# Full Counting Statistics for Twisted XXX Spin Chains

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# Full Counting Statistics (FCS)

- For a state $|K\rangle$ (say, a ground state) and a subsystem of length $\ell < L$

$$Q^{(\ell)}(\bar{\beta}) = \sum_{j=1}^{\ell} (\beta_x \sigma_j^x + \beta_y \sigma_j^y + \beta_z \sigma_j^z)$$

$$\langle \exp Q^{(\ell)}(\bar{\beta}) \rangle_K = \frac{\langle K | \exp Q^{(\ell)}(\bar{\beta}) | K \rangle}{\langle K | K \rangle}$$

  the full distribution of a spin observable
- Periodic chain, $Q \propto \sigma^z$: well known, protected by $SU(2)/U(1)$ symmetry.
- Generic twist ($\sigma^x, \sigma^y$, or non-diagonal boundary): symmetry broken $\Rightarrow$ previously inaccessible.
- **Goal:** closed formula for FCS of the twisted XXX chain

# The model

- Heisenberg chain, $L$ spins $\frac{1}{2}$,

$$H_K = J \sum_{j=1}^{L} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z \right),$$

  with a **non-diagonal twisted boundary**, encoded in

$$K = \begin{pmatrix} \kappa_1 & \kappa_+ \\ \kappa_- & \kappa_2 \end{pmatrix} \quad (K = \mathbb{I} : \text{periodic}).$$

- Transfer matrix $t_K(u)$ built from $K$ commutes at different $u$ and generates $H_K$

$$H_K \sim \partial_u \ln t_K(u)|_{u=0}$$

# Key trick: quantum inverse problem

- Introduce a second twist $\widetilde{K} = Ke^{-Q(\bar{\beta})}$.
- Exact operator identity from the shift property of $t_K(0)$:

$$t_{\widetilde{K}}(0)^{-\ell} t_K(0)^{\ell} = \exp\Big(\sum_{j=1}^{\ell} Q_j(\bar{\beta})\Big).$$

- So $\exp Q^{(\ell)}$ is rewritten via transfer matrices for *two twists*, $K$ and $\widetilde{K}$. Inserting the $\widetilde{K}$-eigenbasis turns FCS into a sum over **overlaps**:

$$\langle \exp Q^{(\ell)} \rangle_K = \sum_{\widetilde{K}\text{-states}} \left(\frac{\Lambda_K(0)}{\Lambda_{\widetilde{K}}(0)}\right)^{\ell} \frac{\langle K|\widetilde{K}\rangle\langle\widetilde{K}|K\rangle}{\langle K|K\rangle\langle\widetilde{K}|\widetilde{K}\rangle}.$$

# Difficulty: $K$ is not diagonal

- Ordinary Bethe ansatz needs a reference state killed by a lowering operator — simple only for **diagonal** twists.
- A generic $K$ (like $\sigma^x, \sigma^y$) breaks $U(1)$, so the standard construction fails
- (Alternative routes potentially can be used: separation of variables)

# MABA: the idea

- Work in a basis adapted to the broken symmetry — the **Modified Algebraic Bethe Ansatz (MABA)** aka *off-diagonal Bethe ansatz*
- Achieved by conjugating the monodromy matrix with two auxiliary matrices built from the twist $K$ itself
- Result: **modified operators** $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}$, playing the role of the usual $A, B, C, D$, but adapted to the twist.
- Modified Bethe vectors act just like ordinary ones, on the *same* reference state $|0\rangle$: $|u, K\rangle = \widehat{B}(\bar{u})|0\rangle$
- Fixed (!) number of the Bethe roots

# Twist decomposition, explicitly

- Factorize the twist matrix as $K = \mathcal{B}\,\mathcal{D}\,\mathcal{A}$:

$$\mathcal{A} = \sqrt{\mu}\begin{pmatrix} 1 & \rho_2/\kappa_- \\ \rho_1/\kappa_+ & 1 \end{pmatrix}, \quad \mathcal{B} = \sqrt{\mu}\begin{pmatrix} 1 & \rho_1/\kappa_- \\ \rho_2/\kappa_+ & 1 \end{pmatrix},$$

$$\mathcal{D} = \begin{pmatrix} \kappa_1 - \rho_1 & 0 \\ 0 & \kappa_2 - \rho_2 \end{pmatrix}, \qquad \mu = \left(1 - \tfrac{\rho_1\rho_2}{\kappa_+\kappa_-}\right)^{-1},$$

with $\rho_1, \rho_2$ tied by one compatibility relation, and one parameter left free

