# New integrable Markov processes from set-theoretical R-matrices

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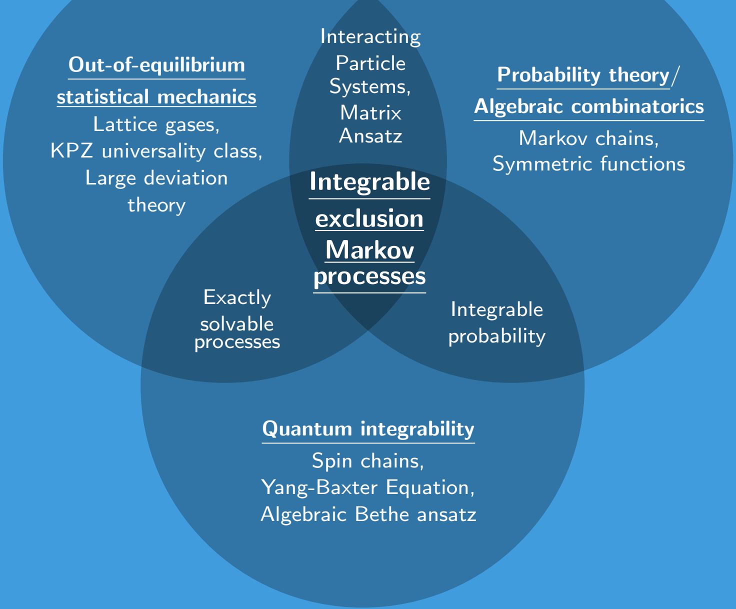
# Outline

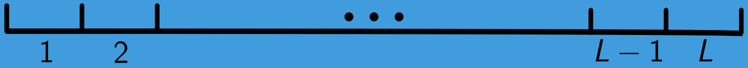
# Introduction : General context



# Introduction : Formalism of exclusion Markov processes

**Framework :**

- Lattice with $L$ sites



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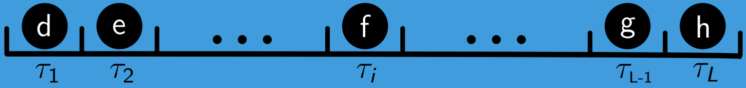
- Lattice with $L$ sites
- Set of species : $X$
- $\tau_i \in X$ : Local configuration variable
- $|\vec{\tau}\rangle = |\tau_1, \cdots, \tau_L\rangle \in \mathfrak{C}$ : Lattice configuration

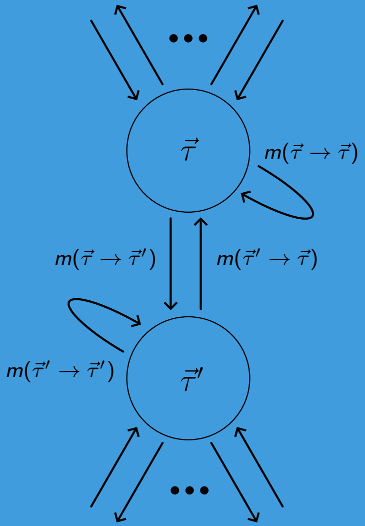


# Introduction : Formalism of exclusion Markov processes

**Dynamics:**

- Stochastic dynamics : Markov matrix

$$M = \sum_{\vec{\tau},\vec{\tau}' \in \mathfrak{C}} m(\vec{\tau} \to \vec{\tau}') |\vec{\tau}'\rangle \langle \vec{\tau}|$$



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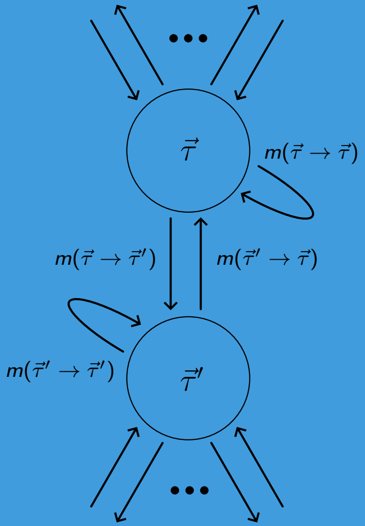

