-\mathrm{i}t}(q-q_0))}{\mathrm{e}^{\mathrm{i}\varphi(q)}}\Big].$$

Here $\varphi(q) = tq^2 - xq$, $q_0 = x/2t$ is the saddle point, $\varphi'(q_0) = 0$, and $A(x,t)$ is an explicit function,

In thermal equilibrium the Fredholm determinant representation was derived in [KS'90] and its long-time, large-distance behaviour studied in the case of thermal equilibrium was studied in [IIKS'90] and [IIKV'92].

$\det(\mathrm{id} + \mathrm{V}) \qquad \leftrightarrow \qquad$ a matrix Riemann–Hilbert problem

Riemann–Hilbert analysis of a generalized sine kernel

In [GKM arXiv:2608.16727] we consider a Fredholm determinant of a generalized sine kernel $\mathrm{V}_{\mathrm{GSK}}$, an the integrable integral operator having four functional parameters:

- the phase function $\varphi(q) = t\varepsilon(q) - xp(q)$;
- the filling fraction $\vartheta(q)$;
- ⋆ two auxiliary functions: $\nu(q)$, $g(q)$.

We derived the asymptotic expansions of the Fredholm determinant $\det_{\mathbb{R}}(\mathrm{id} + \mathrm{V}_{\mathrm{GSK}})$ in physically interesting cases, using the nonlinear steepest descent method [DZ'93] and following [KKMST'09], [S'10], and [K'10].

By setting $\nu(q) = 1/2$, $g(q) = 0$, $p(q) = q$, and $\varepsilon(q) = q^2$, $\mathrm{V}_{\mathrm{GSK}}$ takes the form of V appearing in $g(x, t)$.

Application to the impenetrable Bose gas

The form of the asymptotic expansion of $g(x,t)$ is determined by:

1. the saddle point $q_0 = x/2t$: the solution of $xp'(q_0) - t\varepsilon'(q_0) = 0$;
2. generalized Fermi points defined as solutions of

$$\vartheta(q) = 1/2 \ .$$

E.g., in the thermal equilibrium the Fermi points are given by $\pm\sqrt{h}$ with $h > 0$ being the chemical potential.

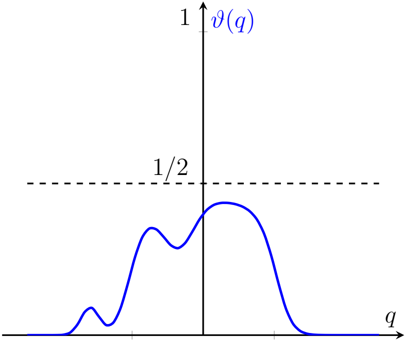
We derived the asymptotic behaviour of $g(x,t)$ for two classes of ϑ having either 0 or 2 distinct generalized Fermi points on $\mathbb{R}$. The asymptotic behaviour of $g(x,t)$ as $x, t \to \infty$ for a fixed $q_0 = x/2t$ has the form

$$g(x,t) = \mathcal{A} t^{\nu} \exp(-\alpha t) \left[1 + \frac{\mathcal{B}}{\sqrt{t}} \big(1 + \mathrm{o}(1)\big) + \mathrm{O}\Big(\frac{(\ln t)^2}{t}\Big) \right],$$

where $\mathcal{A}$, ν, $\alpha > 0$, and $\mathcal{B}$ are explicitly given in terms of q_0 and ϑ. $\mathcal{B} = 0$ in the case of zero generalized Fermi points on $\mathbb{R}$.

Class 1 of filling fractions ϑ

The filling fraction $\vartheta(q)$ is smooth and decaying at least as q^{-8} for $q \to \pm\infty$. It takes values in $[0, 1/2)$.



Remark. Conjecturally $\vartheta(q) = \mathrm{o}(q^{-2})$ as $q \to \pm\infty$.

