R-matrix Dunkl operators and spin Calogero-Moser system

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Chalykh, M.M. R-matrix Dunkl operators and spin Calogero-Moser system, Annales Henri Poincaré 2026

M.M., Mostovskii, Zotov, R-matrix valued Lax pair for elliptic Calogero-Inozemtsev system and associative Yang-Baxter equation of BC type, Letters in Mathematical Physics, 2026

Chalykh, M.M. R-matrix Calogero-Moser system of BC type, (in progress)

Rational Calogero-Moser systems

- x_i, p_i - coordinates and momenta $i=1,\dots,n$

[Calogero 71]

- Hamiltonian

classical: $$H=\sum_{i=1}^{n} p_i^2 - æ^2 \sum_{i\neq j}^{n} \frac{1}{(x_i-x_j)^2}$$

$\hbar \to 0$

quantum: $$\hat{H}=\sum_{i=1}^{n} \hat{p}_i^2 - æ(æ-\hbar) \sum_{i\neq j}^{n} \frac{1}{(x_i-x_j)^2}$$

$\hat{p}_i = \hbar \frac{\partial}{\partial x_i}$

Integrability

classical $\exists\ H_1, H_2, \dots H_n$: $\{H_i, H_j\}=0$

independent

where the Poisson bracket is canonical: $\{x_i, p_j\}=\delta_{ij}$

quantum $\exists\ \hat{H}_1, \hat{H}_2, \dots \hat{H}_n$ $[H_i, H_j]=0$

Solution of the One-Dimensional N-Body Problems with Quadratic and/or Inversely Quadratic Pair Potentials

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(Received 27 May 1970)

1. INTRODUCTION

The motivation for a physicist to study 1-dimensional problems is best illustrated by the story of the man who, returning home late at night after an alcoholic evening, was scanning the ground for his key under a lamppost; he knew, to be sure, that he had dropped it somewhere else, but only under the lamppost was there enough light to conduct a proper search. In a more serious vein, a motivation is perceived[1] in the insight that exact solutions, even of oversimplified models, may provide, and in the possibility to assess the reliability of approximation techniques that can be used in more realistic contexts, by first testing them in exactly solvable cases. Moreover for some physical problems a 1-dimensional schematization may indeed be appropriate. And a final argument emphasizes the formal elegance that exactly solvable models often reveal.

$$H = \sum_{i=1}^{n} p_i^2 - æ^2 \sum_{i \neq j}^{n} \mathcal{U}(x_i - x_j)$$

$$\mathcal{U}(x) = \begin{cases} \dfrac{1}{x^2} + \omega x^2 & \text{rational} \\[2mm] \dfrac{1}{\sin^2 x} & \text{trigonometric} \\[2mm] \dfrac{1}{\sinh^2 x} & \text{hyperbolic} \\[2mm] \wp(x) & \text{elliptic} \end{cases}$$

Integrability via Dunkl operators

- quantum Dunkl operators: [Dunkl 89], [Heckman 91]

$$y_i = \hbar \frac{\partial}{\partial x_i} - \varkappa \sum_{j \neq i} \frac{1}{x_i - x_j} s_{ij} \in \mathcal{D}(V) * S_n \qquad V = \mathbb{C}^n$$

ring of meromorphic functions and diff. operators on V.

$s_{ij} \in S_n$ – standard transposition

$x_i \leftrightarrow x_j \, ; \, \frac{\partial}{\partial x_i} \leftrightarrow \frac{\partial}{\partial x_j}$

- Main properties

1) commutativity $[y_i, y_j] = 0$

2) equivariance $\sigma y_i \sigma^{-1} = y_{\sigma(i)}$

To the Calogero-Moser Hamiltonians

From symmetric combinations of Dunkl operators one can construct the quantum Calogero-Moser Hamiltonians $\hat{H}_k$:

$$\boxed{\hat{H}_k = \operatorname{Res} \sum_{i=1}^{n} y_i^k}$$

Here: $\operatorname{Res} : \sum_{\sigma \in S_n} a_\sigma \sigma \to \sum a_\sigma \in \mathcal{D}(V)$ – restriction on S_n-invariants

From commutativity and equivariance of Dunkl operators we have $[\hat{H}_k, \hat{H}_m] = 0$.