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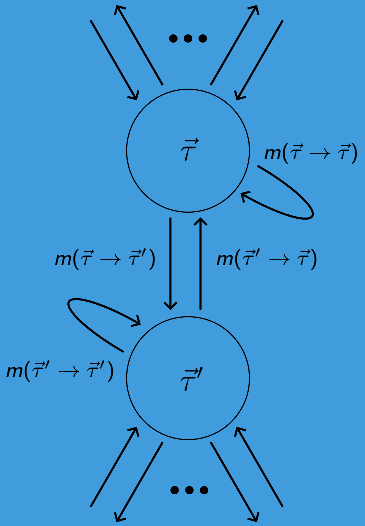
- Probability of $\vec{\tau}$ at time t : $P_t(\vec{\tau})$
- Evolution of $|P_t\rangle = \sum_{\vec{\tau}\in\mathfrak{C}} P_t(\vec{\tau})|\vec{\tau}\rangle$

$$\frac{\mathrm{d}|P_t\rangle}{\mathrm{dt}} = M|P_t\rangle$$



# Introduction : Formalism of exclusion Markov processes

$M$ local : nearest neighbour transitions

$\Rightarrow m$ : local jump operator

# Introduction : Formalism of exclusion Markov processes

**Boundary dynamics:**

- Periodic boundary conditions:

$$M = \sum_{i=1}^{L-1} m_{i,i+1} + m_{L,1}$$

# Introduction : Formalism of exclusion Markov processes

**Boundary dynamics:**

- Periodic boundary conditions:

$$M = \sum_{i=1}^{L-1} m_{i,i+1} + m_{L,1}$$

- Open boundaries:

$$M = B_1 + \sum_{i=1}^{L-1} m_{i,i+1} + \bar{B}_L$$

# Introduction : The multi-species SSEP

**Set of species:**

$$X = \{0, 1, \cdots, p\}$$

**Dynamics:**

- Local jump operator:

$$m = P - \mathrm{id} \text{ with } P|i, j\rangle = |j, i\rangle$$

$\Rightarrow$ Local transition rate:

$$m(ij \to ji) = m(ji \to ij) = 1 \text{ for } i \neq j$$

# Introduction : The multi-species SSEP

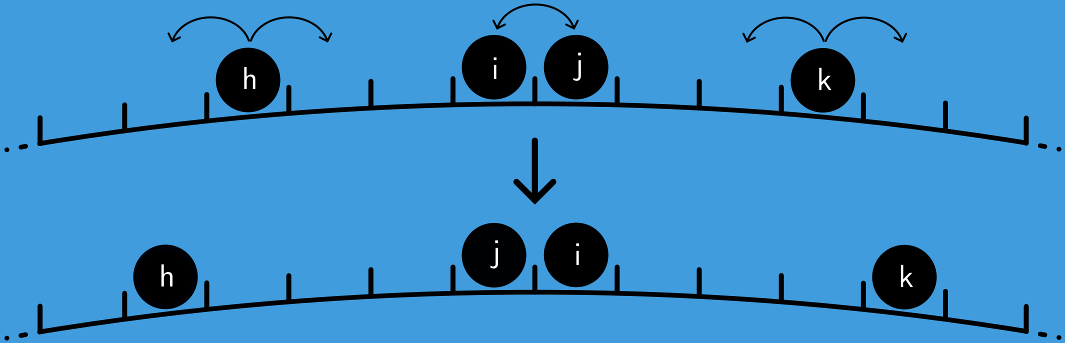



Figure: Multi-species Symmetric Simple Exclusion Process (SSEP) with periodic boundary conditions

# Introduction : The multi-species SSEP

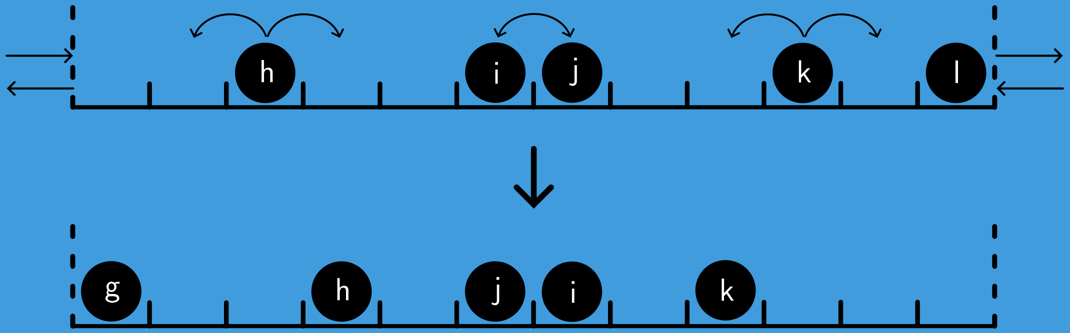



Figure: Multi-species SSEP with open boundary conditions

# The periodic (p,1)-SSEP : Definition

**Set of species:**

$$X = \{0, \bar{0}, 1, \cdots, p\}$$

- $\{1, \cdots, p\}$ : **singlet** particle species
- $\{0, \bar{0}\}$ : **reactive** particle species

