Onsager algebra symmetry in quantum lattice systems

Ref. E. Vernier, Y.M., M. Yamazaki arXiv: 2606.25660
Y.M., J. Lamers, V. Pasquier. SciPost Phys. 11, 067.
Y.M., SciPost Phys. 11, 066.

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Symm. in Quantum Many-body

1. Phases of matter

+ integrability

2. Selection rules, correlation func.
3. Dynamics, transport, thermalisation

Examples.

U(1) symm. in free fermion

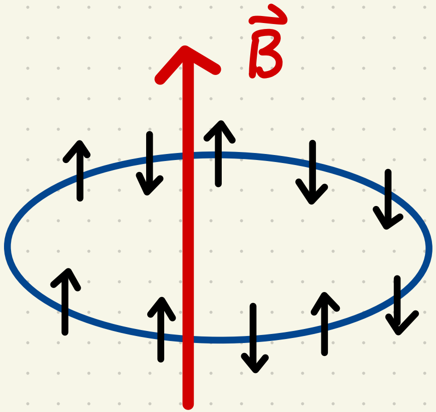
SU(2) symm. in Heisenberg

Yangian symm. in Haldane-Shastry

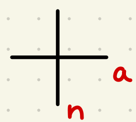
Protagonist : asymm. 6-vertex

$$H = \sum_{n=1}^{L} 2\left(q\, \sigma_n^+ \sigma_{n+1}^- + q^{-1} \sigma_n^- \sigma_{n+1}^+ \right) + \Delta\, \sigma_n^z \sigma_{n+1}^z \qquad \Delta = \frac{q+q^{-1}}{2}$$

spin chain + mag. field



$$R_{an}(t) = \begin{pmatrix} \omega t - 1 & 0 & 0 & 0 \\ 0 & t-1 & \omega - 1 & 0 \\ 0 & t(\omega-1) & (t-1)\omega & 0 \\ 0 & 0 & 0 & \omega t - 1 \end{pmatrix}$$



$\omega = q^2$

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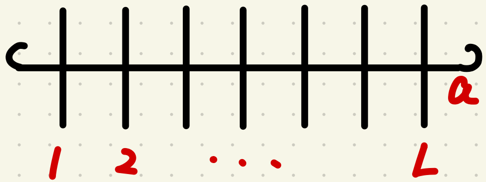
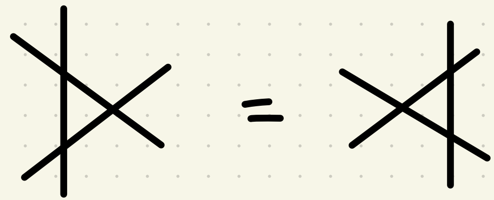
Protagonist : asymm. 6-vertex



transfer mat.

$$T_{6V}(u) = \mathrm{Tr}_a \left(R_{a1}(u) R_{a2}(u) \cdots R_{aL}(u) \right)$$

Yang-Baxter :



$$[T_{6V}(u), T_{6V}(v)] = 0 \quad \forall\ u, v \in \mathbb{C}$$

$$H \sim \partial_u \log T_{6V}(u) \Big|_{u=0}$$

We focus on $\omega^N = 1$
$(q^N = \pm 1)$

root of unity cond.

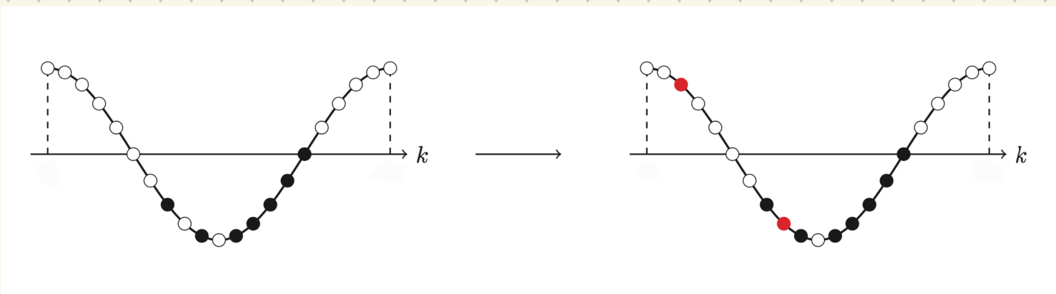
Case I : $q=i$ (free fermion)

$$H_{q=i} = i\sum_{n=1}^{L}\left(\sigma_n^+\sigma_{n+1}^- - \sigma_n^-\sigma_{n+1}^+\right)$$

