Hard Rods, Fredholm Determinants and Random Matrices

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O.G. Miłosz Panfil, Phys. Rev. Lett. **137**, 076502 (2026)

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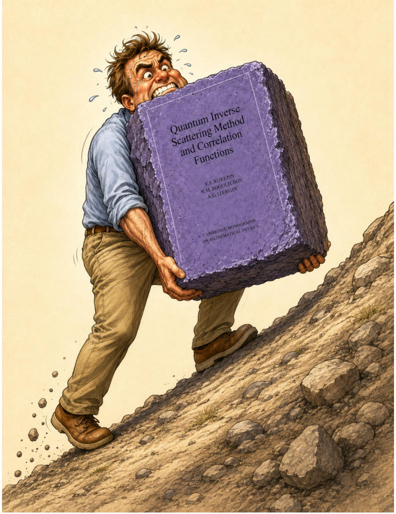
Why hard rods?

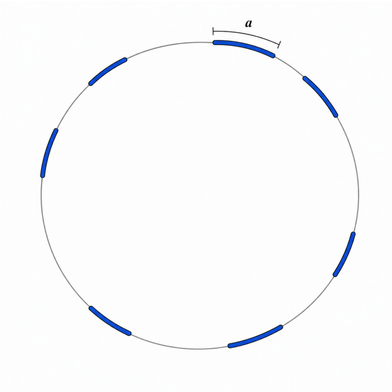
- Accessible correlation functions in quantum integrable models

$$
\begin{tikzcd}
\text{Lieb–Liniger} \arrow[d, "-a \int T\bar{T}"] \arrow[dd, bend right=90, "c \to \infty"'] \\
\text{Hard rods} \arrow[d, "a \to 0"] \\
\text{Tonks–Girardeau}
\end{tikzcd}
$$

- Great toy model
- Possible experiments with Rydberg atoms



Hard rod gas



$$H = \sum_{j=1}^{N} \hat{p}_j^2 + \sum_{i<j}^{N} V_{\mathrm{hr}}(x_i - x_j)$$

$$V_{\mathrm{hr}}(x) = \begin{cases} 0, & |x| > a, \\ \infty, & |x| \leq a. \end{cases}$$

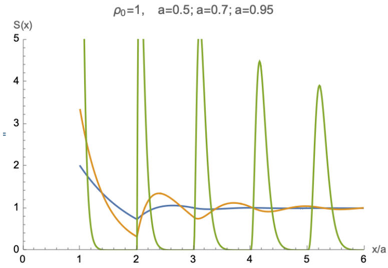
Density $\rho_0 = N/L$
Reduced volume $L_f = L - Na$
Packing factor $0 < \rho_0 a < 1$

- **Classical.** L .Tonks '36; Z. W. Salsburg, R. W. Zwanzig, and J. G. Kirkwood '53. M. Flicker '68; Recently GHD: H .Spohn, B. Doyon, F. Hübner, J .de Nardis . . . [Friedrich's talk this Friday!!!]
- **Quantum.** T. Nagamiya 41'; QMC: F. Mazzanti, et al '08. M. Motta et al 2016. S. Yu, Z. Zhu, and L. Sanchez-Palencia 2026
  ABACUS, S. Kiedrzynski, E. Witkowska, and M. Panfil 2026

Pair correlation (structure factor)

Thermodynamics

$$p = \frac{\rho_0 k_B T}{1-\rho_0 a}, \qquad \int \frac{dk}{2\pi} \frac{1}{e^{\beta(k^2-\mu)}+1} = \frac{\rho_0}{1-\rho_0 a}$$



M. Flicker '68

$$S(x) = \frac{\rho_0^2}{1-\rho_0 a} \sum_{n=1}^{[x/a]} \mathcal{P}_{\rm cl}\left(n-1; \frac{\rho_0(x-na)}{1-\rho_0 a}\right)$$

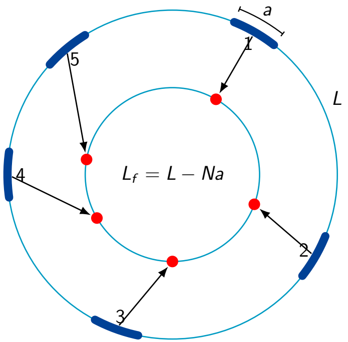
$$\mathcal{P}_{\rm cl}(k;x) = \frac{x^k}{k!} e^{-x}, \qquad \mathcal{P}_{\rm cl} \longrightarrow \mathcal{P}$$