MD, E. Ragoucy, L. Poulain d'Andecy, *Integrable multi-species SSEP with reactive particle species*, arXiv:2607.18959

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**Local configuration changes:**

$$ij \leftrightarrow ji, \;\; 0i \leftrightarrow i0, \bar{0}i \leftrightarrow i\bar{0}, \;\; 00 \leftrightarrow \bar{0}\bar{0}$$

# The periodic (p,1)-SSEP : Interpretation

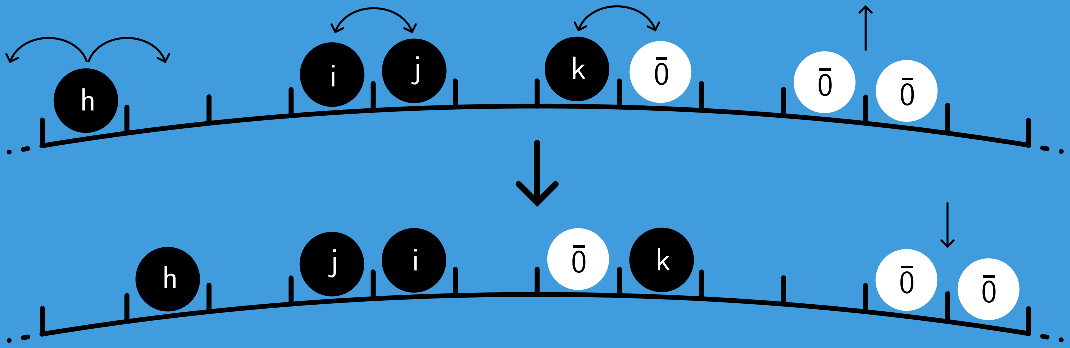


Figure: Evaporation-condensation interpretation of the $(p, 1)$-SSEP with periodic boundary conditions

# The periodic (p,1)-SSEP : Integrability

(p,1)-SSEP constructed from a <u>**set-theoretical solution**</u> to the YBE

# The periodic (p,1)-SSEP : Integrability

For the set $X$, define the bijection $\check{r}$ of $X \times X$ by

$$\check{r}(x, y) = (g_x(y), f_y(x))$$

<u>**Definition**:</u>

$(X, \check{r})$ is a set-theoretical solution to the YBE if

$$\check{r}_{12}\check{r}_{23}\check{r}_{12} = \check{r}_{23}\check{r}_{12}\check{r}_{23}$$

with $\check{r}_{12} = \check{r} \times \mathrm{id} : X^3 \to X^3$ and $\check{r}_{23} = \mathrm{id} \times \check{r} : X^3 \to X^3$.

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For the set $X$, define the bijection $\check{r}$ of $X \times X$ by

$$\check{r}(x, y) = (g_x(y), f_y(x))$$

**Properties**:

- Non-degenerate : $g_x, f_x$ bijections of $X$ for all $x \in X$
- Involutive : $\check{r}^2 = \mathrm{id}$

# The periodic (p,1)-SSEP : Integrability

Linear extension on $V \otimes V$ with $X$ basis of $V$ :

$$\check{r}|x,y\rangle = |g_x(y), f_y(x)\rangle$$

$\check{r}$-matrix of the $(p,1)$-SSEP:

$$\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$$

on $(|0,0\rangle, |0,\bar{0}\rangle, |\bar{0},0\rangle, |\bar{0},\bar{0}\rangle)$
and permutation on the remaining basis vectors

# The periodic (p,1)-SSEP : Integrability

**Baxterization formula**:

$$\check{r}(u) = \frac{u\,\check{r} + \mathrm{id}}{u+1}$$

**Locally**: $m = \check{r}'(0) = \check{r} - \mathrm{id}$
**Globally**: $M$ log-derivative of the *transfer matrix*

$\Rightarrow$ periodic (p,1)-SSEP **integrable**

# The periodic (p,1)-SSEP : Long time evolution

**Goal:** determine stationary states, i.e., non-trivial solutions $|S\rangle$ of $M|S\rangle = 0$

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**Goal:** determine stationary states, i.e., non-trivial solutions $|S\rangle$ of $M|S\rangle = 0$