Example

$$\sum_{i=1}^{n} y_i^2 = \hbar^2 \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \varkappa \sum_{i \neq j}^{n} \frac{1}{(x_i - x_j)^2} (\varkappa - \hbar s_{ij})$$

Res

$$\hat{H}_2 := \operatorname{Res} \sum_{i=1}^{n} y_i^2 = \hbar^2 \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \varkappa(\varkappa - \hbar) \sum_{i \neq j}^{n} \frac{1}{(x_i - x_j)^2}$$ – Calogero-Moser Hamiltonian

Elliptic Dunkl operators

- elliptic Dunkl operators:

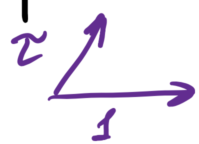
depend on additional variables λ

$$y_i = \hbar \frac{\partial}{\partial x_i} - \varkappa \sum_{j\neq i} \varphi(\lambda_i - \lambda_j,\, x_i - x_j)\, s_{ij}.$$

[Buchstaber, Felder, Veselov 94]

Here $\varphi(z,\mu) = \dfrac{\theta'(0)\,\theta(z-\mu)}{\theta(z)\,\theta(\mu)}$ - Kronecker elliptic function

$\theta(z)$ - odd Jacobi theta-function



- Properties: $[y_i(\lambda),\, y_j(\lambda)] = 0$

$$\sigma\, y_i(\lambda)\, \sigma^{-1} = y_{\sigma(i)}(\sigma(\lambda)) \quad \forall \sigma \in S_n$$

Problem! λ-parameters break S_n-invariance

We need regularisation procedure to construct the Hamiltonians

Elliptic Calogero-Moser systems

The model is described by the Hamiltonian:

$$\hat{H}^{ell} = \hbar^2 \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \varkappa(\varkappa-\hbar) \sum_{i\neq j}^{n} \wp(x_i - x_j)$$

Here $\wp(z) = \frac{1}{z^2} + o(z)$ is the Weierstrass $\wp$-function.

To construct the family of commuting Hamiltonians:

- take the classical rational Hamiltonian $H^{rat}(p,x)$ and put Dunkl operators for momenta, λ for coordinates
- $H^{rat}(y,\lambda)$ – regular at $\lambda \to 0$
- Use $\mathrm{Res}: \sum_{\sigma\in S_n} a_\sigma \to \sum_\sigma a_\sigma \in \mathcal{D}(V)$

[Etingof-Felder-Ma-Veselov 11]

Example

$$\sum_{i=1}^{n} y_i^2 = \hbar^2 \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \varkappa^2 \sum_{i\neq j}^{n} \left(\underbrace{\wp(\lambda_i - \lambda_j)}_{=\frac{1}{(\lambda_i-\lambda_j)^2} + o(\lambda)} - \wp(x_i - x_j)\right) - \varkappa\hbar \sum_{i\neq j} \varphi'(\lambda_i - \lambda_j, x_i - x_j) s_{ij}$$

regular at $\lambda \to 0$

$\varphi'(\mu, z) \xrightarrow[\mu\to 0]{} \wp(z)$

Regularization

$$\mathrm{Res} \lim_{\lambda\to 0} H^{rat}(y,\lambda) = \mathrm{Res} \lim_{\lambda\to 0} \left[\sum_{i=1}^{n} y_i^2 - \varkappa^2 \sum_{i\neq j}^{n} \frac{1}{(\lambda_i - \lambda_j)^2}\right] = \hat{H}^{ell}$$

R-matrix Dunkl operators

Let $\mathcal{U} = \mathcal{U}_1 \otimes \ldots \otimes \mathcal{U}_n \qquad \mathcal{U}_i \cong \mathbb{C}^m. \qquad V = \mathbb{C}^n$

S_n acts by permuting the tensor factors in $\mathcal{U}$:

$P = \sum_{a,b=1}^{m} e_{ab} \otimes e_{ba} \in \operatorname{End}(\mathbb{C}^m \otimes \mathbb{C}^m)$ – permutation operator on $\mathbb{C}^m \otimes \mathbb{C}^m$

standard matrix units

$$P_{ij} = \sum_{a,b=1}^{m} 1\!\!1^{\otimes(i-1)} \otimes e_{ab} \otimes 1\!\!1^{\otimes(j-i-1)} \otimes e_{ba} \otimes 1\!\!1^{\otimes(n-j)}$$

$\mathcal{D}_{\mathcal{U}}(V) := \mathcal{D}(V) \otimes \operatorname{End} \mathcal{U}$ — ring of matrix-valued differential operators on V

$\hat{S}_n$ acts simultaneously on $\mathcal{D}(V)$ and $\operatorname{End}(\mathcal{U})$

s_{ij} $\qquad$ P_{ij}

Let $R(z,\mu) \in \operatorname{End}(\mathbb{C}^m \otimes \mathbb{C}^m)$. Denote by $R_{ij} := R_{ij}(x_i - x_j, \lambda_i - \lambda_j)$

