Exact Three-Point Functions in $\mathcal{N} = 2$ Superconformal Field Theories

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adapted from arXiv:2503.07295 with S. Komatsu, G. Lefundes, and D. Serban

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Context and Motivation

Strongly coupled field theory = very hard to study in general.

Planar $\mathcal{N} = 4$ SYM/strings on AdS_5 = ideal example because of integrability (planar limit).

Integrability remains partially understood ($\approx$ long-range, non-compact spin chains...): no explicit Lax matrix, how to "see" integrability from the Feynman diagrams? etc.

The spectrum of conformal dimensions is known = quantum spectral curve (QSC).

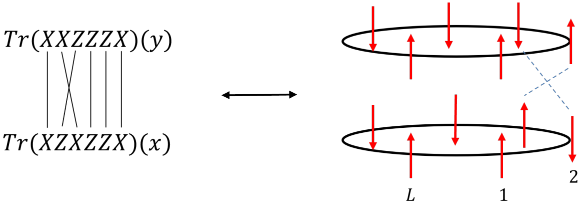
For three- and higher-point correlation functions, there is still work to be done. Various approaches: hexagons, T-functions, QSC or separation of variables (SoV).

In our paper, we applied hexagonalisation to the $\mathbb{Z}_K$ orbifolds of $\mathcal{N} = 4$ SYM and made contact with localisation.

Integrability for the Spectrum of $\mathcal{N} = 4$ SYM

First appearance of integrability via a mapping to a spin chain:

[Minahan and Zarembo (2002)]



Dilatation generator $\longleftrightarrow$ Hamiltonian

Integrability also investigated on the string side of the correspondence.

[Bena, Polchinski, and Roiban (2003)][Kazakov, Marshakov, Minahan, and Zarembo (2004)]

To write the all-loop Bethe equations, one only needs the dispersion relation and S-matrix. The matrix part of the S-matrix is fixed by symmetry, but there remains an overall phase (the dressing factor).

[Beisert, Kristjansen, and Staudacher (2003)]

[Beisert and Staudacher (2005)][Beisert (2005)]

Unitarity and crossing symmetry fix the dressing factor.

[Janik (2006)] [Beisert, Eden, and Staudacher (2006)]

[Arutyunov and Frolov (2009)] [Volin (2009)]

This is valid for any value of the 't Hooft coupling but only asymptotically in the length of the operators. For operators of finite length, the exact result can be formulated in terms of a Y-system and TBA equations. [Arutyunov and Frolov (2007-2009)]

[Gromov, Kazakov, and Vieira (2009)] [Gromov, Kazakov, Kozak, and Vieira (2009)]

[Bombardelli, Fioravanti, and Tateo (2009)]

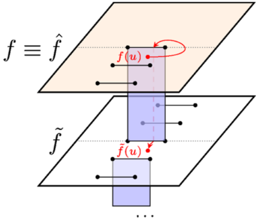
Quantum Spectral Curve

T- and Y-systems = universal structures (integrable spin chains, cluster algebras, q-characters for quantum groups, 2d integrable CFTs, etc.)

QQ-system is universal as well, with an advantage: finite number of Q-functions.

The $\mathcal{N} = 4$ SYM symmetry algebra is $\mathfrak{psu}(2,2|4)$. The highly non-trivial part of QSC is the regularity properties of the Q-functions.

[Gromov, Kazakov, Leurent, and Volin (2013,2014)]

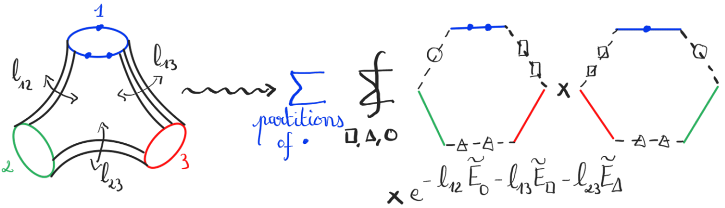


QSC can be solved efficiently to get the anomalous dimensions at finite coupling.

Integrability for n-Point Correlation Functions

- Hexagons: non-perturbative for large operators but difficult to incorporate finite-size corrections.
- TBA-like, i.e. in terms of T- and Y-functions: fully non-perturbative, but not the most practical. Also, not really understood.
  [Jiang, Komatsu, and Vescovi (2019)][Basso, Georgoudis, Klemenchuk Sueiro (2022)]
- QSC-like/SoV-like, i.e. in terms of Q-functions: believed to be the ideal form but remains largely not understood.
  [Cavaglià, Gromov, and Levkovich-Maslyuk (2018)][Bercini, Homrich, and Vieira (2022)]

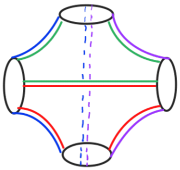
Hexagonalisation



Hexagon form factors = building blocks for n-point correlators.
Gluing along a seam = sum over a complete basis of mirror magnons.

