On the co-property of Bethe vectors

Andrii Liashyk
BIMSA

Based on ongoing work with Bart Vlaar

Outline

Closed spin chain and co-product property

Attenuation

Open spin chain and co-action property

RTT algebra

Consider a bilinear algebra

$$R_{12}(u,v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u,v)$$

on the elements of the monodromy matrix

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$$T_1(u) = T(u) \otimes 1$$

$$T_2(v) = 1 \otimes T(v)$$

and R-matrix is 4×4 matrix of functions that plays role of the structure functions of the algebra

Quantum Integrability

Define transfermatrix $t(u) = \mathrm{tr}\, T(u) = A(u) + D(u)$

RTT equation in the form

$$T_1(u)T_2(v) = R_{12}(u,v)^{-1}T_2(v)T_1(u)R_{12}(u,v)$$

after traces $\mathrm{tr}_1\mathrm{tr}_2$ becomes $t(u)t(v) = t(v)t(u)$

Expansion $t(u) = \sum_{k=1}^{n}(u-u_0)^{k-1}H_k \Longrightarrow [H_k, H_l] = 0, \quad \forall k, l$

Quantum integrable system $\approx$ Family of independent commuting operators $H_1, H_2, \ldots, H_n$ with common simple spectrum $\approx$ Transfermatrix of RTT algebra

Yang-Baxter equation

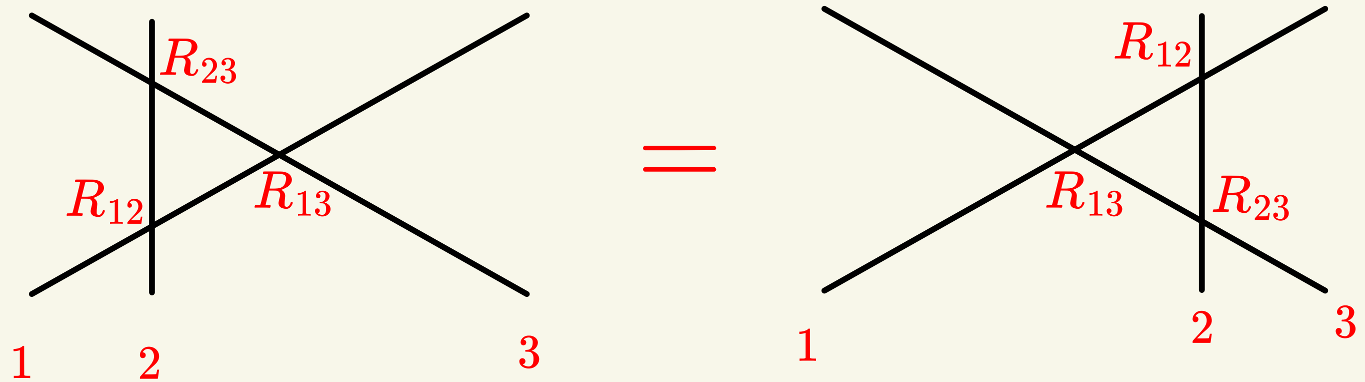
Consistency condition

$$R_{23}\, T_2 T_3 T_1 (R_{23})^{-1} = T_3 T_2 T_1$$

$$\begin{tikzcd}
 & (213) \arrow[r] & (231) \arrow[dr] & \\
(123) \arrow[ur] \arrow[dr] & & & (321) \\
 & (132) \arrow[r] & (312) \arrow[ur] &
\end{tikzcd}$$

is satisfied if we assume the Yang-Baxter equation

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v)$$



Example. XXX Heisenberg spin chain

R-matrix $\quad R(u,v) = 1 + \frac{c}{u-v} P = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & 1 & g & 0 \\ 0 & g & 1 & 0 \\ 0 & 0 & 0 & f \end{pmatrix}$

$$f = \frac{u-v+c}{u-v}$$

$$g = \frac{c}{u-v}$$

Monodromy matrix $\quad T_0(u) = R_{0N}(u) R_{02}(u) \dots R_{01}(u)$

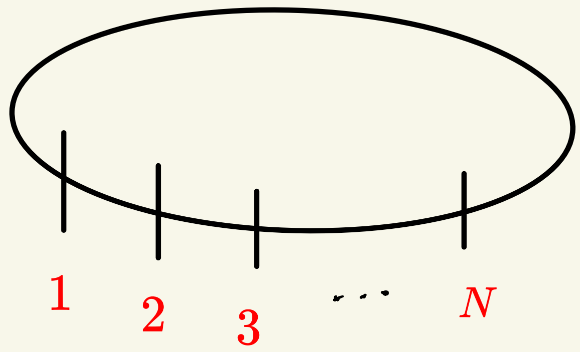
Transfermatrix $\quad t(u) = \mathrm{tr}_0\, T_0(u) = A(u) + D(u)$