Asymptotic expansion for filling fractions from Class 1

As $x,t\to+\infty$ the correlation function $g(x,t)$ behaves as

$$g(x,t)=\mathcal{A}t^{\nu}\exp\big(-\alpha t+\mathrm{i}\psi(q_0)\big)\Big\{1+\mathrm{O}\Big(\frac{(\ln t)^2}{t}\Big)\Big\}.$$

Here $q_0=x/2t$,

$$\nu=-\,\big(1+\mathrm{i}\omega(q_0)\big)^2/2\,,\qquad \omega(q)=-\frac{1}{\pi}\ln|1-2\vartheta(q)|>0,$$

$$\alpha=\int_{\mathbb{R}}\mathrm{d}q\,\omega(q)|q_0-q|,\qquad \psi(q)=xp(q)-t\varepsilon(q)-\varphi(q),$$

$$\varphi(q)=\fint_{\mathbb{R}}\mathrm{d}k\,\mathrm{sgn}(q_0-k)\omega'(k)\ln|k-q|,$$

$$\mathcal{A}=\frac{G^2\big(\frac{1}{2}-\frac{\mathrm{i}\omega(q_0)}{2}\big)G\big(1+\frac{\mathrm{i}\omega(q_0)}{2}\big)G\big(2+\frac{\mathrm{i}\omega(q_0)}{2}\big)}{2G\big(\frac{1}{2}\big)G\big(\frac{3}{2}\big)}$$
$$\times\exp\Bigg\{-\frac{\mathrm{i}\pi}{4}+\omega(q_0)\varphi(q_0)-\frac{1}{4}\int_{\mathbb{R}}\mathrm{d}q\,\mathrm{sgn}(q_0-q)\omega'(q)\varphi(q)\Bigg\},$$

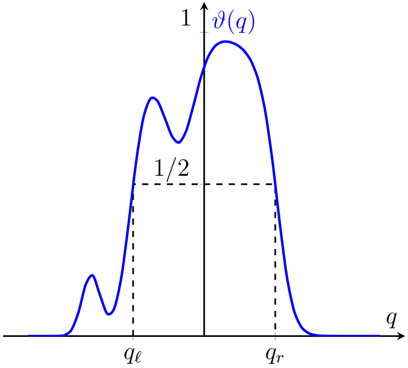
where G is the Barnes G-function.

Class 2 of filling fractions ϑ

$\vartheta(q)$ is smooth and decaying at least as q^{-8} for $q \to \pm\infty$. There are $q_\ell, q_r \in \mathbb{R}$ with $q_\ell < q_r$ such that

$$\vartheta(q) \in \begin{cases} [0,\, 1/2\,) & q \in (-\infty, q_\ell), \\ (\,1/2\,,1] & q \in (q_\ell, q_r), \\ [0,\, 1/2\,) & q \in (q_r, \infty). \end{cases}$$

and $\vartheta'(q_\ell), \vartheta'(q_r) \neq 0$.



Asymptotic expansion for filling fractions from Class 2

As $x, t \to +\infty$ the correlation function $g(x,t)$ behaves as

$$g(x,t) = \mathcal{A} t^{\nu} \exp\big(-\alpha t + \mathrm{i}\psi(q_\ell)\big)$$
$$\times \left\{ 1 + \frac{\mathcal{B} \mathrm{e}^{\mathrm{i}\psi(q_0)}}{t^{1/2 + \mathrm{i}\omega(q_0)}} \left[\frac{\exp\big(-\mathrm{i}\psi(q_\ell)\big)}{|q_0 - q_\ell|^{2+2\mathrm{i}\omega(q_0)}} + \frac{\exp\big(-\mathrm{i}\psi(q_r)\big)}{|q_0 - q_r|^{2+2\mathrm{i}\omega(q_0)}} \right] + \mathrm{o}\Big(\frac{1}{t^{1/2}}\Big) \right\}$$

Here $q_0 = x/2t$ and $q_0 \neq q_\ell, q_r$, and $\omega(q) = -\frac{1}{\pi} \ln |1 - 2\vartheta(q)| > 0$.

$$\nu = \frac{\omega^2(q_0)}{2}, \qquad \psi(q) = xp(q) - t\varepsilon(q) - \varphi(q),$$

$$\alpha = \frac{\mathrm{i}x}{2}\big(\varepsilon(q_r) - \varepsilon(q_\ell)\big) - \frac{\mathrm{i}x^2}{2t}\big(p(q_r) - p(q_\ell)\big) + \int_{\mathbb{R}} \mathrm{d}q\, \omega(q)|q_0 - q|.$$

The coefficients $\mathcal{A}$ and $\mathcal{B}$ are also explicit functions of q_0 and $\vartheta(q)$.