## Example — the modified creation operator

$$\widehat{B}(z) = \mu\Big[B(z) + \tfrac{\rho_1}{\kappa_-}A(z) + \tfrac{\rho_2}{\kappa_-}D(z) + \tfrac{\rho_1\rho_2}{\kappa_-^2}C(z)\Big]$$

# Eigenvalues and Bethe equations

Fix $a(v) = \langle 0|A(v)|0\rangle$, $d(v) = \langle 0|D(v)|0\rangle$, and S-matrix $S(u,v)$

- Eigenvalue of a transfer matrix: $\Lambda_K(u|\bar{u}) = g(u,\bar{u})\,\mathcal{Y}_K(u|\bar{u})$, with $Y_K$ an explicit combination of the vacuum eigenvalues $a(u), d(u)$

$$\begin{aligned}\mathcal{Y}_K(v|\bar{v}) \\ &= (-1)^L(\kappa_1 - \rho_1)a(v)S(\bar{v},v) + (\kappa_2 - \rho_2)d(v)S(v,\bar{v}) \\ &\qquad + (\rho_1 + \rho_2)a(v)d(v)\end{aligned}$$

- Bethe equations: $\mathcal{Y}_K(u_j|\bar{u}_j) = 0$ — same role as usual, adapted to the twist $K$.
- Identical construction for $\widetilde{K}$: its own roots $\bar{v}$ and eigenvalue $\Lambda_K(v|\bar{v})$. Both towers feed the FCS sum
- Off-diagonal BA (Cao, Wang, Yang, Shi)

# The trick: turning a mixed overlap into a scalar product

- Need $\langle K|\widetilde{K}\rangle$: an overlap between eigenstates of *two different* twists. Ordinary Slavnov determinants only give overlaps within a *single* twist — the free MABA parameter closes this gap.

## Step 1 — match the free parameters

Choose $\rho_1, \rho_2$ (twist $K$) and $\widetilde{\rho}_1, \widetilde{\rho}_2$ (twist $\widetilde{K}$) so the modified creation operators become proportional:

$$\frac{\rho_1}{\kappa_-} = \frac{\widetilde{\rho}_1}{\widetilde{\kappa}_-}, \quad \frac{\rho_2}{\kappa_-} = \frac{\widetilde{\rho}_2}{\widetilde{\kappa}_-} \implies \frac{\widehat{B}(u)}{\mu} = \frac{\widetilde{\widehat{B}}(u)}{\widetilde{\mu}}.$$

## Step 2 — substitute inside the overlap

$$\langle \widetilde{K}, \bar{v}|u, K\rangle = \langle 0|\widetilde{\widehat{C}}(\bar{v})\, \widehat{B}(\bar{u})|0\rangle = \left(\frac{\mu}{\widetilde{\mu}}\right)^L \langle 0|\widetilde{\widehat{C}}(\bar{v})\, \widetilde{\widehat{B}}(\bar{u})|0\rangle = \left(\frac{\mu}{\widetilde{\mu}}\right)^L S_{\widetilde{K}}(\bar{v}|\bar{u}).$$

## Step 3 — done

Right-hand side is a **same-twist** scalar product $\Rightarrow$ given directly by the Slavnov determinant (next slide). Swapping $K \leftrightarrow \widetilde{K}$ gives $\langle K, \bar{u}|\bar{v}, \widetilde{K}\rangle \propto S_K(\bar{u}|\bar{v})$ the same way.