XXX spin chain (periodic) $\quad H^{XXX} \propto \left.\frac{\mathrm{d} \ln t(u)}{\mathrm{d}u}\right|_{u=0} = \sum_{i=1}^{N-1} \bar{\sigma}_i \cdot \bar{\sigma}_{i+1} + \bar{\sigma}_N \cdot \bar{\sigma}_1$

$$\bar{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$$

$$\sigma_i^\alpha = 1 \otimes \dots \otimes \underbrace{\sigma^\alpha}_{i\text{-th}} \otimes \dots \otimes 1$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



Algebraic Bethe ansatz framework

We assume that the physical space $\mathcal{H}$ has a special vector $|0\rangle$ called vacuum such that

$$C(u)|0\rangle = 0,$$
$$A(u)|0\rangle = a(u)|0\rangle,$$
$$D(u)|0\rangle = d(u)|0\rangle,$$

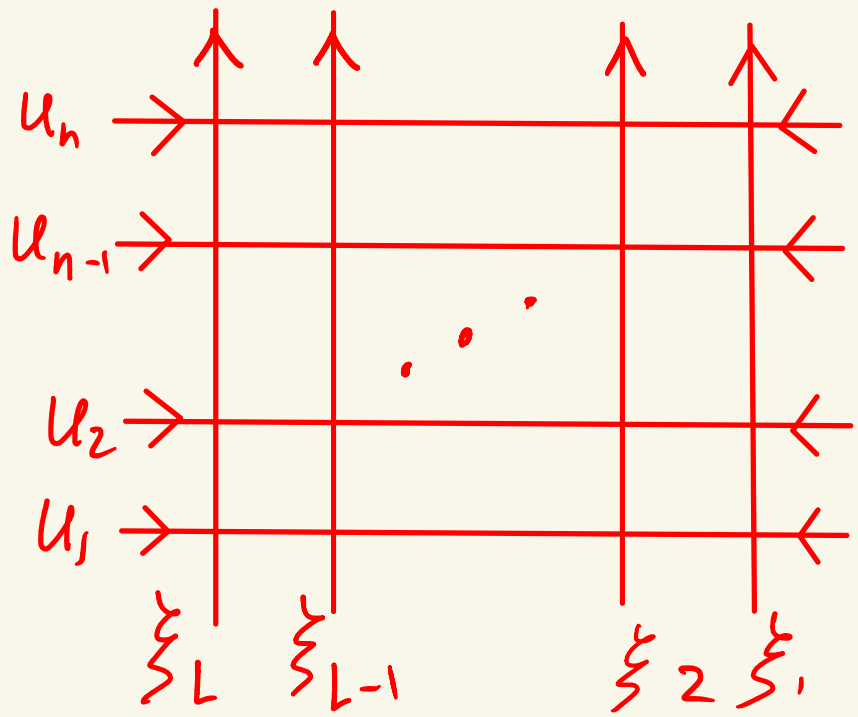
where $a(u), d(u)$ are complex-valued functions.

Bethe vector

We are looking for the eigenvector as a polynomial in terms of the upper-triangular elements of the monodromy matrix

$$\mathbb{B}(\overline{u}) = B(u_1)B(u_2)\cdot\ldots\cdot B(u_n)\,|0\rangle.$$

In the case of inhomogeneous model, we can think of Bethe vector as set of partition functions:



Inhomogeneous model:

$$T_0(u) = R_{0L}(u-\xi_L)R_{02}(u)\ldots R_{01}(u-\xi_1)$$

Eigenvector

One can prove that Bethe vector becomes eigenvector of transfermatrix $t(z)$

$$t(z)\mathbb{B}(\overline{u}) = \tau(z|\overline{u})\mathbb{B}(\overline{u})$$

with eigenvalue

$$\tau(z|\overline{u}) = a(z)\prod_{j=1}^{n}\frac{u_j - z + c}{u_j - z} + d(z)\prod_{j=1}^{n}\frac{u_j - z - c}{u_j - z}$$