In this talk

$$\boxed{S(x,t) = \lim_{\rm th} \langle \boldsymbol{\lambda} | \hat{\rho}(x,t) \hat{\rho}(0) | \boldsymbol{\lambda} \rangle}$$

$$S(x,t) = \lim_{\rm th} \sum_{\boldsymbol{\mu}} e^{it(E_{\boldsymbol{\mu}} - E_{\boldsymbol{\lambda}}) - ixP_{\boldsymbol{\mu}}} |\langle \boldsymbol{\mu} | \rho | \boldsymbol{\lambda} \rangle|^2.$$

QHR wave-functions



Hard-rods coordinates $x_j = r_j - ja$

$$H = -\sum \partial_{x_i}^2, \qquad \Psi(x_1, x_2, \ldots x_N)\Big|_{x_i = x_{i+1}} = 0$$

$$\Psi(x_1, x_2, \ldots x_N) = \det(e^{i\lambda_j x_k})$$

$$S(\lambda_1, \lambda_2) = e^{-ia(\lambda_1 - \lambda_2)}$$

$$\boxed{\lambda_j = \frac{2\pi}{L_f}\left(n_j - \nu_{\boldsymbol{\lambda}}\right), \quad \nu_{\boldsymbol{\lambda}} = \frac{aP_{\boldsymbol{\lambda}}}{2\pi}}.$$

$$L_f = L - Na$$

$$P_{\boldsymbol{\lambda}} = \sum_{j=1}^{N} \lambda_j = \frac{2\pi}{L} \sum_{j=1}^{N} n_j, \qquad E_{\boldsymbol{\lambda}} = \sum_{j=1}^{N} \lambda_j^2.$$

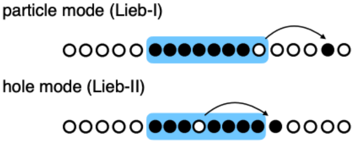
Excitations and matrix elements

$$P_{\boldsymbol{\lambda}} = \sum_{j=1}^{N} \lambda_j, \qquad E_{\boldsymbol{\lambda}} = \sum_{j=1}^{N} \lambda_j^2, \qquad \lambda_j = \frac{2\pi}{L_f} n_j - \frac{a P_{\boldsymbol{\lambda}}}{L_f}.$$

Particle-hole excitations around ground state

$$\omega_{\pm}(\lambda) = \frac{2\lambda k_F \pm \lambda^2}{K}$$

$$\boxed{K = (1 - \rho_0 a)^2}$$



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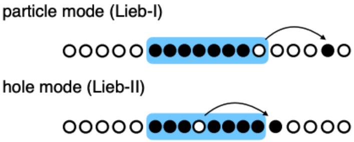


$$S(x,t) = \sum_{\boldsymbol{\mu}} e^{it(E_{\boldsymbol{\mu}} - E_{\boldsymbol{\lambda}}) - ix(P_{\boldsymbol{\mu}} - P_{\boldsymbol{\lambda}})} |\langle \boldsymbol{\mu} | \rho | \boldsymbol{\lambda} \rangle|^2.$$

Matrix elements **[S. Kiedrzynski, E. Witkowska, and M. Panfil (2026)]**

$$\langle \boldsymbol{\mu} | \rho | \boldsymbol{\lambda} \rangle = \frac{P_{\boldsymbol{\mu}} - P_{\boldsymbol{\lambda}}}{L} \left(\frac{2 \sin \pi \left(\nu_{\boldsymbol{\mu}} - \nu_{\boldsymbol{\lambda}} \right)}{L_f} \right)^{N-1} C[\boldsymbol{\mu}, \boldsymbol{\lambda}], \qquad C[\boldsymbol{\mu}, \boldsymbol{\lambda}] = \det \left(\frac{1}{\mu_i - \lambda_j} \right).$$

Taming Backreaction

$$\mu_j = \frac{2\pi}{L_f}\left(n_j - \nu_{\boldsymbol{\mu}}\right), \qquad \nu_{\boldsymbol{\mu}} = \frac{aP_{\boldsymbol{\mu}}}{2\pi}.$$

To disentangle the quasimomenta we employ the following trick:

$$\sum_{\boldsymbol{\mu}}(\dots) = \sum_{\boldsymbol{\mu}}\sum_{P}\int_{-L_f/2}^{L_f/2}\frac{ds}{L_f}\, e^{is(P-\sum_{j=1}^{N}\mu_j)}(\dots) = \sum_{P}\sum_{\boldsymbol{\mu}}\int_{-L_f/2}^{L_f/2}\frac{ds}{L_f}\, e^{is(P-\sum_{j=1}^{N}\mu_j)}(\dots)$$