Analytical cross-check

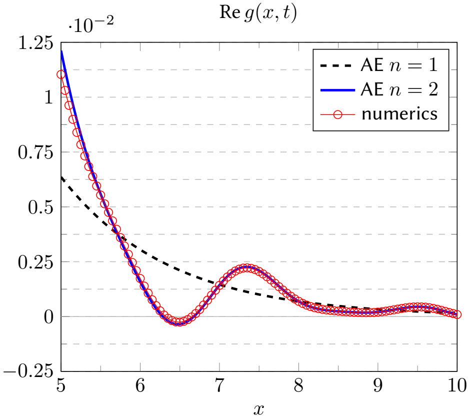
In particular, in thermal equilibrium, i.e., for

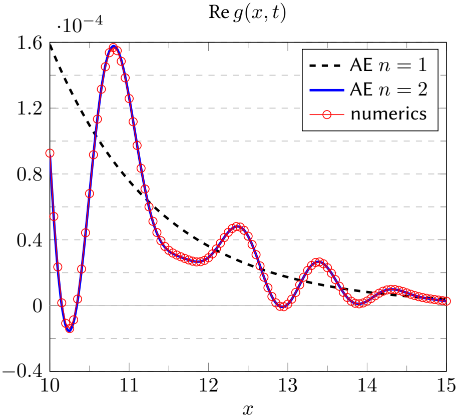
$$\vartheta_T(q) = \frac{1}{1 + \exp\left(\frac{q^2 - h}{T}\right)},$$

$\vartheta_T(q) \in [0,\, 1/2\,)$ for $h < 0$; and $\vartheta_T(\pm\sqrt{h}) = 1/2$ for $h > 0$, and we reproduce asymptotic expansions of $g(x, t)$ from [IIKV'92].
Our expression for the overall constant is complete and simpler than one in [IIKV'92].

Numerical cross-check: $h > 0$, space-like regime

$g(x,t)$ as a function of x for $t = 1$, $h = 1$ and $T = 1$: the leading term ($n = 1$, dashed) and the leading + the sub-leading terms ($n = 2$, blue); the numerical data (red circles).





Summary

1. The asymptotic analysis of the Fredholm determinant of the generalized sine kernel V_{GSK} is performed in [GKM arXiv:2608.16727].
2. The long-time and large-distance behaviour of the field-field correlation function of the impenetrable Bose gas is derived for the two classes of filling fractions ϑ (in preparation for publication).
The resulting asymptotic expansions:

- are 'complete': the leading and sub-leading terms, logarithmic corrections, and the overall constant;
- in particular, cover both thermal and non-thermal equilibria (GGE) on the same level of complexity;
- depend on the filling fraction ϑ, the saddle-point $q_0 = x/2t$, and the generalized Fermi points (the solutions of $\vartheta(q) = 1/2$);
- are compared with literature [IIKV'92] and numerical analysis for the case of thermal equilibrium.

Outlook

The asymptotic analysis of the generalized sine kernel using Riemann–Hilbert techniques [GKM arXiv:2608.16727] allows one to study:

1. other correlation functions of the impenetrable Bose gas, e.g., the generating function for the density-density $\langle \rho(x,t)\rho(0,0)\rangle$ and the current density-current density $\langle j(x,t)j(0,0)\rangle$ correlation functions (work in progress);
2. other models having the same kernel: the impenetrable Fermi gas, the impenetrable anyon gas, the two-component Bose gas, etc;
3. other kernels: e.g., the transverse correlation function $\left\langle \sigma_{m+1}^{+}\sigma_{1}^{-}\right\rangle_{T}$ of the XX spin chain (modification of the analysis is needed);

$\star$ the Lieb–Liniger model with finite coupling constant $c > 0$, using the auxiliary functional parameter $\nu(q)$ and $g(k)$ following [KMS'11, KT'11, K'15] (work in progress).

Thank you for your attention!