# Scalar product (Slavnov determinant)

$$g(x,y) = \frac{1}{x-y}$$

- **Overlap** (on-shell $\bar{v}$, off-shell $\bar{u}$, same twist):

$$S_K(\bar{u};\bar{v}) \;\propto\; \frac{1}{g(\bar{v},\bar{u})}\;\det\left[c\,\frac{\partial \Lambda_K(v_k|\bar{u})}{\partial u_j}\right],$$

  with $\Delta(\bar{u}) = \prod_{i>j} g(u_i,u_j)$, $\Delta'(\bar{u}) = \prod_{i<j} g(u_i,u_j)$.
- **Norm** of an eigenstate: same formula with $\bar{v} \to \bar{u}$ and $\Lambda_K \to Y_K$.
- This is the twisted-chain analogue of the classic Slavnov determinant for scalar products in the ordinary Bethe ansatz — and, combined with the previous slide's trick, it is all we need to build the *mixed* overlaps $\langle K|\widetilde{K}\rangle$ that enter the FCS.

# Main result: FCS as a form-factor sum

## Form factor expansion

$$\langle \exp Q^{(\ell)}(\bar{\beta})\rangle_K = \sum_{\bar{v}} \frac{1}{g(\bar{u},\bar{v})\, g(\bar{v},\bar{u})} \left(\frac{\Lambda_K(0|\bar{u})}{\Lambda_{\widetilde{K}}(0|\bar{v})}\right)^{\ell} \frac{\det[\,\partial\Lambda_K/\partial u\,]\ \det[\,\partial\Lambda_{\widetilde{K}}/\partial v\,]}{\det[\,\partial Y_K/\partial u\,]\ \det[\,\partial Y_{\widetilde{K}}/\partial v\,]}$$

- $\bar{u}$: ground-state Bethe roots for twist $K$ (fixed).
- Sum over $\bar{v}$: all solutions of the Bethe equations for $\widetilde{K} = Ke^{-Q(\bar{\beta})}$.
- Fully algebraic — no explicit wavefunctions needed.

$$g(x,y) = \frac{1}{x-y}$$

# Equivalent form: contour integral

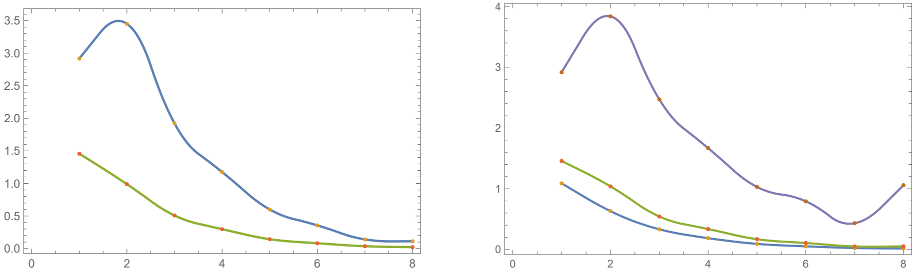
- Trade the sum over $\bar{v}$ for a contour integral around the Bethe roots:

$$\sum_{\bar{v}} F(\bar{v}) = \frac{1}{L!} \oint \frac{d\bar{z}}{(2\pi)^L} \ \det\left[\frac{\partial Y_{\widetilde{K}}(z_j|\bar{z})}{\partial z_k}\right] \frac{F(\bar{z})}{\prod_j Y_{\widetilde{K}}(z_j|\bar{z})}.$$

- Two equivalent representations of the FCS:
  - form-factor sum $\rightarrow$ long-range / large-$\ell$ asymptotics;
  - contour integral $\rightarrow$ short-range / small-$\ell$ computations.

# Numerical checks

- $L = 8$; Bethe roots via a McCoy-type $TQ$-equation, cross-checked against exact diagonalization.
- Exact agreement between the MABA sum and direct diagonalization in all tested cases.




$K = \sigma^x$ (left) and $K = \sigma^y$ (right): FCS vs. block length $\ell$; dots = exact diagonalization, curves = MABA.

# Summary & Outlook

- First closed-form FCS for **generic non-diagonal** twisted boundaries in the XXX chain.
- Key tools: quantum inverse problem + MABA + Slavnov-type overlap determinants.
- Two equivalent forms: form-factor sum and contour integral.

**Open directions:**

- Completeness of Bethe states for generic twist (only checked in analogous open-boundary cases).
- Thermodynamic limit of the form-factor series.
- Higher spin, XXZ chain, other symmetry groups ($SU(N)$, Hubbard).