Define the R-matrix Dunkl operators:

$$\boxed{y_i^{(R)} = \hbar \frac{\partial}{\partial x_i} - \varkappa \sum_{j(\neq i)}^{n} R_{ij} \hat{s}_{ij}}$$

[Chalykh, M.M. '26]

Associative Yang-Baxter equation

Let $R(z,\mu)$ satisfy the following properties:

- skew-symmetry : $R_{12}(z,\mu) = -R_{21}(-z,-\mu)$
- associative Yang-Baxter equation : [Fomin, Kirillov 99]

[Polishchuk 02] (parameter depended form)

$$R_{12}(z,\mu)R_{23}(z',\mu') = R_{13}(z+z',\mu')R_{12}(z,\mu-\mu') + R_{23}(z',\mu'-\mu)R_{13}(z+z',\mu)$$

Then R-matrix Dunkl operators commute $[y_i^{(R)}, y_j^{(R)}] = 0$

For $m=1$ the AYBE is Fay's trisecant identity:

$$\varphi(z,\mu)\varphi(z',\mu') = \varphi(z+z',\mu')\varphi(z,\mu-\mu') + \varphi(z',\mu'-\mu)\varphi(z+z',\mu)$$

Properties of $R(z,\mu)$

(1) skew-symmetry $R_{12}(z,\mu) = -R_{21}(-z,-\mu)$

(2) associative Yang-Baxter equation

(3) unitarity $R_{12}(z,\mu)R_{21}(-z,\mu) = (\wp(\mu)-\wp(z))\,Id$

(or any degenerations of $\wp(z) \to \frac{\pi^2}{\sin^2 \pi z} \to \frac{1}{z^2}$)

(4) regularity: $R(z,\mu)$ has the first order pole at $\mu=0$.
$\operatorname{res}_{\mu=0} R(z,\mu)$ is independent of z

From (1), (2), (3) it follows the quantum YB:

$$R_{12}(z,\mu)R_{13}(z+z',\mu)R_{23}(z',\mu) = R_{23}(z',\mu)R_{13}(z+z',\mu)R_{12}(z,\mu)$$

Elliptic Baxter-Belavin R-matrix

All the properties are satisfied by quantum elliptic Baxter-Belavin R-matrix [Belavin 81]

For $m=2$ it is known as 8-vertex R-matrix [Baxter 72]

$$R(z,\mu)=\frac{1}{2}\begin{pmatrix} \varphi_{00}+\varphi_{10} & 0 & 0 & \varphi_{01}-\varphi_{11} \\ 0 & \varphi_{00}-\varphi_{10} & \varphi_{01}+\varphi_{11} & 0 \\ 0 & \varphi_{01}+\varphi_{11} & \varphi_{00}-\varphi_{10} & 0 \\ \varphi_{01}-\varphi_{11} & 0 & 0 & \varphi_{00}+\varphi_{10} \end{pmatrix}$$

Here
$$\varphi_{00}=\phi\left(z,\frac{\mu}{2}\right)$$
$$\varphi_{01}=e^{\pi i z}\phi\left(z,\frac{\tau}{2}+\frac{\mu}{2}\right)$$
$$\varphi_{10}=\phi\left(z,\frac{1}{2}+\frac{\mu}{2}\right)$$
$$\varphi_{11}=e^{\pi i z}\phi\left(z,\frac{1+\tau}{2}+\frac{\mu}{2}\right)$$

Expansion

$$R(z,\mu)=\frac{1}{\mu}\,\mathrm{Id}+r(z)+O(\mu) \quad \text{as } \mu\to 0$$

classical elliptic r-matrix [Belavin, Drienfeld 82]

R-matrix Calogero-Moser Hamiltonians

The recipe is the same as for elliptic Cell system:

- take R-matrix Dunkl operators $y_i^{(R)}$ and put in the classical rational Hamiltonians $H^{rat}(y^R, \lambda)$
- then $\lambda \to 0$
- Apply Res : $\sum_{\hat\sigma \in \hat S_n} a_\sigma \hat\sigma \longrightarrow \sum a_\sigma$

The Hamiltonian

$$\hat H^R = \sum_{i=1}^n \hbar^2 \frac{\partial^2}{\partial x_i^2} - \hbar \varkappa \sum_{i\neq j}^n \wp_{ij} - \varkappa^2 \sum_{i\neq j}^n \wp(x_i - x_j)$$