"let's ignore b.c. for now"

$$\overset{\text{J-W}}{\rightsquigarrow} \quad H_f = i\sum_{n=1}^{L}\left(c_n^\dagger c_{n+1} - c_{n+1}^\dagger c_n\right)$$

$$\sim \sum_k \sin k \; c_k^\dagger c_k$$



Lots of degeneracies

e.g. $[c_k^\dagger c_{-k}^\dagger, H_f] = 0$

Vernier, O'Brien, Fendley 16

Ising in a6v

"Ising model"

2 U(1) charges of $H_{q=i}$

$$A_0 = \sum_{n=1}^{L} \sigma_n^z \qquad A_1 = \sum_{n=1}^{L} \sigma_n^x \sigma_{n+1}^x$$

$$[H_{q=i}, A_0] = [H_{q=i}, A_1] = 0$$

$H_{q=i}$ in the centre of $\mathcal{A} = \mathrm{span}_{\mathbb{C}}\{A_0, A_1\}$

Moreover, $[A_0,[A_0,[A_0,A_1]]] = 16[A_0,A_1]$

$$[A_1,[A_1,[A_1,A_0]]] = 16[A_1,A_0]$$

Dolan-Grady relation $\longleftrightarrow$ Onsager alg.

Interlude: Onsager alg.

$\{A_m, G_n \mid m,n \in \mathbb{Z}\}$ $\quad [A_m, A_n] = 4G_{m-n}$

$$[G_m, A_n] = 2(A_{n+m} - A_{n-m})$$

∞-dim. Lie alg. (also fixed-point Lie subalg. of $\widehat{sl_2}$)

isomorphic to D-G $\quad \mathrm{span}_{\mathbb{C}}\{A_0, A_1\}$

$A_m, G_m \rightsquigarrow$ nested commutators of A_0 & A_1

"Ising rep." $\quad A_0 = \sum_{n=1}^{L} \sigma_n^z \qquad A_1 = \sum_{n=1}^{L} \sigma_n^x \sigma_{n+1}^x$

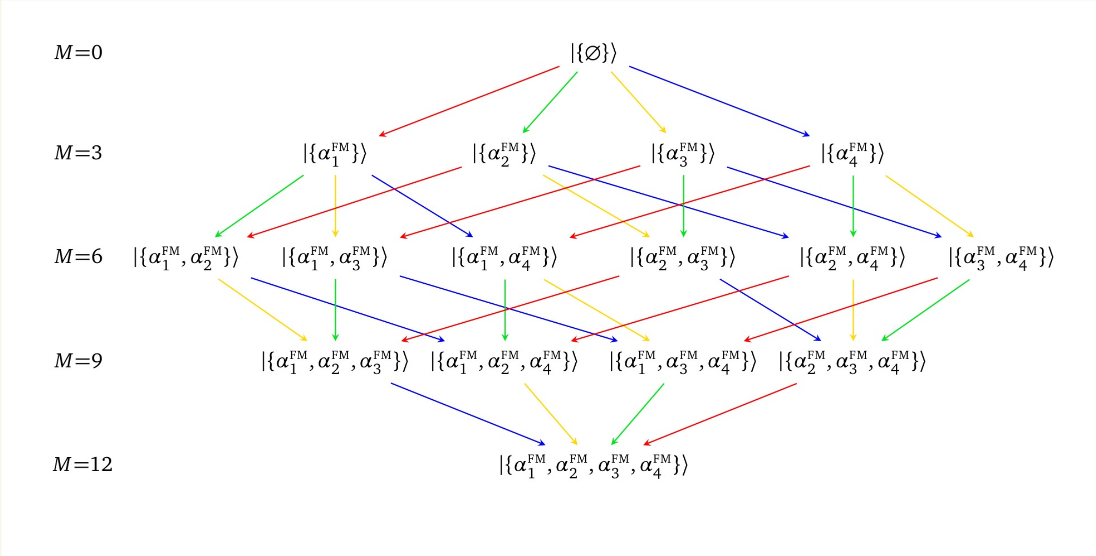
Onsager 44, Davies 90

Spec. of 6-vertex @ r.o.u.