if Bethe equations satisfied

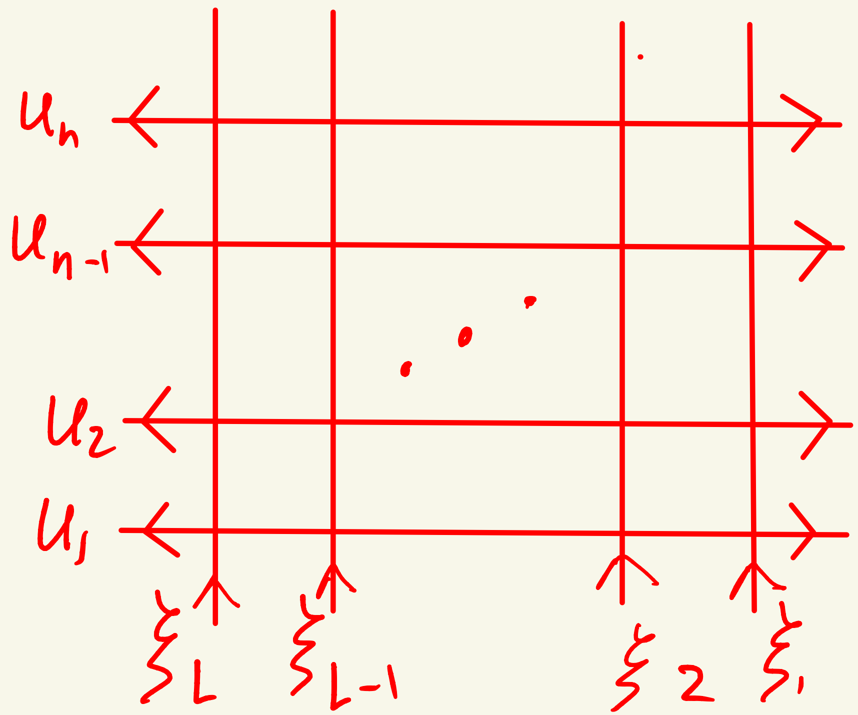
$$\frac{a(u_i)}{d(u_i)} = \prod_{j\neq i}^{n}\frac{u_i - u_j + c}{u_i - u_j - c}, \qquad i = 1, \dots, n$$

Dual Bethe vector

We are looking for the eigenvector as a polynomial in terms of the lower-triangular elements of the monodromy matrix

$$\mathbb{C}(\overline{u}) = \langle 0|C(u_1)C(u_2)\cdot\ldots\cdot C(u_n)$$

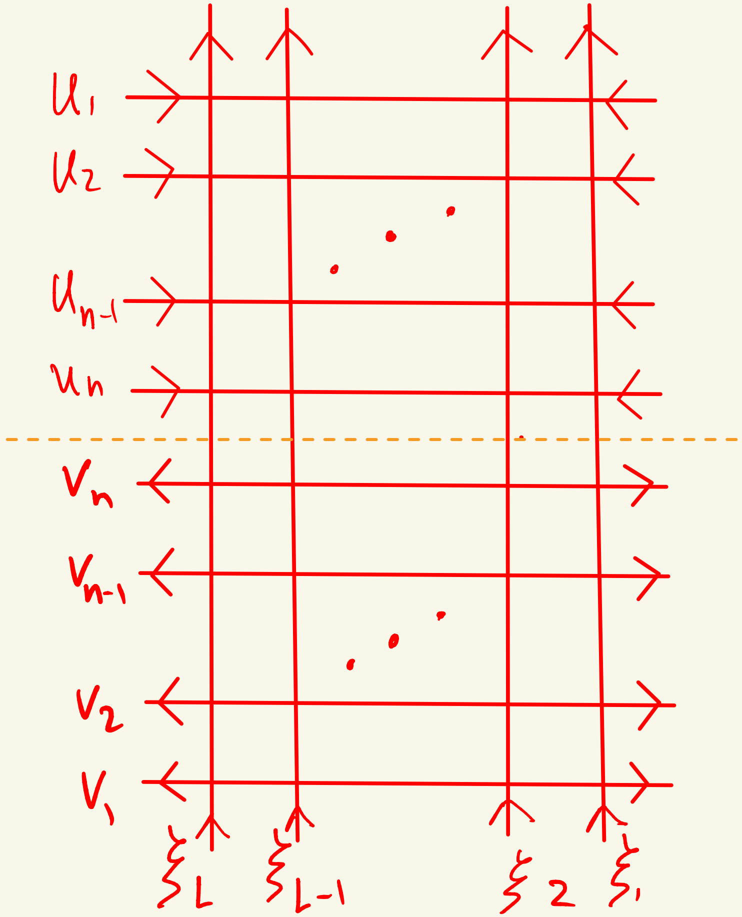
In the case of inhomogeneous model, we can think of dual Bethe vector as set of partition functions:



Scalar product of Bethe vectors

We define a scalar product of Bethe vectors as overlap of Bethe vector with dual Bethe vector

$$S(\overline{v}|\overline{u}) = \mathbb{C}(\overline{v})\mathbb{B}(\overline{u})$$



Combinatorial expression the scalar product is known

$$S(\overline{v}|\overline{u}) = \sum_{\substack{\bar{u} \vdash \{\bar{u}_I, \bar{u}_{II}\} \\ \bar{v} \vdash \{\bar{v}_I, \bar{v}_{II}\}}} a(\bar{v}_I) d(\bar{u}_I) a(\bar{u}_{II}) d(\bar{v}_{II}) \times K(\bar{u}_I|\bar{v}_I) K(\bar{v}_{II}|\bar{u}_{II}) f(\bar{v}_{II}, \bar{v}_I) f(\bar{u}_I, \bar{u}_{II})$$