In the classical limit $\hbar \to 0$: $\hat H^R \to H^{ell}$ (classical elliptic CM Hamiltonian)

Quantum spin Calogero-Moser system

$R(z,\mu) = \varphi(z,\mu)P$ satisfies all the properties, it leads to the Hamiltonian of the quantum spin elliptic Calogero-Moser system

$$H^{\text{spin CM}} = \sum_{i=1}^{n} \hbar^2 \frac{\partial^2}{\partial x_i^2} - \sum_{i\neq j}^{n} \wp(x_i - x_j)\,\varkappa(\varkappa - \hbar P_{ij})$$

$\rightsquigarrow$ freezing trick : put the particles in the equilibrium positions of the classical model $x_j = j/n$:

Inozemtsev spin chain

[Inozemtsev 90, Dittrich, Inozemtsev, 08,09]

$$\boxed{\mathcal{H}^{In} = \sum_{i<j} \wp\left(\frac{i-j}{n}\right) P_{ij}}$$

integrability via Dunkl [Chalykh 24]

Anisotropic long-range spin chain

Using freezing trick one can obtain from $\hat{H}^R_k$ the Hamiltonians $\mathcal{H}_k$ of the long-range spin chain [Sechin - Zotov 18]

$$x_k = \frac{k}{n}$$

$$\hat{H}^R_2 = \sum_{i=1}^n \hbar^2 \frac{\partial^2}{\partial x_i^2} - \hbar \varkappa \sum_{i\neq j}^n \partial r_{ij} - \varkappa^2 \sum_{i \neq j}^n \wp(x_i - x_j) \xrightarrow{\text{freezing}} \mathcal{H}_2 = \sum_{i \neq j} \partial r_{ij}$$

$$\hat{H}^R_3 = \sum_{i<j<k} \Big[\hbar^3 \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} + \hbar \varkappa^2 \Big(\wp(x_i - x_j) \frac{\partial}{\partial x_k} + \wp(x_i - x_k) \frac{\partial}{\partial x_j} + \wp(x_j - x_k) \frac{\partial}{\partial x_i} \Big)$$

$$+ \hbar^2 \varkappa \Big(\partial r_{ij} \frac{\partial}{\partial x_k} + \partial r_{jk} \frac{\partial}{\partial x_i} + \partial r_{ki} \frac{\partial}{\partial x_j} \Big) + \hbar \varkappa^2 \big[\partial r_{ij}, r_{ik} + r_{jk} \big] \Big]$$

$$\xrightarrow{\text{freezing}} \mathcal{H}_3 = \sum_{i<j<k} \big[\partial r_{ij}, r_{ik} + r_{jk} \big]$$

R-matrices and spin chains

$R(z,\mu)$		Spin chain
elliptic $\varphi(z,\mu)P$	$\rightsquigarrow$	Inozemtsev
$\downarrow$ trig $\operatorname{Im}\tau \to \infty$		
$(\cot \pi z + \cot \pi\mu)P$	$\rightsquigarrow$	Haldane-Shastry
elliptic Baxter-Belavin (8-vertex)	$\rightarrow$	[Sechin-Zotov 18]
trig 6-vertex	$\rightarrow$	[Fukui, Kawakami 96]

+ Classification of trigonometric solutions of AYBE
[Schedler 03, Polishchuk 10] (7-vertex, for example)

graded trigonometric R-matrix $gl(N|M)$ $\rightarrow$ [M.M., Zotov 24]

Further direction: relativistic generalization

Calogero-Moser Sutherland model		Ruijsenaars-Shneider model Macdonald operators
Dunkl operators	← non relativistic	Dunkl-Cherednik operators
Jack polynomials		Macdonald polynomials

trig. spin Ruijsenaars model [Lamers, Pasquier, Serban 22]
q-Haldane-Shastry [Lamers 18], [Uglov 95]

elliptic version Baxter-Belavin R-matrix [M.M., Zotov 23]
dynamical R-matrix [Klabbers, Lamers 24]

trig. graded R-matrix [M.M., Zotov 24]
overview [Klabbers, Lamers 25]

Generalization to root systems

W - Weyl group, $\Phi \subset V$ - root system of W ; $\xi \in V$, $\lambda \in V$

$R \in \mathrm{End}_{\mathbb{C}}\, \mathcal{U}$ satisfying $R_\alpha(z,\mu) = -R_{-\alpha}(-z,-\mu)$

R-matrix Dunkl operator:

$$y_\xi = \partial_\xi - \frac{1}{2}\sum_{\alpha\in\Phi} \varkappa_\alpha (\alpha,\xi)\, R_\alpha\big((\alpha,\xi),(\alpha^\vee,\lambda)\big)\hat{s}_\alpha$$

[Chalykh, M.M.]