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Associated matrix Riemann–Hilbert problem

One can associate to an integrable integral operator V,

$$V(p,q) = \frac{\mathbf{E}_L^{\mathsf{T}}(p) \cdot \mathbf{E}_R(q)}{p-q}, \qquad \mathbf{E}_L^{\mathsf{T}}(q) \cdot \mathbf{E}_R(q) = 0,$$

where $\mathbf{E}_{L,R}(q) : \mathbb{C} \to \mathbb{C}^2$, the following matrix Riemann–Hilbert problem. Determine $\chi(q) \in \mathbb{C}^{2\times 2}$ such that

1. $\chi(q)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$;
2. $\chi(q)$ satisfies the jump condition $\chi(q - \mathrm{i}0) = \chi(q + \mathrm{i}0) G_\chi(q)$ for $q \in \mathbb{R}$ with the jump matrix

$$G_\chi(q) = \mathrm{I}_2 + 2\pi\mathrm{i}\, \mathbf{E}_R(q) \cdot \mathbf{E}_L^{\mathsf{T}}(q);$$

3. $\chi(q) = \mathrm{I}_2 + \mathrm{O}\big(q^{-1}\big)$ as $q \to \infty$.

Let $\beta = x$ or q_0 and $\Gamma(\mathbb{R})$ be a narrow loop around $\mathbb{R}$, then

$$\partial_\beta \ln \det_{\mathbb{R}}(\mathrm{id} + \mathrm{V}) = \int\limits_{\Gamma(\mathbb{R})} \frac{\mathrm{d}q}{4\pi} \operatorname{tr}\Big\{ \chi'(q) \begin{pmatrix} 1 & f(q) \\ 0 & -1 \end{pmatrix} \chi^{-1}(q) \Big\} \partial_\beta \Big\{ x \Big(q - \frac{q^2}{2q_0} \Big) \Big\}.$$

Generalized sine kernel I

Consider an integrable integral operator V acting on $L^2(\mathcal{C})$, which is a generalization of V depending on 4 functional parameters, u, ϑ, ν, and g,

$$V(q,k) = \frac{\mathbf{E}_L^{\intercal}(q)\cdot\mathbf{E}_R(k)}{q-k}, \qquad \mathbf{E}_L^{\intercal}(q)\cdot\mathbf{E}_R(q) = 0,$$

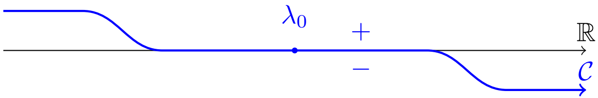
where $\mathbf{E}_L, \mathbf{E}_R$ are vector-valued functions

$$\mathbf{E}_L(q) = \sin\left(\pi\nu(q)\right)\begin{pmatrix}-e(q)\\ E(q)\end{pmatrix}, \quad \mathbf{E}_R(q) = \frac{2\vartheta(q)\sin\left(\pi\nu(q)\right)}{\pi\mathrm{i}}\begin{pmatrix}E(q)\\ e(q)\end{pmatrix}.$$

Here

$$e(q) = \exp\left[-\frac{\mathrm{i}x}{2}u(q) - \frac{g(q)}{2}\right], \quad E(q) = \frac{e^{-1}(q)}{\exp(-2\pi\mathrm{i}\nu(q))-1} - e(q)\mathrm{C}_{-}(q).$$

The function $\mathrm{C}(q)$ is the Cauchy transform of $e^{-2}(q)$ with respect to the contour $\mathcal{C}$. The integration contour $\mathcal{C}$ is a slight deformation of the contour along the real axis.



Generalized sine kernel II

The four functional parameters u, g, ν and ϑ are defined by their analytic properties in a finite strip Ω

$$\Omega = \left\{ z \in \mathbb{C} \middle| |\mathrm{Im}\, z| < w, w > 0 \right\},$$

containing the integration contour $\mathcal{C}$. The functions u, g and ν are holomorphic and ϑ is meromorphic in the strip Ω. In particular, we assume that the "monodromy conditions" are satisfied

$$\int_{\mathcal{C}} \mathcal{L}'_\ell(q)\, \mathrm{d}q = 0 = \int_{\mathcal{C}} \mathcal{L}'_r(q)\, \mathrm{d}q$$

with

$$\mathcal{L}_\ell(q) = -\frac{1}{2\pi \mathrm{i}} \ln\left[1 + \vartheta(q)\left(\mathrm{e}^{2\pi \mathrm{i}\nu(q)} - 1\right)\right], \quad \mathcal{L}_r(q) = \frac{1}{2\pi \mathrm{i}} \ln\left[1 + \vartheta(q)\left(\mathrm{e}^{-2\pi \mathrm{i}\nu(q)} - 1\right)\right],$$