$L=12$

$\Delta = \frac{1}{2}$

$\left(q = e^{i\pi/3}\right)$



∞-many non-Abelian charges?

degeneracies, "free fermionic"

"exact Bethe string"

Conj. Onsager symm. for any 6-v. @ r.o.u.

YM, Lamers, Pasquier 21

YM 21

Remark. Deguchi et al ~'00

"sl_2 loop symm. of XXZ @ root of unity"

Difference : $L(sl_2)$ acts only on sectors ($S^z = 0 \bmod N$)

symm. of the Hamiltonian only at sectors

(from Lusztig's divided power of $U_q(sl_2)$)

$\rightsquigarrow$ different origin as Onsager story

acting on the whole $\mathcal{H}$, can reproduce

degeneracies from $L(sl_2)$ story....

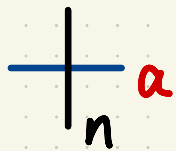
KBBS τ_2 model

$\omega = q^2$

Weyl pair $\quad ZX = \omega XZ$

$Z^{\ell_2} = X^{\ell_2} = 1 \qquad \omega^{\ell_2} = 1$

$$Z = \mathrm{diag}(1, \omega, \cdots, \omega^{\ell_2 - 1}) \qquad X = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \vdots \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$



ℓ_2-dim aux. space

$$L_{an}(u, v, s, y, y') =$$

$$\begin{pmatrix} e^{y+y'}(e^{-u-s} + e^{u+s} X_a) & i\omega e^{y-y'}(e^{-v-s} - e^{v+s} X_a) Z_a \\ -i e^{y'-y}(e^{v-s} - \omega^{-1} e^{-v+s} X_a) Z_a^{-1} & e^{-y-y'}(\omega e^{u-s} + e^{-u+s} X_a) \end{pmatrix}_n$$

Factorisation I

$$L_{an}(u,v,s,y,y')$$

$$= \exp(y\,\sigma_n^z)\cdot\begin{pmatrix} Z_a & 0 \\ 0 & \mathbb{1}\end{pmatrix}_n \cdot U_n(t_+)\cdot\exp(s\,\sigma_n^z)\cdot\begin{pmatrix} X_a & 0 \\ 0 & \mathbb{1}\end{pmatrix}_n \cdot$$

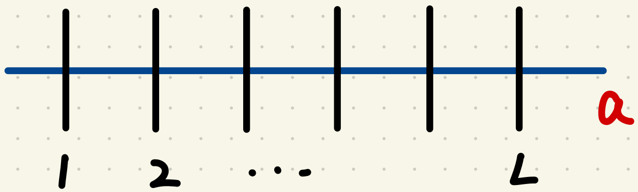
$$V_n(t_-)\cdot\begin{pmatrix} Z_a^{-1} & 0 \\ 0 & \mathbb{1}\end{pmatrix}_n \cdot \exp(y'\sigma_n^z)$$

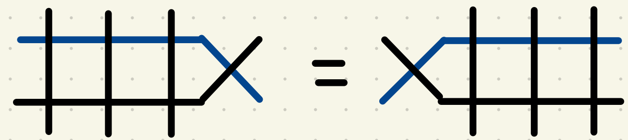
$$U_n(t_+) = \begin{pmatrix} t_+ & t_+^{-1} \\ i t_+^{-1} & -i t_+ \end{pmatrix} \qquad V_n(t_-) = \begin{pmatrix} \omega^{-1} t_- & -i t_-^{-1} \\ t_-^{-1} & i\omega t_- \end{pmatrix}$$

$$T_c(u,v,s,y,y') = \mathrm{Tr}_a\left(\prod_{n=1}^{L} L_{an}\right) \qquad t_\pm = e^{\frac{u\pm v}{2}}$$

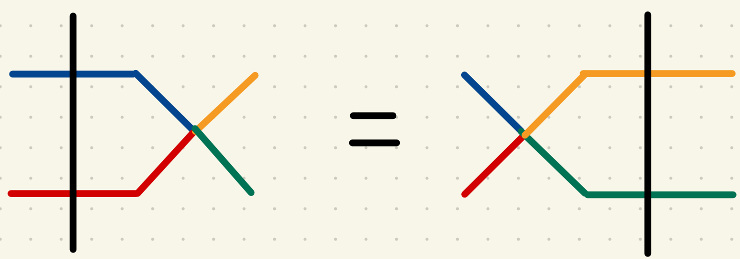
T_2 Lax operator

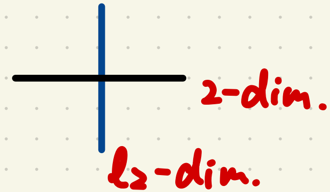
$$[T_{6v}(u), T_c] = 0$$





However, $[T_c(u,v,s,y,y'), T_c(\tilde{u},\tilde{v},\tilde{s},\tilde{y},\tilde{y}')] \neq 0$ in general.