Here

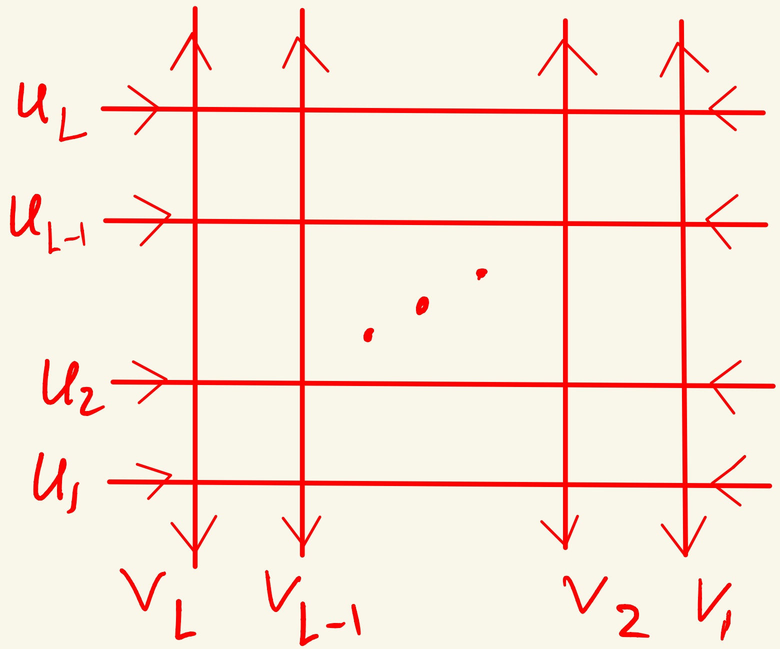
$$a(\bar{u}) = \prod_{u_i \in \bar{u}} a(u_i), \quad f(\bar{u}, \bar{v}) = \prod_{u_i \in \bar{u}} \prod_{v_j \in \bar{v}} f(u_i, v_j).$$

Izergin determinant $K(\bar{u}|\bar{v})$

Izergin determinant describes partition function with domain wall boundary condition in inhomogeneous model

Explicit expression is known

$$K(\bar{u}|\bar{v}) = \Delta(\bar{u})\Delta'(\bar{v})h(\bar{u},\bar{v}) \det\left(\frac{g(u_i,v_j)}{h(u_i,v_j)}\right)$$



Here

$$\Delta(\bar{v}) = \prod_{j>i} g(v_i,v_j), \qquad \Delta'(\bar{v}) = \prod_{j<i} g(v_i,v_j)$$

and

$$h(u,v) = \frac{u-v+c}{c}, \qquad g(u,v) = \frac{c}{u-v}.$$

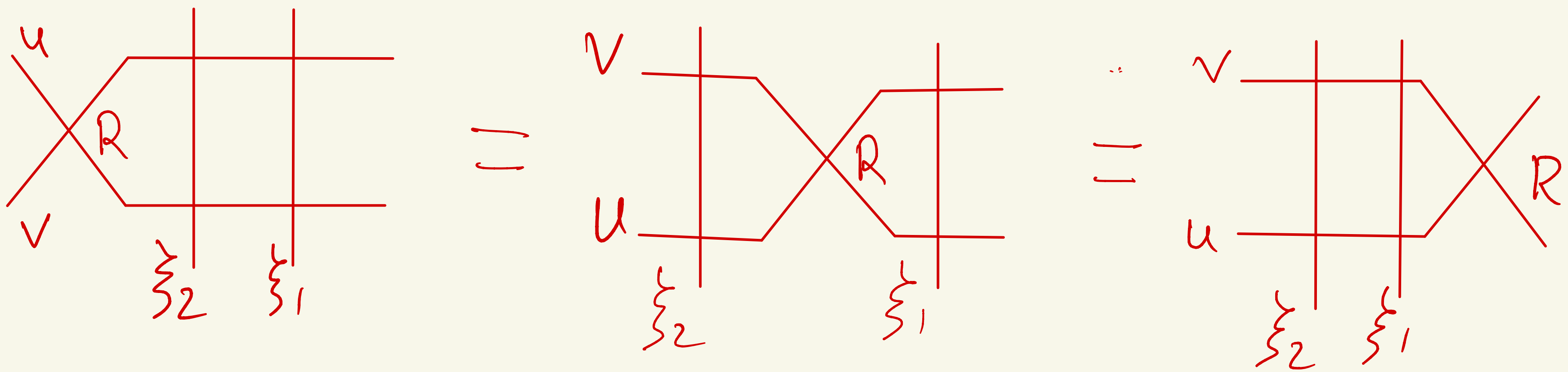
L-operators and two site model

Let $L, \tilde{L}$ be two solutions of RTT eq. with mutual commutative elements

$$[L_{ij}(u), \tilde{L}_{k\ell}(v)] = 0 \quad \Leftrightarrow \quad L_1(u)\tilde{L}_2(v) = \tilde{L}_2(v)L_1(u)$$