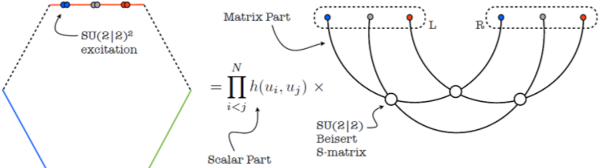
[Basso, Komatsu, and Vieira (2015)] [Fleury and Komatsu (2016,2017)]

[Eden and Sfondrini (2016)]

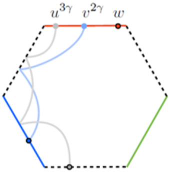


$\longrightarrow$ 4 hexagons, 6 cuts

Start with all excitations on the same edge:



Move excitations from one edge to another using mirror transformation



[Basso, Komatsu, and Vieira (2015)]

The Octagon

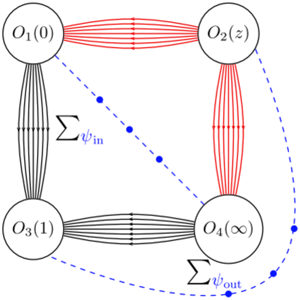
Take $\mathcal{O}_1 = \mathrm{Tr}\big(Z^K (X^\dagger)^K\big) + \ldots$, $\mathcal{O}_2 = \mathrm{Tr}\big((Z^\dagger)^{2K}\big)$, $\mathcal{O}_3 = \mathrm{Tr}\big(X^{2K}\big)$.
Then, when $K \to +\infty$, one has

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_1(x_4)\rangle \sim \frac{\mathbb{O}_0^2(z,\bar{z})}{(x_{12}^2 x_{24}^2 x_{13}^2 x_{34}^2)^K}\,.$$

Conformal cross-ratios:

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad \frac{z}{\bar{z}} = e^{2\,\mathrm{i}\,\phi}\,,$$

$$(1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,.$$



[Coronado (2018)]

Generalisation with arbitrary bridge length ℓ: [Coronado (2018)]

$$\mathbb{O}_\ell(z,\bar{z}) = \det(1-\lambda K_\ell)\,,$$

where $\lambda = (\cos\phi - \mathrm{ch}(\sqrt{z\bar{z}}))/2$ and K_ℓ is a semi-infinite matrix: for $m, n \geqslant 0$,

$$(K_{\ell-1}(g))_{mn} = 4\sqrt{(\ell+2m)(\ell+2n)}\int_0^{+\infty} \frac{J_{\ell+2m}(2gt)J_{\ell+2n}(2gt)}{\cos\phi - \mathrm{ch}\sqrt{t^2+z\bar{z}}}\frac{\mathrm{d}t}{t}\,.$$

[Kostov, Petkova, and Serban (2019)] [Belitsky and Kortchemsky (2019)]

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This object occurs in several other situations.
[Beisert, Eden, and Staudacher (2006)] [Basso, Dixon, and Papathanasiou (2020)]
[Sever, Tumanov, and Wilhelm (2020,2021)] [Basso and Tumanov (2024)]

$\mathbb{Z}_K$ orbifolds of $\mathcal{N} = 4$ SYM

- $\mathcal{N} = 2$ SCFT with gauge group $\mathrm{SU}(N)_0 \times \cdots \times \mathrm{SU}(N)_{K-1}$, K vector multiplets and K bifundamental hypermultiplets.
- In $\mathcal{N} = 4$ language, fields are $KN \times KN$ matrices that satisfy

$$[A_\mu, \gamma] = [Z, \gamma] = \{X, \gamma\} = \{Y, \gamma\} = 0\,,$$

where the $\mathbb{Z}_K$ twist is $\gamma = \mathrm{Diag}(I_N, \rho I_N, \ldots, \rho^{K-1} I_N)$ for $\rho = \mathrm{e}^{2\,\mathrm{i}\,\pi/K}$. Thus, $Z = \mathrm{Diag}(Z_0, Z_1, \ldots, Z_{K-1})$ for instance.