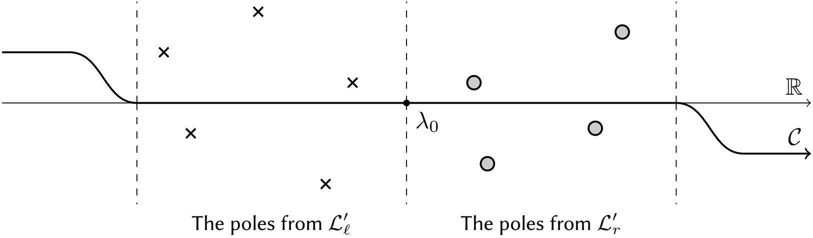
Setting

$$u(q) = q - \frac{q^2}{2q_0}, \qquad \nu(q) = 1/2\ , \qquad g(q) = 0$$

in $V(q,k)$ with $q_0 = x/2t$, we obtain the kernel $V(q,k)$.

Contribution of the poles

Certain poles of $\mathcal{L}'_\ell$, which are to the left from q_0, and certain poles of $\mathcal{L}'_r$, which are to the right from it, contribute to the asymptotic expansion. We denote the set of the poles as $\mathcal{S}$.



- The contribution of the poles away from $\mathbb{R}$ is $\mathrm{O}(x^{-\infty})$.
- The contribution of the poles on $\mathbb{R}$ is $\mathrm{O}(1)$, the contour should be deformed.

We consider three cases:

1. there are no poles on $\mathbb{R}$ (leading + sub-leading orders);
2. there are n poles in Ω (leading order);
3. there are exactly two poles on $\mathbb{R}$ (leading + sub-leading orders).

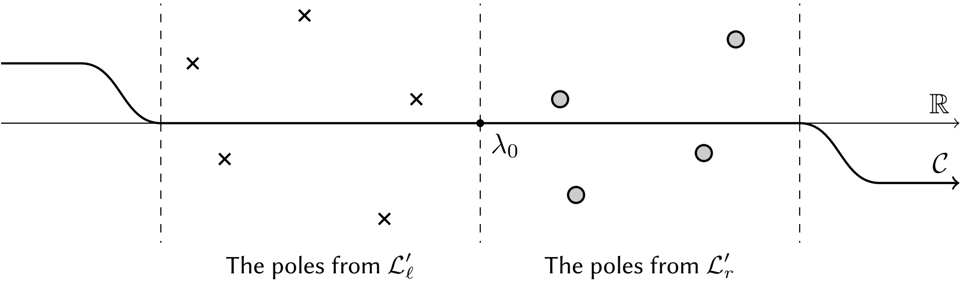
Theorem 1: generalized sine kernel, no poles on $\mathbb{R}$

Let $\mathcal{S} \cap \mathbb{R} = \emptyset$, then the Fredholm determinant of the integrable integral operator V has the large-x asymptotic expansion

$$\det_{\mathcal{C}}(\mathrm{id} + \mathrm{V}) = A(q_0) x^{-\frac{\tau^2(q_0)}{2}} \times \exp\left\{ -\int_{\mathcal{C}} \mathcal{L}(q|q_0) \left(\mathrm{i}xu'(q) + g'(q)\right) \mathrm{d}q \right\} \cdot \left(1 + \mathrm{O}\left(\frac{(\ln x)^2}{x}\right)\right).$$

Here the functions $\tau(q)$, $\mathcal{L}(q)$, and $A(q_0)$ are explicitly given in terms of $u(q)$, $\vartheta(q)$ and $\nu(q)$.

Theorem 2: generalized sine kernel, n poles in Ω



We split the set of poles $\mathcal{S}$ into two sets $\mathcal{S}^\pm$,

$$\mathcal{S}^\pm = \left\{ q \in \mathcal{S} \cap \Omega^\pm \,\middle|\, \operatorname{Re} q < q_0 \right\} \cup \left\{ q \in \mathcal{S} \cap \Omega^\mp \,\middle|\, \operatorname{Re} q > q_0 \right\}.$$

The cardinalities of the sets $\mathcal{S}^\pm$ are $n^\pm \in \mathbb{N}_0$.