Then their product $\tilde{\tilde{L}}(u) = L(u)\tilde{L}(u)$ also satisfies RTT

$$\begin{aligned} R_{12}\tilde{\tilde{L}}_1\tilde{\tilde{L}}_2 = R_{12}L_1\tilde{L}_1L_2\tilde{L}_2 = R_{12}L_1L_2\tilde{L}_1\tilde{L}_2 &= L_2L_1R_{12}\tilde{L}_1\tilde{L}_2 \\ = L_2L_1\tilde{L}_2\tilde{L}_1R_{12} = L_2\tilde{L}_2L_1\tilde{L}_1R_{12} &= \tilde{\tilde{L}}_2\tilde{\tilde{L}}_1R_{12} \end{aligned}$$



- Way to construct new integrable models $\simeq$ representations of Yangian

Co-product

We define co-product as

$$\Delta T(u) = T(u)_2 T(u)_1 = \begin{pmatrix} A(u)_2 & B(u)_2 \\ C(u)_2 & D(u)_2 \end{pmatrix} \begin{pmatrix} A(u)_1 & B(u)_1 \\ C(u)_1 & D(u)_1 \end{pmatrix}$$

and opposite co-product as

$$\Delta^{op} T(u) = T(u)_1 T(u)_2 = \begin{pmatrix} A(u)_1 & B(u)_1 \\ C(u)_1 & D(u)_1 \end{pmatrix} \begin{pmatrix} A(u)_2 & B(u)_2 \\ C(u)_2 & D(u)_2 \end{pmatrix}$$

In particular, we have

$$\Delta B(u) = A(u)_2 B(u)_1 + B(u)_2 D(u)_1$$

Co-product property of Bethe vectors

If we define co-product for the vacuum $\Delta|0\rangle = |0\rangle_2 \otimes |0\rangle_1$

then we can extend co-product to Bethe vector

$$\Delta\mathbb{B}(\bar{u}) = \sum_{\text{part}} d^{(1)}(\bar{u}_{II})a^{(2)}(\bar{u}_I)f(\bar{u}_I,\bar{u}_{II})\,\mathbb{B}^{(1)}(\bar{u}_I)\cdot\mathbb{B}^{(2)}(\bar{u}_{II})$$

and similarly for dual Bethe vector

$$\Delta\mathbb{C}(\bar{u}) = \sum_{\text{part}} a^{(1)}(\bar{u}_{II})d^{(2)}(\bar{u}_I)f(\bar{u}_I,\bar{u}_{II})\,\mathbb{C}^{(1)}(\bar{u}_I)\cdot\mathbb{C}^{(2)}(\bar{u}_{II}).$$

Note, that

$a(z) = a^{(1)}(z)a^{(2)}(z)$ and $d(z) = d^{(1)}(z)d^{(2)}(z)$

Co-product and Bethe equation

using co-product property

$$\Delta\mathbb{B}(\bar{u}) = \sum_{\text{part}} d^{(1)}(\bar{u}_{II})a^{(2)}(\bar{u}_I)f(\bar{u}_I,\bar{u}_{II})\,\mathbb{B}^{(1)}(\bar{u}_I)\cdot\mathbb{B}^{(2)}(\bar{u}_{II})$$

opposite co-product property

$$\Delta\mathbb{B}(\bar{u}) = \sum_{\text{part}} d^{(2)}(\bar{u}_{II})a^{(1)}(\bar{u}_I)f(\bar{u}_I,\bar{u}_{II})\,\mathbb{B}^{(2)}(\bar{u}_I)\cdot\mathbb{B}^{(1)}(\bar{u}_{II})$$

and taking into account Bethe equations

$$\frac{a(\bar{u}_I)}{d(\bar{u}_I)} = \frac{f(\bar{u}_I,\bar{u}_{II})}{f(\bar{u}_{II},\bar{u}_I)}$$

one can notice

$$\boxed{\Delta\mathbb{B}(\bar{u})|_{\text{BE}} \propto \Delta^{\text{op}}\,\mathbb{B}(\bar{u})|_{\text{BE}}}$$

Co-product and Izergin determinant

Consider the representation of Izergin determinant

$$K(\bar{u}|\bar{\xi}) = \langle\tilde{0}|B(u_1)B(u_2)\cdot\ldots\cdot B(u_n)\,|0\rangle = \langle\tilde{0}|\mathbb{B}(\bar{u}),$$

here variables $\bar{\xi}$ are hidden as inhomogeneities,
and $\langle\tilde{0}|$ is the state with all spin flipped.