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- Orbifold point $g_0 = \cdots = g_{K-1}$, theory (expected to be) integrable in the planar limit.
- The twist breaks $\mathrm{PSU}(2,2|4)$ down to $\mathrm{PSU}(2,2|2) \times \mathrm{SU}(2)$ (for $K = 2$) or $\mathrm{PSU}(2,2|2) \times \mathrm{U}(1)$ (for $K > 2$).

Twisted BPS Operators

We focus on correlation functions of the following BPS operators:

$$\mathcal{O}_\ell^{(\alpha)} = \frac{1}{\sqrt{K}} \operatorname{Tr}\big(\gamma^\alpha Z^\ell\big)\,,$$

where $\alpha \in \{0, \ldots, K-1\}$. The operator $\mathcal{O}_\ell^{(0)}$ is called untwisted, the others are twisted operators.

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$$\langle \mathcal{O}_\ell^{(\alpha),\dagger}(x)\mathcal{O}_\ell^{(\alpha)}(0)\rangle = \frac{\ell\, G_\ell^{(\alpha)}(g)}{x^{2\ell}}\,,$$

where the normalisation is expressed in terms of the octagon

$$G_\ell^{(\alpha)}(g) = \frac{\det(1 - s_\alpha K_{\ell+1})}{\det(1 - s_\alpha K_{\ell-1})}\,,$$

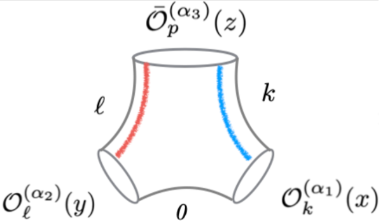
where $s_\alpha = \sin^2(\pi\alpha/K)$.

[Galvagno and Preti (2020)]

[Billò, Frau, Galvagno, Lerda, and Pini (2021)]

Three-Point Functions

$$\frac{\langle \mathcal{O}_k^{(\alpha_1)}(x)\mathcal{O}_\ell^{(\alpha_2)}(y)\mathcal{O}_{k+\ell}^{(\alpha_3),\dagger}(z)\rangle}{\sqrt{G_k^{(\alpha_1)}(g)G_\ell^{(\alpha_2)}(g)G_{k+\ell}^{(\alpha_3)}(g)}} = \frac{k\ell(k+\ell)}{\sqrt{K}N}\,\frac{C_k^{(\alpha_1)}C_\ell^{(\alpha_2)}C_{k+\ell}^{(\alpha_3)}}{|x-z|^{2k}|y-z|^{2\ell}}$$



$$C_\ell^{(\alpha)}(g) = \sqrt{1 + \frac{g}{2\ell}\partial_g \ln \frac{\det(1 - s_\alpha K_{\ell+1})}{\det(1 - s_\alpha K_{\ell-1})}}$$

[Billò, Frau, Lerda, Pini, and Vallarino (2022)]

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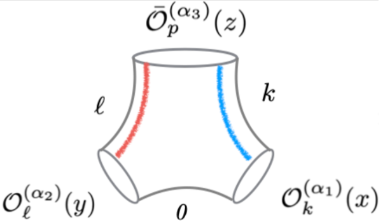


$$C_\ell^{(\alpha)}(g) = \sqrt{1+\frac{g}{2\ell}\partial_g \ln\frac{\det(1-s_\alpha K_{\ell+1})}{\det(1-s_\alpha K_{\ell-1})}}$$

[Billò, Frau, Lerda, Pini, and Vallarino (2022)]

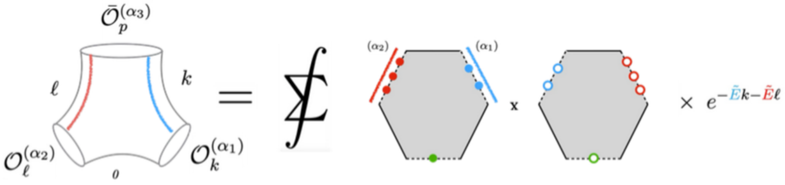
$$C_\ell^{(\alpha)}(g) = \frac{\det(1-s_\alpha K_\ell)}{\sqrt{\det(1-s_\alpha K_{\ell-1})\det(1-s_\alpha K_{\ell+1})}}$$

[Ferrando, Komatsu, Lefundes, and Serban (2025)]

[Korchemsky and Testa (2025)] [Korchemsky (2025)]

How can we recover this result in the hexagon framework?

Twisted Hexagons



Twisting is implemented by inserting powers of $1_L \times \mathrm{Diag}(\rho, \rho^{-1}, 1, 1)_R$.