- state space partitioned into sectors
- existence of a unique stationary state per sector
- stationary state uniform for each sector

# The open (p,1)-SSEP :

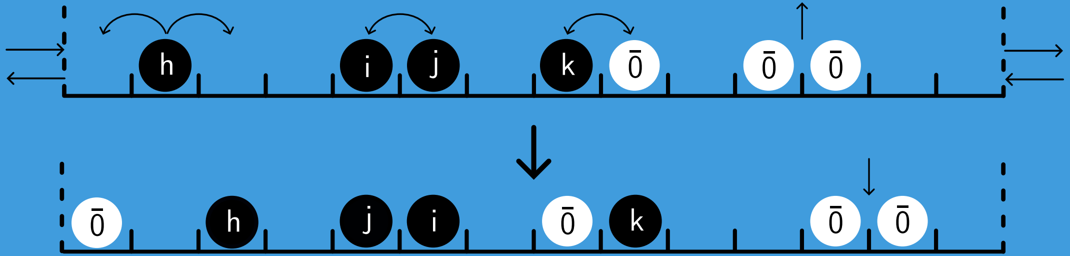


Figure: Evaporation-condensation interpretation of the $(p, 1)$-SSEP with open boundaries

# The open (p,1)-SSEP : Symmetry preserving boundaries

Boundary preserving bulk symmetry : $0 \leftrightarrow \bar{0}$:

$$B^{(1)} = \begin{pmatrix} \gamma & \gamma & \dots & \dots & \gamma \\ \gamma & \gamma & \dots & \dots & \gamma \\ \alpha_1 & \alpha_1 & \dots & \dots & \alpha_1 \\ \alpha_2 & \alpha_2 & \dots & \dots & \alpha_2 \\ \vdots & \vdots & & & \vdots \\ \alpha_p & \alpha_p & \dots & \dots & \alpha_p \end{pmatrix} - \mathrm{Id}\,, \qquad \bar{B}^{(1)} = \begin{pmatrix} \beta & \beta & \dots & \dots & \beta \\ \beta & \beta & \dots & \dots & \beta \\ \delta_1 & \delta_1 & \dots & \dots & \delta_1 \\ \delta_2 & \delta_2 & \dots & \dots & \delta_2 \\ \vdots & \vdots & & & \vdots \\ \delta_p & \delta_p & \dots & \dots & \delta_p \end{pmatrix} - \mathrm{Id}$$

with the constraints

$$\sum_{i=1}^{p} \alpha_i + 2\gamma = \sum_{i=1}^{p} \delta_i + 2\beta = 1$$

# The open (p,1)-SSEP : Symmetry preserving boundaries

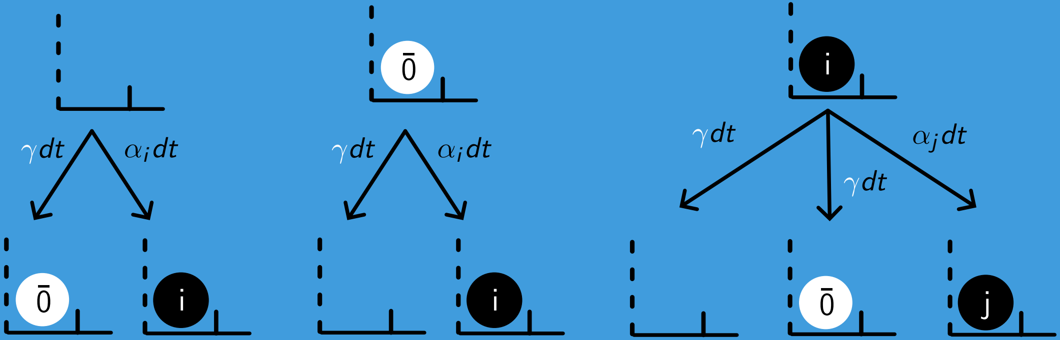


Figure: Injection/extraction rates for the integrable boundary $B^{(1)}$

# The open (p,1)-SSEP : Symmetry breaking boundaries

Boundary breaking bulk symmetry : $0 \leftrightarrow \bar{0}$:

$$B^{(2)} = \begin{pmatrix} \gamma_1 & \gamma_1 & 0 & \dots & 0 \\ \gamma_2 & \gamma_2 & 0 & \dots & 0 \\ 0 & 0 & \alpha_1 & \dots & \alpha_1 \\ 0 & 0 & \alpha_2 & \dots & \alpha_2 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \alpha_p & \dots & \alpha_p \end{pmatrix} - \mathrm{Id}\,, \qquad \bar{B}^{(2)} = \begin{pmatrix} \beta_1 & \beta_1 & 0 & \dots & 0 \\ \beta_2 & \beta_2 & 0 & \dots & 0 \\ 0 & 0 & \delta_1 & \dots & \delta_1 \\ 0 & 0 & \delta_2 & \dots & \delta_2 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \delta_p & \dots & \delta_p \end{pmatrix} - \mathrm{Id}$$

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$$\sum_{i=1}^{p} \alpha_i = \gamma_1 + \gamma_2 = \sum_{i=1}^{p} \delta_i = \beta_1 + \beta_2 = 1$$

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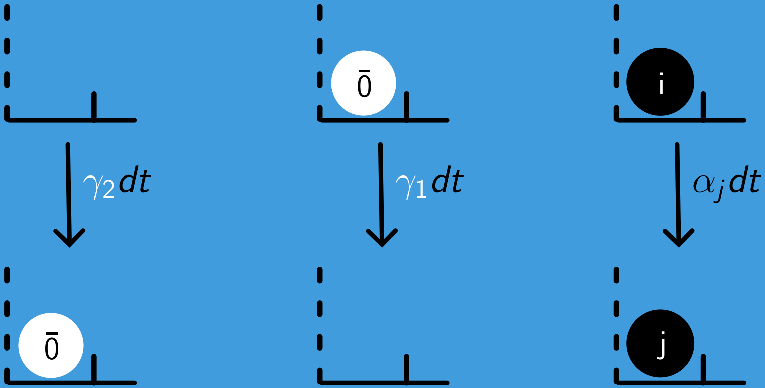


Figure: Injection/extraction rates for the integrable boundary $B^{(2)}$

# Physical quantities : Densities, currents, reaction rates

**Definition**:

The mean reaction rate of the reactive species $\bar{0}$ in the stationary state is

$$\langle \nu_{\bar{0}} \rangle_i = \langle \nu_{\bar{0}} \rangle_{i-1,i} + \langle \nu_{\bar{0}} \rangle_{i,i+1} \, ,$$

with

$$\langle \nu_{\bar{0}} \rangle_{i,i+1} = \langle 00 \rangle_{i,i+1} \, m(00 \to \bar{0}\bar{0}) - \langle \bar{0}\bar{0} \rangle_{i,i+1} \, m(\bar{0}\bar{0} \to 00)$$

# Physical quantities : Densities, currents, reaction rates

**Symmetry preserving boundaries:**

**Densities:**

$$\langle 0\rangle_i = \langle \bar{0}\rangle_i = \frac{\gamma(L-i+1)+\beta i}{L+1}$$

$$\langle s\rangle_i = \frac{\alpha_s(L-i+1)+\delta_s i}{L+1}$$

# Physical quantities : Densities, currents, reaction rates

**Symmetry preserving boundaries:**

**Currents:**

$$\langle j_0 \rangle_{i,i+1} = \langle j_{\bar{0}} \rangle_{i,i+1} = \frac{\gamma - \beta}{L+1}$$

$$\langle j_s \rangle_{i,i+1} = \frac{\alpha_s - \delta_s}{L+1}$$

# Physical quantities : Densities, currents, reaction rates

**Symmetry preserving boundaries:**

**Reaction rate:**

$$\langle \nu_{\bar{0}} \rangle_i = 0$$

# Physical quantities : Densities, currents, reaction rates

**Symmetry breaking boundaries:**

- Mean reaction rate $\langle \nu_{\bar{0}} \rangle_i \neq 0$
- Mean currents non uniform

# The (p,q)-SSEP :

Set of species : $X = \{1, \cdots, p, 1^-, 1^+, \cdots, q^-, q^+\}$

- $\{1, \ldots, p\}$ : p singlet particle species
- $\{1^-, 1^+, \ldots, q^-, q^+\}$ : $q$ pairs of reactive species