Here all μ_j are independent!

$$\boxed{\sum_{P} \to \int d\nu_{\boldsymbol{\mu}}}$$

Remark. Similar technique is applicable to the multicomponent models.
Mobile impurity (Fermionic Yang-Gaudin) **[O.G., A.G. Pronko, M.B. Zvonarev (2014)]**

$$e^{ik_j L} = \frac{k_j - \Lambda + ig/2}{k_j - \Lambda - ig/2} \equiv e^{-i\nu(k_j)}, \qquad k_j = \frac{2\pi}{L}\left(n_j - \frac{\nu(k_j,\Lambda)}{\pi}\right)$$

TD limit

$$\nu(k_j,\Lambda) = \frac{\pi}{2} - \arctan(\Lambda - 2k_j/g)$$

Fredholm determinant

$$\sum_{\boldsymbol{\mu}} e^{it(E_{\boldsymbol{\mu}}-E_{\boldsymbol{\lambda}})-isP_{\boldsymbol{\mu}}} \left(\frac{2\sin(\pi\nu)}{L_f}\right)^{2N} C[\boldsymbol{\mu},\boldsymbol{\lambda}]^2 = \det(1+n\hat{V}) \equiv \mathcal{D}_\nu(s,t),$$

The thermal factor $n(\lambda) = 1/(e^{\beta(\lambda^2-k_F^2)}+1)$. Kernel:

$$V(\lambda,\mu) = \frac{1}{\pi}\frac{e_+(\mu)e_-(\lambda)-e_+(\lambda)e_-(\mu)}{\mu-\lambda}$$

with $e_-(\lambda) = e^{-it\lambda^2/2+i\lambda s/2}$ and

$$e_+(\lambda) = -\sin(\pi\nu)e^{it\lambda^2/2-i\lambda s/2}\left[\cos(\pi\nu)+i\sin(\pi\nu)\mathrm{Erf}\left(\frac{s-2\lambda t}{2\sqrt{-it}}\right)\right].$$

$$\boxed{S(x,t) = \int_{-\infty}^{\infty}\frac{dP}{2\pi}\frac{\sqrt{K}P^2e^{-iPx}}{(2\sin(\pi\nu))^2}\int_{-\infty}^{\infty} ds\, e^{isP}\mathcal{D}_\nu(s,t)},$$

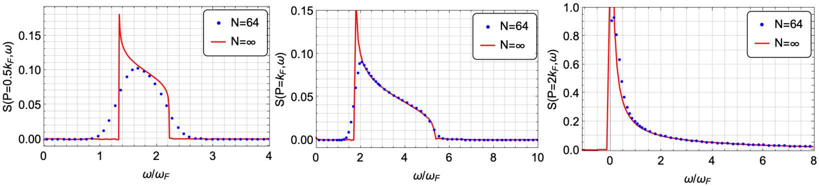
The phase shift is not compact! $\mathcal{D}_{\nu+1}(x,t) = \mathcal{D}_\nu(x,t)$

$$\boxed{S(x,t) = \sqrt{K}\sum_{n\in\mathbb{Z}}\sigma_n(x-na,t), \qquad \sigma_n(x,t) = -\int_0^1 \frac{e^{-2\pi i\nu n}}{4\sin^2(\pi\nu)}\partial_x^2\mathcal{D}_\nu(x,t)d\nu.}$$

Results

$$S(P,\omega) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \, \frac{\sqrt{K} P^2 e^{isP - i\omega t}}{(2\sin(\pi\nu))^2} \mathcal{D}_\nu(s,t),$$

Zero temperature $k_F = \pi\rho_0$, and $a\rho_0 = 1/4$.



- $\mathcal{D}_\nu(s,t) = \mathcal{D}_{-\nu}(-s,t) = \mathcal{D}^*_{-\nu}(s,-t) = \mathcal{D}^*_\nu(-s,-t)$.
  $S(P,\omega) = S(-P,\omega)$
- $\mathcal{D}_\nu(s,t) = \mathcal{D}_\nu(s,-t+i/T)$
  $S(P,-\omega) = S(P,\omega)e^{-\omega/T}, \qquad S(x,t) = S(x,-t+i/T)$.
- $\int \frac{d\omega}{2\pi} \omega S(P,\omega) = \rho_0 P^2$.
- $C = \int \mathrm{d}x \, \left(\langle \hat\rho(x,0)\hat\rho(0)\rangle - \langle\hat\rho(0)\rangle^2 \right) = (1-\rho_0 a)^3 \int \frac{\mathrm{d}k}{2\pi} n(k)(1-n(k))$.