Apply the co-product to both sides

$$K(\bar{u}|\bar{\xi}) = \sum_{\bar{u}\vdash\{\bar{u}_I,\bar{u}_{II}\}} d^{(1)}(\bar{u}_{II})a^{(2)}(\bar{u}_I)f(\bar{u}_I,\bar{u}_{II})\,K(\bar{u}_I|\bar{\xi}_I)\cdot K(\bar{u}_{II}|\bar{\xi}_{II})$$

where

$$d^{(1)}(z) = 1, \qquad a^{(2)}(z) = \prod_{\xi_i\in\bar{\xi}_{II}} \frac{z-\xi_i+c}{z-\xi_i}$$

Co-product and Scalar product

Consider the expression for scalar product

$$S(\bar{v}|\bar{u}) = \mathbb{C}(\bar{v})\mathbb{B}(\bar{u}) = \sum_{\substack{\bar{u}\vdash\{\bar{u}_I,\bar{u}_{II}\} \\ \bar{v}\vdash\{\bar{v}_I,\bar{v}_{II}\}}} \boxed{a(\bar{v}_I)}\boxed{d(\bar{u}_I)a(\bar{u}_{II})}\boxed{d(\bar{v}_{II})}$$

$$K(\bar{u}_I|\bar{v}_I)K(\bar{v}_{II}|\bar{u}_{II})\boxed{f(\bar{v}_{II},\bar{v}_I)}\boxed{f(\bar{u}_I,\bar{u}_{II})}$$

and compare with expressions for co-product of Bethe vector

$$\Delta\mathbb{B}(\bar{u}) = \sum_{\text{part}} \boxed{d^{(1)}(\bar{u}_{II})a^{(2)}(\bar{u}_I)f(\bar{u}_I,\bar{u}_{II})}\,\mathbb{B}^{(1)}(\bar{u}_I)\cdot\mathbb{B}^{(2)}(\bar{u}_{II})$$

and dual Bethe vector

$$\Delta\mathbb{C}(\bar{u}) = \sum_{\text{part}} \boxed{a^{(1)}(\bar{u}_{II})d^{(2)}(\bar{u}_I)f(\bar{u}_I,\bar{u}_{II})}\,\mathbb{C}^{(1)}(\bar{u}_I)\cdot\mathbb{C}^{(2)}(\bar{u}_{II}).$$

RKRK algebra

Consider a bilinear algebra

$$R_{12}(u-v)K_1(u)R_{12}(u+v)K_2(v) = K_2(v)R_{12}(u+v)K_1(u)R_{12}(u-v)$$

on the elements of the boundary monodromy matrix

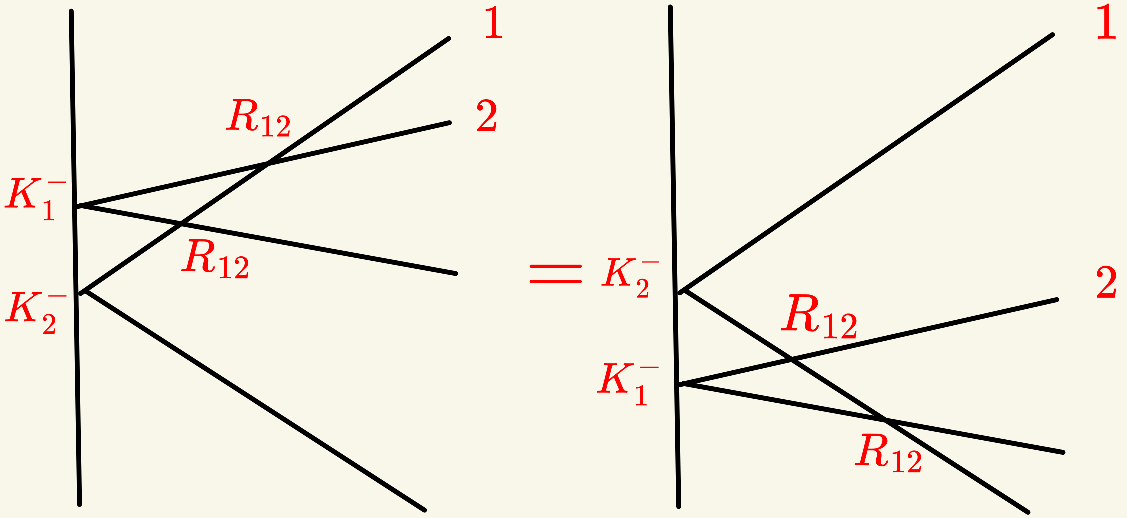
$$K(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix}$$

