ENTANGLEMENT HAMILTONIAN AFTER A LOCAL QUENCH

RAQIS26 - LAPTh, Annecy
07/09/2026

R.B. and V. Eisler, *JHEP02(2026)232*

MŰEGYETEM 1782

OUTLINE

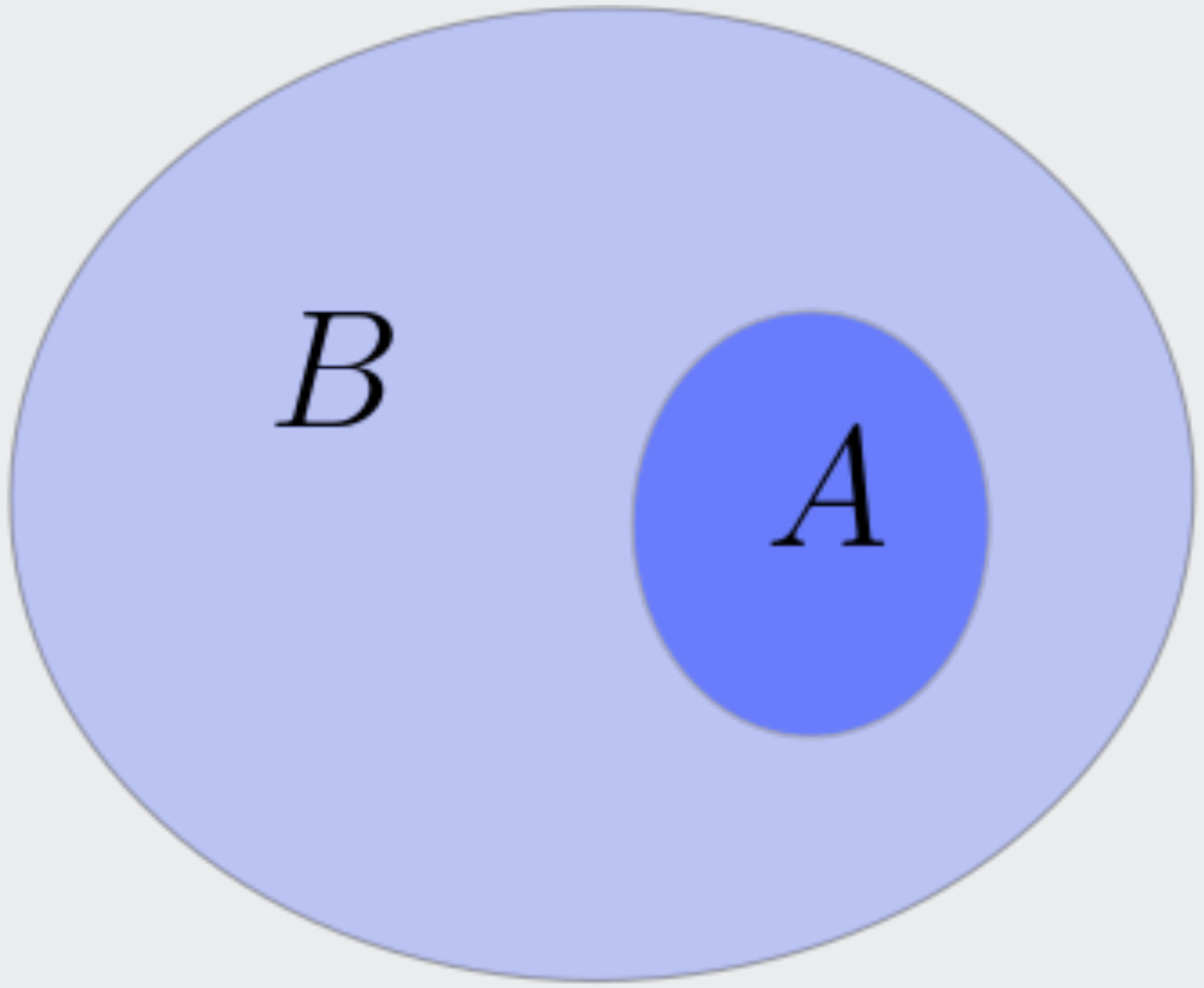
- Entanglement Hamiltonian
  - Definitions and motivations
  - EH in (1+1)D CFTs
  - EH for free fermions on a lattice
- EH after a local quench
  - CFT derivation
  - Continuum limit
- Conclusions

DEFINITIONS

ENTANGLEMENT HAMILTONIAN: DEFINITION

- Let us consider a bipartite quantum system $S = A \cup B$ in a pure state $\rho = |\psi\rangle\langle\psi|$

- Reduced density matrix of A: $\rho_A = \mathrm{Tr}_B \rho$



$$\rho_A = \frac{\mathrm{e}^{-\boxed{\mathscr{H}}}}{Z}$$

- $\mathscr{H}$ is the *entanglement hamiltonian*.