This easily explains part of the result:

$$\frac{\det(1 - s_{\alpha_1} K_k)\,\det(1 - s_{\alpha_2} K_\ell)\,\det(1 - s_{\alpha_3} K_{k+\ell})}{\prod_{i=1}^{3} \sqrt{\det(1 - s_{\alpha_i} K_{\ell_i - 1})\,\det(1 - s_{\alpha_i} K_{\ell_i + 1})}}$$

Regularisation

The naïve hexagon formula contains some divergences when the quantum numbers of magnons living on different bridges coincide. These divergences stem from the infinite volume of the mirror theory.

Difficult to regularise in general. [Basso, Gonçalves, and Komatsu (2017)]
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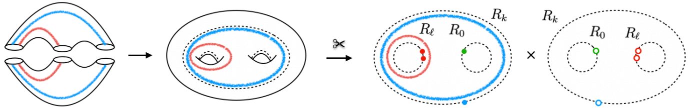
In our case, the operators are protected and we can do something similar to the usual torus regularisation for the ground state energy.

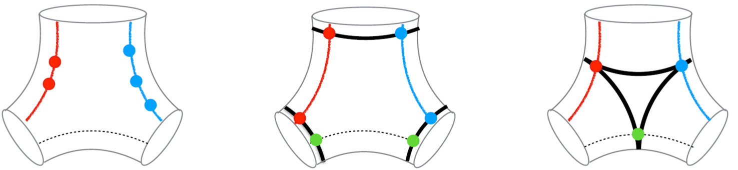
$C^2_{\ell,k+\ell} =$



$$\left(C_k^{(\alpha_1)} C_\ell^{(\alpha_2)} C_{k+\ell}^{(\alpha_3)} \right)^2 = \sum_{n_u, n_v, n_w \geqslant 0} \mathcal{C}_{(n_u, n_v, n_w)}$$

Complete Result

$$\frac{\det(1 - s_{\alpha_1} K_k)\ \det(1 - s_{\alpha_2} K_\ell)\ \det(1 - s_{\alpha_3} K_{k+\ell})}{\prod_{i=1}^{3} \sqrt{\det(1 - s_{\alpha_i} K_{\ell_i - 1})\,\det(1 - s_{\alpha_i} K_{\ell_i + 1})}}$$



We explained the appearance of each building block separately, but we still need to make sure that the full result is indeed factorised.

Conclusion and Outlook

We proposed an extension of the hexagon formalism to the $\mathbb{Z}_K$ orbifold $\mathcal{N} = 2$ SCFT, and a regularisation applicable to 3-pt functions of operators corresponding to (twisted) vacua.

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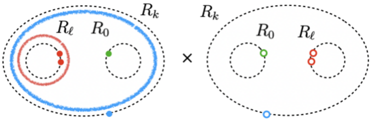
We proposed an extension of the hexagon formalism to the $\mathbb{Z}_K$ orbifold $\mathcal{N} = 2$ SCFT, and a regularisation applicable to 3-pt functions of operators corresponding to (twisted) vacua.

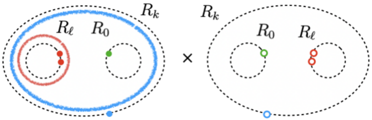
Further possible directions:

- Some of our results are still conjectural, how can we prove them?
- 3-pt functions of non-BPS operators or higher-point correlation functions [le Plat and Skrzypek (2025)]
- Can we interpret the result in the language of T-functions? of Q-functions? [Basso, Georgoudis, and Klemenchuk Sueiro (2022)] [Bercini, Homrich, and Vieira (2022)][Bargheer, Bercini, Cavaglià, Lai, and Ryan (2025)]
- Extension to other twisted theories?

Thank you!



The two new objects are themselves 3-pt functions in the mirror theory. And we only need them in the infinite volume limit = asymptotic 3-pt functions.



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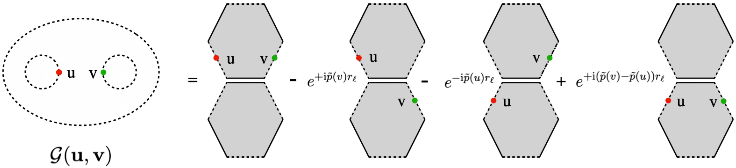
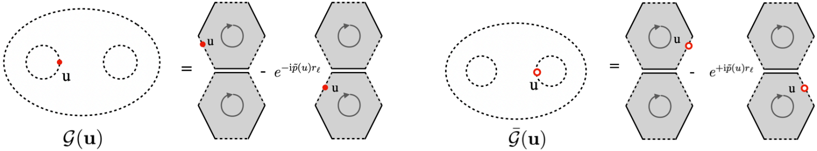
We can simply compute them using the asymptotic hexagon formula: sum over partitions of the (mirror) excitations between 2 hexagons.