# The (p,q)-SSEP :

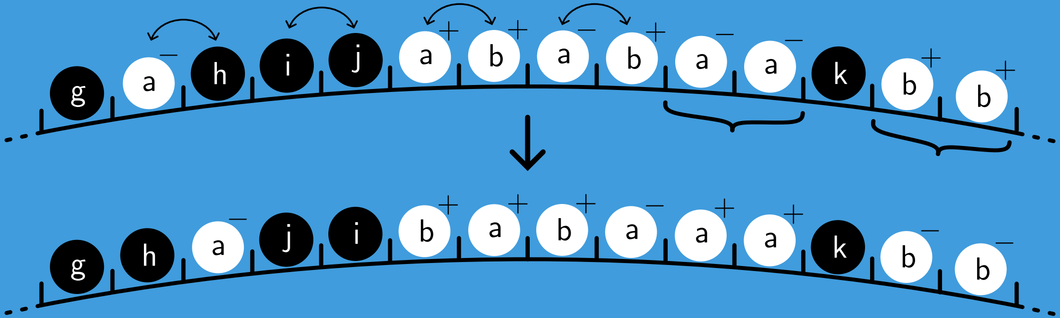


Figure: The (p,q)-SSEP with periodic boundary conditions

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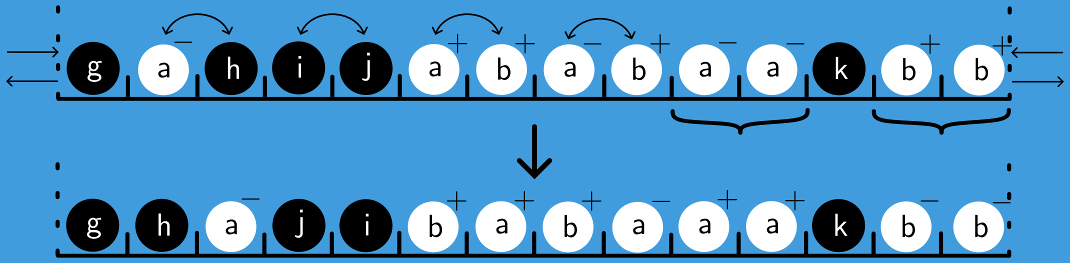


Figure: The (p,q)-SSEP with open boundaries

# Perspectives

- Determine the stationary distribution of the $(p, 1)$-SSEP with symmetry breaking boundaries and deduce physical quantities (using the Matrix Ansatz)

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# Perspectives

- Determine the stationary distribution of the $(p, 1)$-SSEP with symmetry breaking boundaries and deduce physical quantities (using the Matrix Ansatz)
- Explore the thermodynamic limit $L \to \infty$
- Classify the solutions of the reflection equation and study the associated processes
- Construct new processes from more elaborate set-theoretical solutions

Thank you for your attention !

# Supplementary material : Integrability with boundaries

**Reflection equation:**

$$\check{r}_{12}(u-v)k_1(u)\check{r}_{12}(u+v)k_1(v) = k_1(v)r_{12}(u+v)k_1(u)r_{12}(u-v)$$

**Reversed reflection equation:**

$$(\check{r}_{12}(u-v))^{-1}\bar{k}_1(u)(\check{r}_{12}(u+v))^{-1}\bar{k}_1(v) = \bar{k}_1(v)(r_{12}(u+v))^{-1}\bar{k}_1(u)(r_{12}(u-v))^{-1}$$

**Dual reflection equation:**

$$r_{12}(v-u)\tilde{k}_1(u)(r_{12}^{t_2}(u+v)^{-1})^{t_2}\tilde{k}_2(v) = \tilde{k}_2(v)(r_{21}^{t_1}(u+v)^{-1})^{t_1}\tilde{k}_1(u)r_{21}(v-u)$$

# Supplementary material : Integrability with boundaries

**Relation $\bar{k}(u) \leftrightarrow \tilde{k}(u)$:**

$$\tilde{k}_1(u) = \mathrm{tr}_0 \left( \bar{k}_0(-u) \left( \left( r_{01}^{t_1}(2u) \right)^{-1} \right)^{t_1} P_{01} \right)$$

$$\bar{k}_1(u) = \mathrm{tr}_0 \left( \tilde{k}_0(-u) r_{01}(-2u) P_{01} \right)$$