Remark. Baxter '88 $\rightsquigarrow$ 

Bazhanov-Stroganov '90 $\Rightarrow$ bridge between 6-v. & chiral Potts

Similar construction :

Date - Jimbo - Miki - Miwa '91

Weston '26

$\Rightarrow$ 5-parameter $\longrightarrow$ 2-parameter $\rightarrow$ Q-op. "$[T, T] = 0$"

restrict on "CP curve"

Factorisation of τ_2 T

$$\boxed{\begin{aligned} T_0 \cdot T_c(u, v, s, y, y') = \\ \exp\left(\frac{y\ell_2}{2} A_0\right) T_+(t_+) \exp\left(\frac{s\ell_2}{2} A_1\right) T_-(t_-) \exp\left(\frac{y'\ell_2}{2} A_0\right) \end{aligned}}$$

$$T_\pm(e^x) = T_c(x, \pm x, 0, 0, 0) \qquad A_0 = \frac{2}{\ell_2} \sum_{n=1}^{L} \sigma_n^z$$

$$T_0 = T_c(0, 0, 0, 0, 0)$$

N.B. $A_0, A_1, T_\pm$ NOT COMMUTE in general

Back to Onsager

From the properties of T_2 transfer mat.

D-G

we can prove

$$\boxed{\begin{aligned} [A_0,[A_0,[A_0,A_1]]] &= 16[A_0,A_1] \\ [A_1,[A_1,[A_1,A_0]]] &= 16[A_1,A_0] \end{aligned}}$$

$\{A_0, A_1\}$ as a rep. of Onsager alg. in $(\mathbb{C}^2)^{\otimes L}$.

6-vertex & Onsager

$[T_{6v}(u), T_{\tau_2}] = 0$ $\qquad$ $\{A_0, A_1\}$ span Onsager alg.

$\Rightarrow [T_{6v}(u), A_n] = 0 \quad , \forall n \in \mathbb{Z}$

$([H, A_n] = 0)$

(a) 6-v. model at any root of unity (XXZ chain)

has Onsager alg. symm. !

App: Non-invertible Symm.

Thorngren, Wang, 24

Duality between A_0 & A_1

$$\mathcal{D} A_n = A_{1-n} \mathcal{D} \qquad \Rightarrow \qquad [H_{6v}, \mathcal{D}] = 0$$

$$\mathcal{D}_\pm = T_0^{-1/2} \cdot \lim_{u \to \pm\infty} e^{\mp u \cdot L} T_\pm(u)$$

$$L \bmod (2l_z) = 0 \,, \qquad \mathcal{D}_\pm \cdot \mathcal{D}_\pm^\dagger = \mathcal{D}_\pm^\dagger \cdot \mathcal{D}_\pm = \underbrace{\mathbb{1} + e^{\frac{2i\pi}{l_z} Z} + e^{\frac{4i\pi}{l_z} Z} + \cdots}_{\text{projector}}$$

e.g. $l_z = 2$ $(\Delta = 0)$ KW duality

$$\mathcal{D}: \quad \sigma_n^z \to \sigma_n^x \sigma_{n+1}^x \qquad\qquad [i \sum_{n=1}^{L} (\sigma_n^+ \sigma_{n+1}^- - h.c.), \mathcal{D}] = 0.$$

$$\sigma_n^x \sigma_{n+1}^x \to \sigma_{n+1}^z$$

$\rightsquigarrow$ $c=1$ FCB CFT w. $r \sim \sqrt{l_z}$ ✓

Conclusion & Outlook

I. Hidden Onsager alg. symm. in 6-vertex model at root of unity $\Rightarrow$ higher-spin generalisation

II. Proof relies on KBBS τ_2 model (rep. theoretical?)
$\Rightarrow$ relation to quantum group

III. Non-invertible symm. @ root of unity
$\Rightarrow$ orbifolding $c=1$ CFT

IV. Physical consequence of non-Abelian charges?
GGE / GHD / correlation func. ⋯