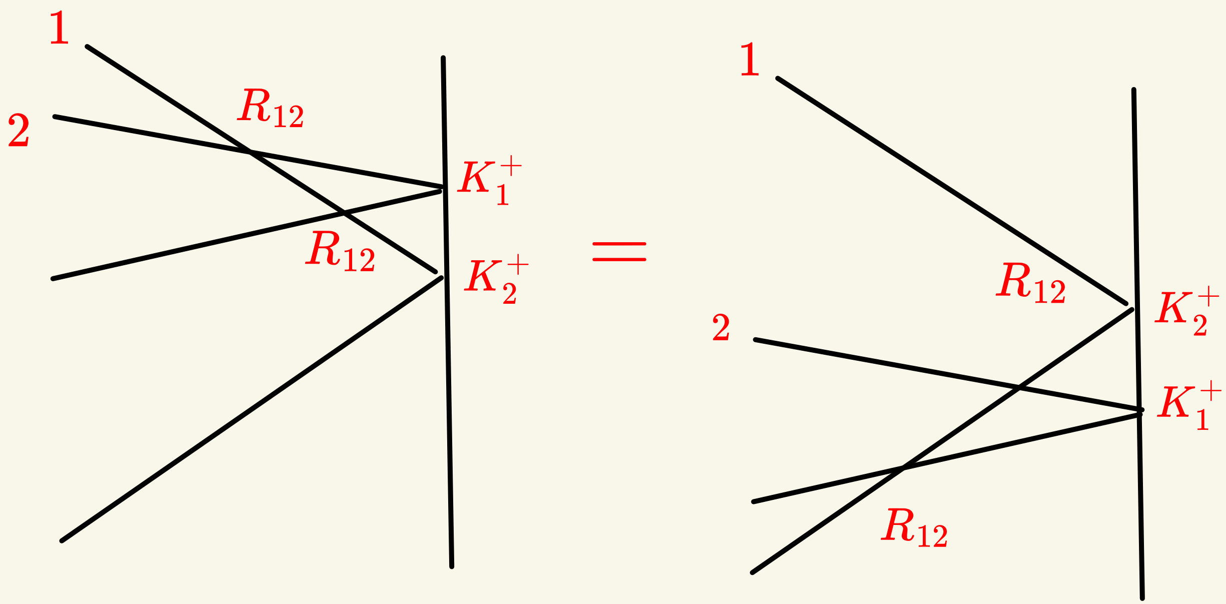
$$K_1(u) = K(u) \otimes 1$$

$$K_2(v) = 1 \otimes K(v)$$

and R-matrix is 4×4 matrix of functions that plays role of the structure functions of the algebra

Left and right reflection equation





Quantum Integrability

Define transfermatrix $t(u) = \mathrm{tr}\, K^-(u)K^+(u) = k_1^-(u)\mathcal{A}(u) + k_2^-(u)\mathcal{D}(u)$

Left and right RKRK equations imply

$$t(u)t(v) = t(v)t(u)$$

Expansion $t(u) = \sum_{k=1}^{n} (u-u_0)^{k-1} H_k \implies [H_k, H_l] = 0, \quad \forall k, l$

Example. XXX Heisenberg spin chain

K^+-matrix $$K^+(u) = \begin{pmatrix} 1 & 0 \\ 0 & \kappa^+(u) \end{pmatrix} \qquad \kappa^+ = \frac{\xi^+ - u}{\xi^+ + u}$$

K^--matrix $$K^-(u) = \begin{pmatrix} 1 & 0 \\ 0 & \kappa^-(u) \end{pmatrix} \qquad \kappa^- = \frac{\xi^- + c + u}{\xi^- - c - u}$$

Monodromy matrix $$\mathcal{T}_0(u) = R_{0N}(u-\xi_N)\dots R_{01}(u-\xi_1)K^+(u)R_{01}(u+\xi_1)\dots R_{01}(u+\xi_N)$$

Transfermatrix $$t(u) = \mathrm{tr}_0\, K^-(u)\mathcal{T}_0(u) = \mathcal{A}(u) + \kappa^-(u)\mathcal{D}(u)$$

XXX spin chain (open) $$H^{XXX} \propto \left.\frac{\mathrm{d}\ln t(u)}{\mathrm{d}u}\right|_{u=0} = \sum_{i=1}^{N-1} \overline{\sigma}_i \cdot \overline{\sigma}_{i+1} + \text{boundary terms}$$

Co-action

We define co-action as

$$\Delta_B \mathcal{T}(u) = T(u)_2 \mathcal{T}(u)_1 \hat{T}(u)_2 =$$

$$\begin{pmatrix} A(u)_2 & B(u)_2 \\ C(u)_2 & D(u)_2 \end{pmatrix} \begin{pmatrix} \mathcal{A}(u)_1 & \mathcal{B}(u)_1 \\ \mathcal{C}(u)_1 & \mathcal{D}(u)_1 \end{pmatrix} \begin{pmatrix} D(-c-u)_2 & -B(-c-u)_2 \\ -C(-c-u)_2 & A(-c-u)_2 \end{pmatrix}$$