Theorem 2: generalized sine kernel, n poles in Ω

The Fredholm determinant of the integrable integral operator V has the large-x asymptotic expansion

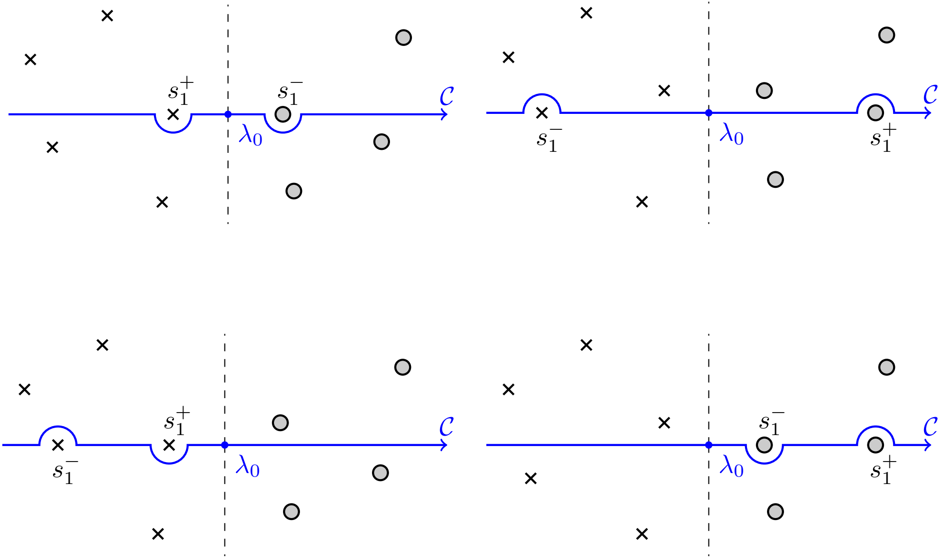
$$\det_{\mathcal{C}}\big(\mathrm{id}+\mathrm{V}\big) = A(q_0)\det\big(\mathrm{I}_{n^+}-\mathrm{A}\big)x^{-\frac{\tau^2(q_0)}{2}} \\ \times \exp\Bigg\{-\int_{\mathcal{C}}\mathcal{L}(q|q_0)\left(\mathrm{i}xu'(q)+g'(q)\right)\mathrm{d}q\Bigg\}\big(1+\mathrm{o}(1)\big),$$

where the leading order contribution from the poles in $\mathcal{S}^{\pm}$ is comprised in the matrix $\mathrm{A}=\mathrm{A}^{-}\mathrm{A}^{+}$,

$$A^{-}_{jk}=\frac{h^{-}_{k}}{s^{+}_{j}-s^{-}_{k}},\quad A^{+}_{kj}=\frac{h^{+}_{j}}{s^{-}_{k}-s^{+}_{j}},\quad j=1,\dots,n^{+},\quad k=1,\dots,n^{-}.$$

Here the functions $\tau(q)$, $\mathcal{L}(q)$, $A(q_0)$ and the coefficients $h^{\pm}_{k}$ for $k=1,\dots,n^{\pm}$ are explicitly given in terms of $u(q)$, $\vartheta(q)$, $\nu(q)$ and $g(q)$.
Remark. $h^{\pm}_{j}=\mathrm{O}(1)$ for $s^{\pm}_{j}\in\mathbb{R}$ and $h^{\pm}_{j}=\mathrm{O}(x^{-\infty})$ for $s^{\pm}_{j}\notin\mathbb{R}$.

Theorem 3: generalized sine kernel, 2 poles on $\mathbb{R}$



Theorem 3: generalized sine kernel, 2 poles on $\mathbb{R}$

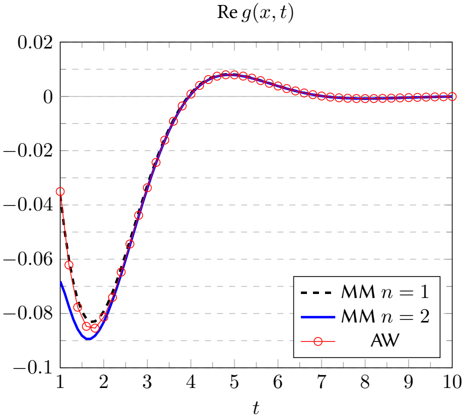
Let the sets $\mathcal{S}^{\pm} \cap \mathbb{R}$ both have cardinality one and set $\{s^{\pm}\} = \mathcal{S}^{\pm} \cap \mathbb{R}$. Then the Fredholm determinant of the integrable integral operator V has the large-x asymptotic expansion