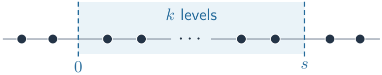
Static case and RMT

$$\hat{V}\Big|_{t=0} = (e^{2\pi i \nu} - 1)\hat{\mathcal{K}}, \quad \mathcal{K}(q,p) = \frac{\sin \frac{s(q-p)}{2}}{\pi(q-p)},$$

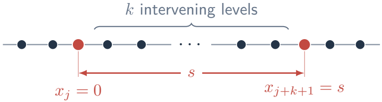
$$S(x) = \frac{\rho_0^2}{1-\rho_0 a} \sum_{n=1}^{[x/a]} \mathcal{P}\left(n-1; \frac{\rho_0(x-na)}{1-\rho_0 a}\right)$$

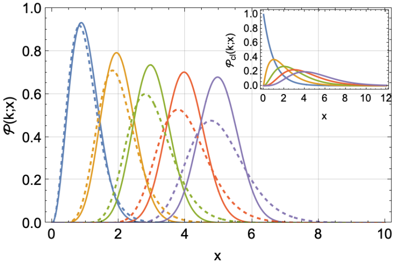
Random matrices (GUE) $\cdots < x_{j-1} < x_j < x_{j+1} < \dots$

$$E(k;s) = \Pr[\mathcal{N}_{(0,s)} = k]$$



$$p(k;s)ds = \Pr(s \le x_{j+k+1} - x_j < s + ds)$$

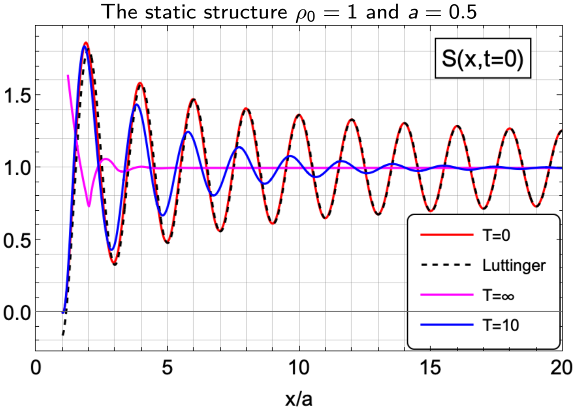




$$E(k;s) = \frac{\partial_\xi^k}{k!} \det\left(1 + (\xi - 1)n\hat{\mathcal{K}}\right)\Big|_{\xi=0}, \qquad p(k;x) = \partial_x^2 \sum_{j=0}^{k} (k+1-j)E(j;x)$$

$$\mathcal{P}(k;x) = \mathcal{N}^{-2} p(k; x/\mathcal{N}), \qquad \mathcal{N} = \rho_0/(1-\rho_0 a), \qquad \int_0^\infty \mathcal{P}(k;x)dx = 1, \qquad \int_0^\infty x\mathcal{P}(k;x)dx = k+1.$$

Results



Luttinger liquid prediction

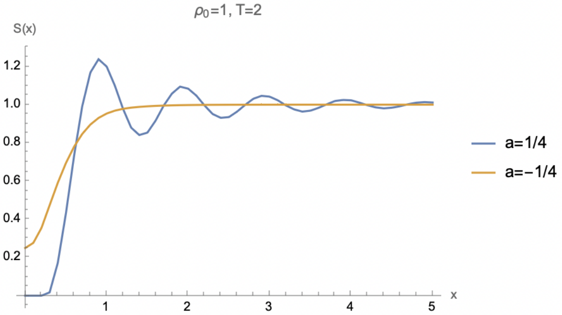
$$\frac{S(x)}{\rho_0^2} = 1 - \frac{K}{2(k_F x)^2} + \sum_{m=1}^{\infty} \frac{A_m \cos(2mk_F x)}{(\rho_0 x)^{2m^2 K}},$$

Results (Negaive a)

$$S(x) \overset{a \geq 0}{=} \sqrt{K} \sum_{n=1}^{[x/a]} \sigma_n(x - na).$$

$$S(x) = \sqrt{K} \sum_{n=1}^{\infty} \sigma_n(x + n|a|) + \sqrt{K} \sum_{n=[x/|a|]+1}^{\infty} \sigma_n(n|a| - x)$$

The static structure $\rho_0 = 1$ and $a = 0.5$



Outlook

- Exact expressions for the DSF
- Relation to RMT

Summary

- Non-equilibrium (without the quench action?)
- Multicomponent (extra averaging?)
- Other correlation function (one-body, FCS?)