Bethe vector

We are looking for the eigenvector as a polynomial in terms of the upper-triangular elements of the monodromy matrix

$$\mathfrak{B}(\overline{u}|\xi^{+}) = \mathcal{B}(u_1)\mathcal{B}(u_2)\cdot\ldots\cdot\mathcal{B}(u_n)\,|0\rangle.$$

With co-action property:

$$\Delta\mathcal{B}(\bar{u}) = \sum_{\text{part}}(-1)^{\#\bar{u}_{\mathrm{III}}}\delta^{(1)}(\bar{u}_{\mathrm{II}})\alpha^{(1)}(\bar{u}_{\mathrm{III}})a^{(2)}(\{\bar{u}_{\mathrm{I}},-\bar{u}_{\mathrm{I}}-c,-\bar{u}_{\mathrm{II}}-c,\bar{u}_{\mathrm{III}}\})$$
$$\frac{f(\bar{u}_{\mathrm{I}},-c-\bar{u}_{\mathrm{II}})}{f(\bar{u}_{\mathrm{I}},c+\bar{u}_{\mathrm{II}})}\frac{f(\bar{u}_{\mathrm{I}},\bar{u}_{\mathrm{III}})}{f(\bar{u}_{\mathrm{I}},-\bar{u}_{\mathrm{III}})}\Delta_f(\{\bar{u}_{\mathrm{II}},-c-\bar{u}_{\mathrm{III}}\})\mathcal{B}^{(1)}(\bar{u}_{\mathrm{I}})\otimes B^{(2)}(\{\bar{u}_{\mathrm{II}},-\bar{u}_{\mathrm{III}}-c\})$$

where

$$\Delta_f(\bar{v}) = \prod_{j>i} f(v_i,-\eta-v_j).$$

Co-action and Bethe equation

$$\boxed{\Delta \mathfrak{B}(\bar{u}\,|\,\xi^-)\Big|_{\mathrm{BE}} \propto \Delta\, \hat{\mathfrak{B}}(\bar{u}\,|\,\xi^+)\Big|_{\mathrm{BE}}}$$

Co-action and Tsuchiya determinant

$$\mathbb{K}(\bar{u}|\bar{v}) = \langle\tilde{\emptyset}|\mathcal{B}(\bar{v})|0\rangle$$

Co-action property implies

$$\frac{h(\bar{u},\xi^+)}{g(2\bar{v},-c)}\Delta_f(\bar{v})\ \mathbb{K}(\bar{u}|\bar{v}) = \sum_{\bar{v}\vdash\{\bar{v}_{\mathrm{I}},\bar{v}_{\mathrm{II}}\}}(-1)^{\#\bar{v}_{\mathrm{II}}}h(\{\bar{v}_{\mathrm{I}},-c-\bar{v}_{\mathrm{II}}\},\xi)\Delta_f(\{\bar{v}_{\mathrm{I}},-c-\bar{v}_{\mathrm{II}}\})K(\bar{u}|\{\bar{v}_{\mathrm{I}},-c-\bar{v}_{\mathrm{II}}\})$$

Co-action and scalar product

Co-action property implies

$$\mathcal{S}(\bar{v}|\bar{u}) = \langle\emptyset|\mathcal{C}(\bar{v})\mathcal{B}(\bar{u})|\emptyset\rangle = \sum \alpha(\bar{u}_{\rm I})\delta(\bar{u}_{\rm II})\alpha(\bar{v}_{\rm I})\delta(\bar{v}_{\rm II})(-1)^{\#\bar{u}_{\rm I}+\#\bar{v}_{\rm II}}$$

$$K(\{-\bar{u}_{\rm I}-c,\bar{u}_{\rm II}\}\mid\{\bar{v}_{\rm I},-\bar{v}_{\rm II}-c\})\;\Delta_f(\{-\bar{u}_{\rm I}-c,\bar{u}_{\rm II}\})\;\Delta_f(\{-\bar{v}_{\rm I}-c,\bar{v}_{\rm II}\})$$

Admit that here $K(\bar{u}|\bar{v})$ is Izergin (not Tsuchiya) determinant.

And $\Delta_f(\bar{v}) = \prod_{j>i} f(v_i,-\eta-v_j).$

Attenuation

For inhomogeneous spin chain consider the shift

$$u \mapsto u + \lambda, \qquad \xi_i \mapsto \xi_i + \lambda$$

and then take limit

$$\lambda \to \infty.$$

So,

$$\mathcal{T}_0(u) = R_{0N}(u - \xi_N) \dots R_{01}(u - \xi_1) K^+(u) R_{01}(u + \xi_1) \dots R_{01}(u + \xi_N) \to$$

$$\to R_{0N}(u - \xi_N) \dots R_{01}(u - \xi_1) = T_0(u)$$

As a consequence, all identities for periodic case follow from the open one!

The last slide

- Co-product property allows describing various parts of the "periodic" case

- In "open" case co-property also helps

- Attenuation allows get "periodic" quantities from "open" ones.