**Baxterisation of $k$:** With

$$\check{r}_{12} k_1 \check{r}_{12} k_1 = k_1 \check{r}_{12} k_1 \check{r}_{12}$$

and

$$k^2 = a_1 k + a_0 \text{ for some } a_0, a_1 \in \mathbb{C}$$

Then

$$k(u) = f(u) \left( k + \frac{c - a_1}{2u} \mathrm{id} \right)$$

solution of the reflection equation

# Supplementary material : k matrices of the $(p, 1)$-SSEP

$$k^{(1)} = 2\begin{pmatrix} \gamma & \gamma & \dots & \dots & \gamma \\ \gamma & \gamma & \dots & \dots & \gamma \\ \alpha_1 & \alpha_1 & \dots & \dots & \alpha_1 \\ \alpha_2 & \alpha_2 & \dots & \dots & \alpha_2 \\ \vdots & \vdots & & & \vdots \\ \alpha_p & \alpha_p & \dots & \dots & \alpha_p \end{pmatrix} \quad \text{with } 2\gamma + \sum_{i=1}^{p} \alpha_i = 1$$

$$k^{(1)}(u) = \Big(1 - u\Big)\Big(uk^{(1)} + \Big(1 - u\Big)\mathrm{Id}\Big)$$

# Supplementary material : k matrices of the $(p, 1)$-SSEP

$$\bar{k}^{(1)} = 2 \begin{pmatrix} \beta & \beta & \dots & \dots & \beta \\ \beta & \beta & \dots & \dots & \beta \\ \delta_1 & \delta_1 & \dots & \dots & \delta_1 \\ \delta_2 & \delta_2 & \dots & \dots & \delta_2 \\ \vdots & \vdots & & & \vdots \\ \delta_p & \delta_p & \dots & \dots & \delta_p \end{pmatrix} \qquad \text{with } 2\beta + \sum_{i=1}^{p} \delta_i = 1$$

$$\bar{k}^{(1)}(u) = -\Big(1+u\Big)\Big(u\bar{k}^{(1)} - \Big(1+u\Big)\mathrm{Id}\Big)$$

# Supplementary material : k matrices of the $(p, 1)$-SSEP

$$k^{(2)} = 2 \begin{pmatrix} \gamma_1 & \gamma_1 & 0 & \dots & 0 \\ \gamma_2 & \gamma_2 & 0 & \dots & 0 \\ 0 & 0 & \alpha_1 & \dots & \alpha_1 \\ 0 & 0 & \alpha_2 & \dots & \alpha_2 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \alpha_p & \dots & \alpha_p \end{pmatrix} \qquad \text{with } \gamma_1 + \gamma_2 = \sum_{i=1}^{p} \alpha_i = 1.$$

$k^{(2)}(u)$ similar expression as $k^{(1)}(u)$

# Supplementary material : k matrices of the $(p, 1)$-SSEP

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$\bar{k}^{(2)}(u)$ similar expression as $\bar{k}^{(1)}(u)$

# Supplementary material : Definition mean densities/currents

The mean density of a particle species $s \in X$ in the stationary state $S$ is

$$\langle s \rangle_i = \sum_{\tau_1,\dots,\tau_{i-1},\tau_{i+1},\dots,\tau_L \in X} S(\tau_1, \dots, \tau_{i-1}, s, \tau_{i+1}, \dots, \tau_L)\,.$$

The mean current of a particle species $s \in X$ in the stationary state $S$ is

$$\langle j_s \rangle_{i,i+1} = \langle j_s \rangle_{i\to i+1} - \langle j_s \rangle_{i\leftarrow i+1}\,,$$

with

$$\langle j_s \rangle_{i\to i+1} = \sum_{\substack{s' \in X\\ s' \neq s}} \langle s s' \rangle_{i,i+1} m(ss' \to s's), \quad \langle j_s \rangle_{i\leftarrow i+1} = \sum_{\substack{s' \in X\\ s' \neq s}} \langle s' s \rangle_{i,i+1} m(s's \to ss')\,.$$