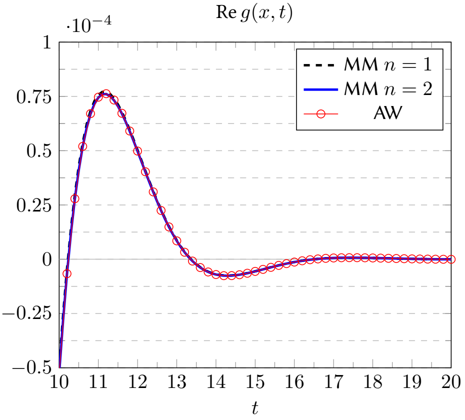
$$\det_{\mathcal{C}}(\mathrm{id} + \mathrm{V}) = A(q_0) x^{-\frac{\tau^2(q_0)}{2}} \exp\left\{ -\int_{\mathcal{C}} \mathcal{L}(q|q_0) \left(\mathrm{i}xu'(q) + g'(q) \right) \mathrm{d}q \right\}$$
$$\times \left\{ 1 + \frac{h^+ h^-}{(s^+ - s^-)^2} + \frac{\sqrt{2}(1 + \mathrm{o}(1))}{x^{\frac{1}{2}} \sqrt{-u''(q_0)}} \left(\frac{h^+ b_{21}(q_0)}{(q_0 - s^+)^2} + \frac{h^- b_{12}(q_0)}{(q_0 - s^-)^2} \right) \right\}$$

Here the functions $\tau(q)$, $\mathcal{L}(q)$, $b_{12}(q)$ and $b_{21}(q)$, as well as the coefficients h_ℓ and h_r, and the constant $A(q_0)$, are explicitly given and depend on the saddle point q_0 and on the filling fraction ϑ. In general, the coefficients h_ℓ and h_r also depend on a “regime”, i.e., on the relative position of the saddle point q_0 with respect to the poles $s^{\pm}$.

Numerical cross-check: $h > 0$, time-like regime

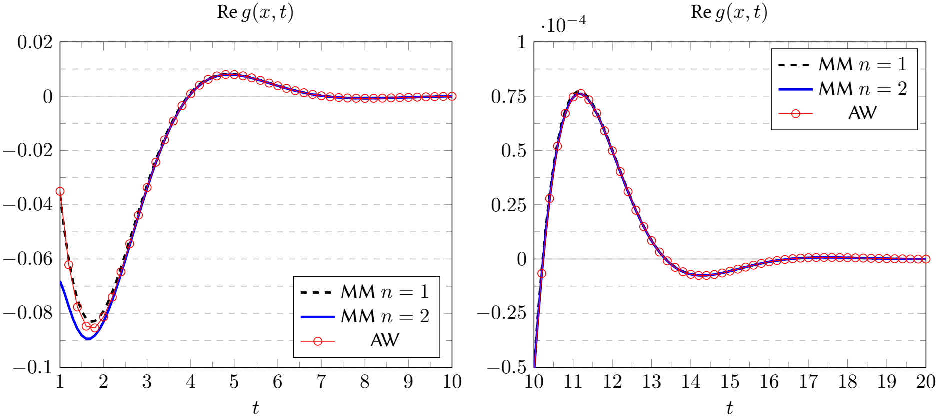
$g(x,t)$ as a function of t for $x = 1$, $h = 1$ and $T = 0.5$: the leading term ($n = 1$, dashed) and the leading + the sub-leading terms ($n = 2$, blue); the numerical data [AW] (red circles).