Bridge Contributions

$$\mathcal{C}_{(1,0,0)} = \lim_{r_\ell \to \infty} \sum_{a=1}^{\infty} \int \frac{\mathrm{d}u}{2\pi} \mu_a(u) e^{-\ell \widetilde{E}_a(u)} \, \mathrm{STr}_a \left[\tau_a^{\alpha_2} \, \mathcal{G}(u) \, \overline{\mathcal{G}}(u) \right]$$



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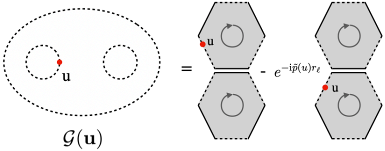

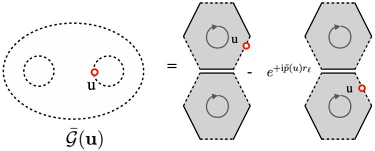

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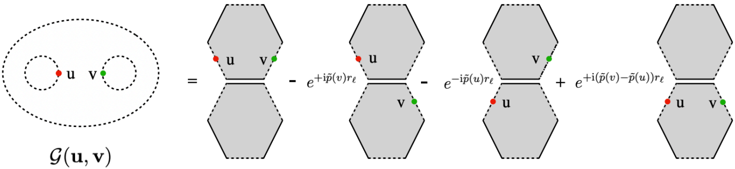
$$= 2 \sum_{a=1}^{\infty} \int \frac{\mathrm{d}u}{2\pi}\, e^{-\ell \widetilde{E}_a(u)}\, T_a^{(\alpha_2)}\,, \qquad T_a^{(\alpha)} = \mathrm{STr}\ \tau_a^{\alpha} = 4 a s_\alpha\,.$$

First contribution to $(\det(1 - s_{\alpha_2} K_\ell))^2$. Easily generalised to

$$\sum_{n=0}^{\infty} \mathcal{C}_{(n,0,0)} = (\det(1 - s_{\alpha_2} K_\ell))^2\ .$$

Wrapping Contributions



$$\mathcal{C}_{(1,1,0)} = \lim_{r_\ell \to \infty} \sum_{a,b=1}^{\infty} \int \frac{\mathrm{d}u\,\mathrm{d}v}{(2\pi)^2} \mu_a(u)\,\mu_b(v)\, e^{-\ell \widetilde{E}_a(u)}\, \mathrm{STr}_{ab}[\tau_a^{\alpha_2}\, \mathcal{G}(u,v)\, \bar{\mathcal{G}}(u,v)]$$

Compute (one of) the integrals by closing the contour in the lower/upper half-plane. We choose the order $\mathrm{Im}(u) > \mathrm{Im}(v) > \mathrm{Im}(w)$.
Only decoupling poles $(a,u) = (b,v)$ survive:

$$\mathcal{C}_{(1,1,0)} = \sum_{a=1}^{\infty} \int \frac{\mathrm{d}u}{2\pi} e^{-\ell \widetilde{E}_a(u)} \left(-i\,\partial_v\, \mathrm{STr}_{ab}\ \tau_a^{\alpha_2}\, (\mathcal{S}_{ab}(u,v))^2 \right) \Big|_{v \to u}$$

We conjecture that

$$\sum_{n=0}^{\infty} \mathcal{C}_{(n,n,0)} = \frac{1}{\det(1 - s_{\alpha_2} K_{\ell-1}) \det(1 - s_{\alpha_2} K_{\ell+1})} \,,$$

$$\mathcal{C}_{(n,n,0)} = \frac{\mathrm{i}^{-n}}{n!} \prod_{k=1}^{n} \left(\sum_{a_k=1}^{\infty} \int \frac{\mathrm{d}u_k}{2\pi} e^{-\ell \widetilde{E}_{a_k}(u_k)} \right) \partial_{\vec{v}} \, \mathrm{STr}\big[\tau_{\vec{a}}^{\alpha_2} \, \mathcal{S}_{\vec{a}\vec{b}}(\vec{u}, \vec{v}) \, \mathcal{S}_{\vec{a}\vec{b}}(\vec{u}, \vec{v}) \big] \Big|_{\vec{v} \to \vec{u}} \,,$$

where

$$\tau_{\vec{a}}^{\alpha} = \prod_{k=1}^{n} \tau_{a_k}^{\alpha} \,, \quad \mathcal{S}_{\vec{a}\vec{b}}(\vec{u}, \vec{v}) = \prod_{i=1}^{n} \prod_{j=1}^{n} \mathcal{S}_{a_i, b_j}(u_i, v_j) \,.$$