# Supplementary material : k matrices of the (p,q)-SSEP

$$k = 2\left(\begin{array}{c|c} k' & 0 \\ \hline 0 & k'' \end{array}\right) ,$$

with

$$k' = \begin{pmatrix} \alpha_1 & \alpha_1 & \cdots & \cdots & \cdots & \cdots & \alpha_1 \\ \vdots & \vdots & & & & & \vdots \\ \vdots & \vdots & & & & & \vdots \\ \alpha_p & \alpha_p & \cdots & \cdots & \cdots & \cdots & \alpha_p \\ \gamma_1 & \gamma_1 & \cdots & \cdots & \cdots & \cdots & \gamma_1 \\ \gamma_1 & \gamma_1 & \cdots & \cdots & \cdots & \cdots & \gamma_1 \\ \vdots & \vdots & & & & & \vdots \\ \gamma_{q_1} & \gamma_{q_1} & \cdots & \cdots & \cdots & \cdots & \gamma_{q_1} \\ \gamma_{q_1} & \gamma_{q_1} & \cdots & \cdots & \cdots & \cdots & \gamma_{q_1} \end{pmatrix} , \quad k'' = \begin{pmatrix} \begin{matrix} \gamma_1^- & \gamma_1^- \\ \gamma_1^+ & \gamma_1^+ \end{matrix} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \begin{matrix} \gamma_{q_2}^- & \gamma_{q_2}^- \\ \gamma_{q_2}^+ & \gamma_{q_2}^+ \end{matrix} \end{pmatrix}$$

# Supplementary material : Boundaries of the (p,q)-SSEP

$$B = \left(\begin{array}{c|c} B' & 0 \\ \hline 0 & B'' \end{array}\right) , \quad \bar{B} = \left(\begin{array}{c|c} \bar{B}' & 0 \\ \hline 0 & \bar{B}'' \end{array}\right) ,$$

with

$$B' = \begin{pmatrix} \alpha_1 & \alpha_1 & \cdots\cdots & \alpha_1 \\ \vdots & \vdots & & \vdots \\ \alpha_p & \alpha_p & \cdots\cdots & \alpha_p \\ \gamma_1 & \gamma_1 & \cdots\cdots & \gamma_1 \\ \gamma_1 & \gamma_1 & \cdots\cdots & \gamma_1 \\ \vdots & \vdots & & \vdots \\ \gamma_{q_1} & \gamma_{q_1} & \cdots\cdots & \gamma_{q_1} \\ \gamma_{q_1} & \gamma_{q_1} & \cdots\cdots & \gamma_{q_1} \end{pmatrix} - \mathrm{Id}, \quad B'' = \begin{pmatrix} \begin{matrix} \gamma_1^- & \gamma_1^- \\ \gamma_1^+ & \gamma_1^+ \end{matrix} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \begin{matrix} \gamma_{q_2}^- & \gamma_{q_2}^- \\ \gamma_{q_2}^+ & \gamma_{q_2}^+ \end{matrix} \end{pmatrix} - \mathrm{Id},$$

and similar expressions for $\bar{B}$

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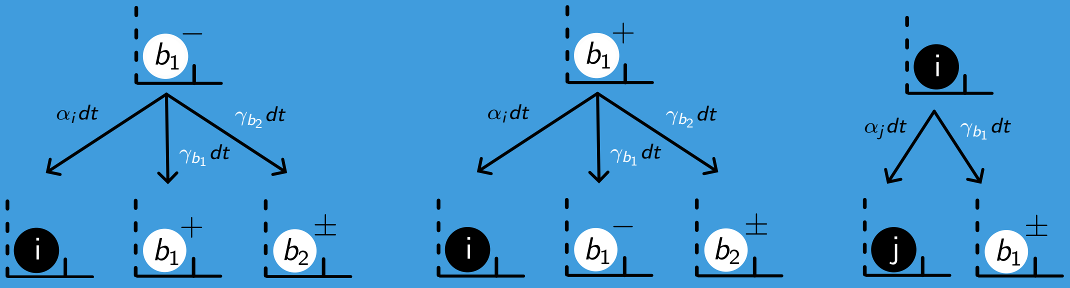


Figure: Injection/extraction rates corresponding to $B'$

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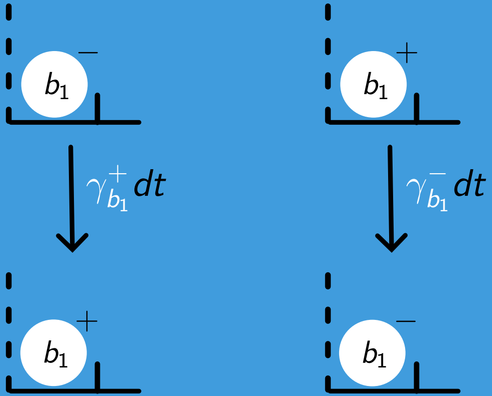


Figure: Injection/extraction rates corresponding to $B''$