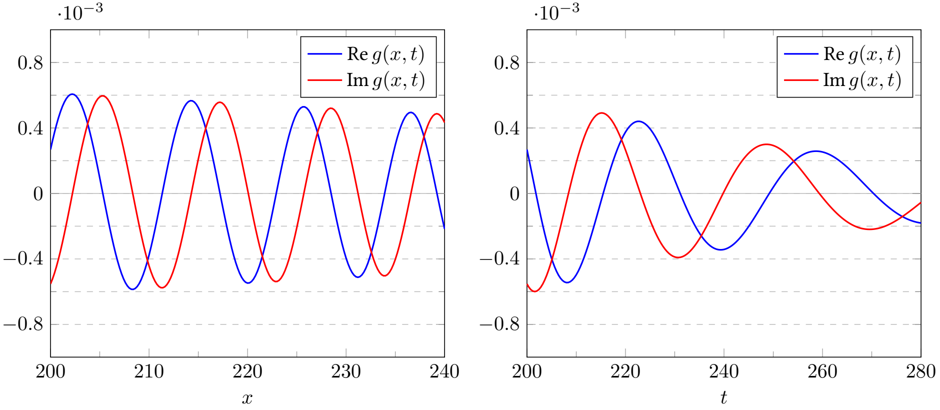
Numerical cross-check: $h > 0$, time-like regime

Here, the plots with $n = 1$ and $n = 2$ are indistinguishable, but an additional verification shows that the sign in front of the sub-leading term is correct.



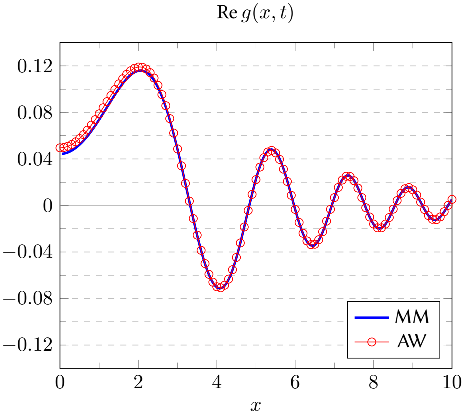
Very large distance x and time t

Correlation function $g(x,t)$ for $h=-5$ and $T=1.25$: as a function of x for $x \in [200,240]$ at $t=200$ on the left and as function of t for $t \in [200,280]$ at $x=200$ on the right.



Numerical cross-check: $h < 0$, x-dependence

Asymptotic expansion of the correlation function $g(x,t)$ (blue) and the numerical analysis data [AW] (red circles, every 5th point).



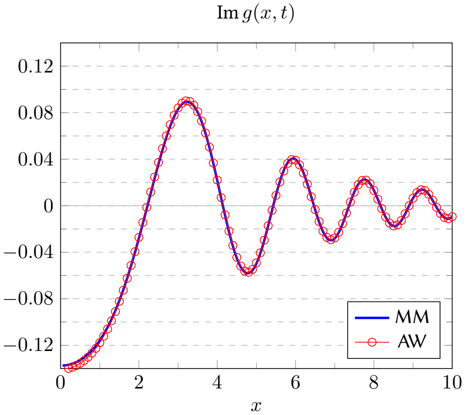


Figure: Correlation function as a function of x for $t = 1$, $h = -5$ and $T = 4$.

Numerical cross-check: $h < 0$, t-dependence

Asymptotic expansion of the correlation function $g(x,t)$ (blue) and the numerical analysis data [AW] (red circles, every 5th point).

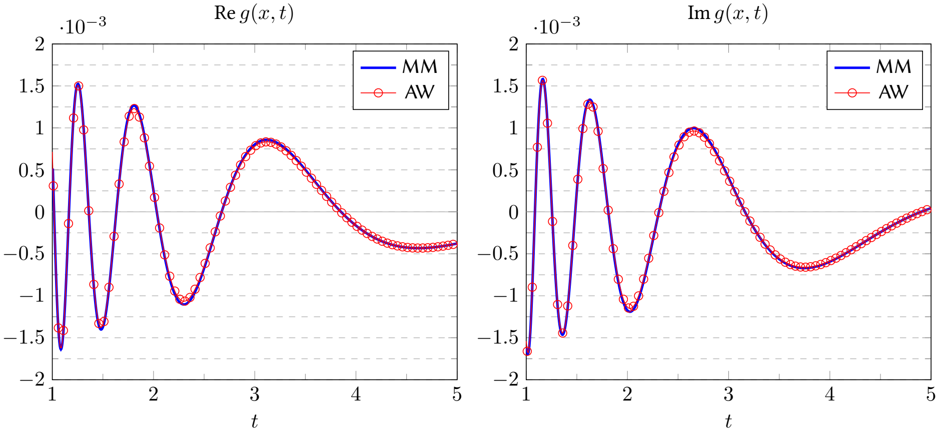


Figure: Correlation function as a function of t for $x = 10$, $h = -1$ and $T = 2$.