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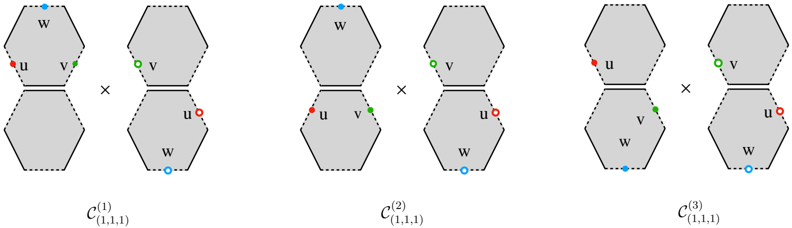
For comparison, recall that $\sum_{n=0}^{\infty} \mathcal{C}_{(n,0,0)} = \left(\det(1 - s_{\alpha_2} K_\ell) \right)^2$,

$$\mathcal{C}_{(n,0,0)} = \frac{\mathrm{i}^{-n}}{n!} \prod_{k=1}^{n} \left(\sum_{a_k=1}^{\infty} \int \frac{\mathrm{d}u_k}{2\pi} e^{-\ell \widetilde{E}_{a_k}(u_k)} \mu_{a_k}(u_k) 4 a_k s_{\alpha_2} \right) \prod_{i<j} H_{a_i, a_j}(u_i, u_j) ,$$

Bridge-Like Contribution

There are 3-magnon contact terms, simplest example $= \mathcal{C}_{(1,1,1)}$.



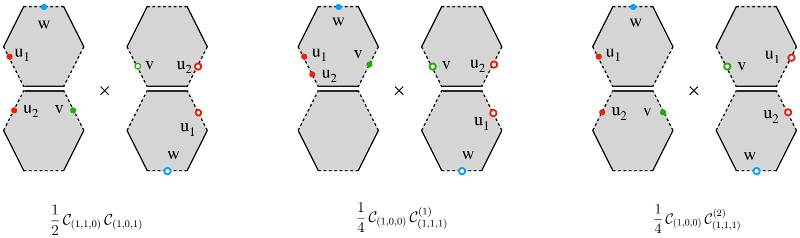
$$\mathcal{C}^{(2)}_{(1,1,1)} = \lim_{r_\ell, r_p \to \infty} \sum_{a,b,c=1}^{\infty} \int \frac{du\, dv\, dw}{(2\pi)^3}\, \mu_a(u)\, \mu_b(v)\, \mu_c(w)\; e^{-\ell \widetilde{E}_a(u) - k\widetilde{E}_c(w)}$$

$$\times\; e^{i(\tilde{p}_c(w)-\tilde{p}_a(u))r_p + i(\tilde{p}_b(v)-\tilde{p}_a(u))r_\ell}\; \frac{\mathrm{STr}_{abc}\;\; \tau_a^{\alpha_2}\, \mathcal{S}_{ab}(u,v)\, \tau_c^{\alpha_1}\, \mathcal{S}_{ac}(u,w)}{h_{ab}(v,u)\, h_{ca}(w,u)}\,,$$

Decoupling poles at $(c, w) = (a, u)$ and $(b, v) = (a, u)$. Eventually,

$$\mathcal{C}_{(1,1,1)} = 2\left(\mathcal{C}^{(1)}_{(1,1,1)} + \mathcal{C}^{(2)}_{(1,1,1)} + \mathcal{C}^{(3)}_{(1,1,1)}\right) = 2\sum_{a=1}^{\infty} \int \frac{du}{2\pi}\, e^{-(k+\ell)\widetilde{E}_a(u)} \mu_a(u) 4 a s_{\alpha_3}\,.$$

Factorisation: an Example



$$\mathcal{C}_{(2,1,1)} = \mathcal{C}_{(1,1,0)}\,\mathcal{C}_{(1,0,1)} + \mathcal{C}_{(1,0,0)}\,\mathcal{C}_{(1,1,1)}$$