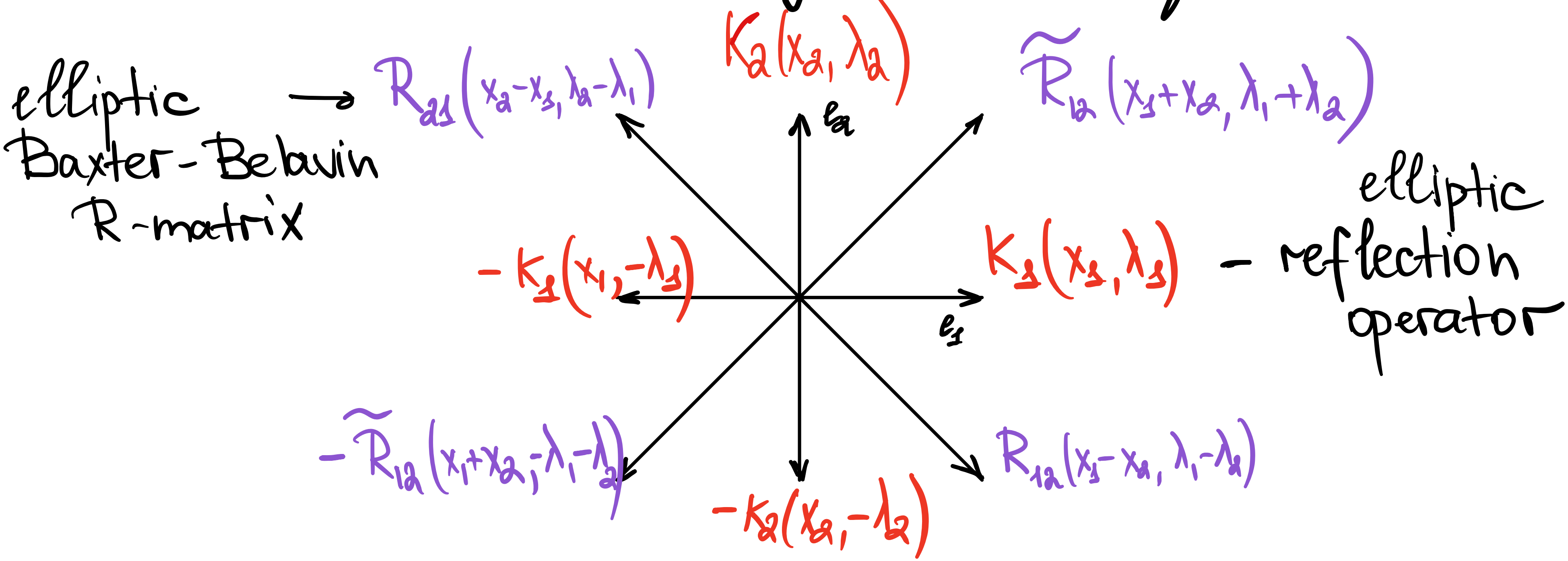
The commutativity condition $[y_\xi, y_\eta] = 0$ is equivalent to the system of functional equations:

$$\sum_{\alpha,\beta\in\Phi^0:\ s_\alpha s_\beta = r} \varkappa_\alpha \varkappa_\beta\, R_\alpha\big((\alpha,x),(\alpha^\vee,\lambda)\big)\, R_{s_\alpha(\beta)}\big((s_\alpha(\beta),x),(\beta^\vee,\lambda)\big) = 0,$$

where Φ^0 denotes a two-dimensional root subsystem.

$(\Phi^0 \cong A_1 \times A_1,\ A_2,\ B_2,\ G_2)$

BC-associative Yang-Baxter equation



Commutativity of y_g leads to the set of quadratic relations on K_i, R_{ij}, $\tilde{R}_{ij}$, which we call the BC-associative Yang-Baxter equation

$$R_{12}(x_1-x_2, \lambda+\mu)K_2(x_2, \mu) = K_1(x_1, \mu)R_{12}(x_1-x_2, \lambda-\mu) + \tilde{R}_{12}(x_1+x_2, \mu-\lambda)K_1(x_1, \lambda) + K_2(x_2, -\lambda)\tilde{R}_{12}(x_1+x_2, \lambda+\mu)$$

[Mostovskii, M.M., Zotov 26]

Thank you for your attention!