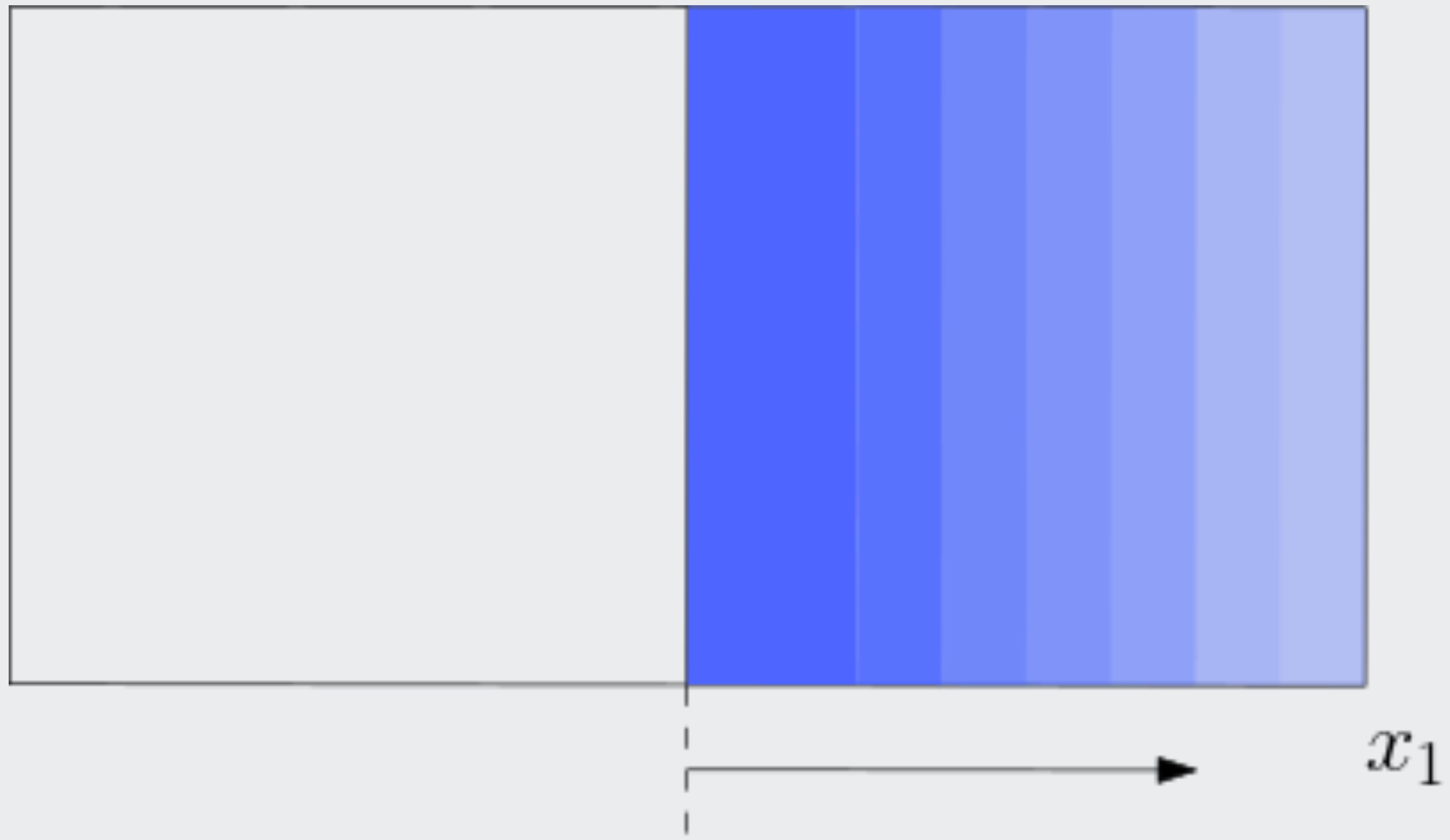
- The spectrum of $\mathscr{H}$: *entanglement spectrum*
  - Entanglement Entropy (EE): $S = -\mathrm{Tr}\rho_A \log \rho_A$

BISOGNANO-WICHMANN THEOREM

- Relativistic QFT in (D+1)dimensions, spatial coordinates $x = \{x_1, x_2, \cdots, x_D\}$, hamiltonian density $H(x)$.
- The partition A is the (right) half plane $x_1 > 0$.

- *Bisognano-Wichmann theorem:* the EH of the vacuum state is

$$\mathscr{H} = \frac{2\pi}{c} \int_{x \in A} \mathrm{d}x x_1 H(x)$$

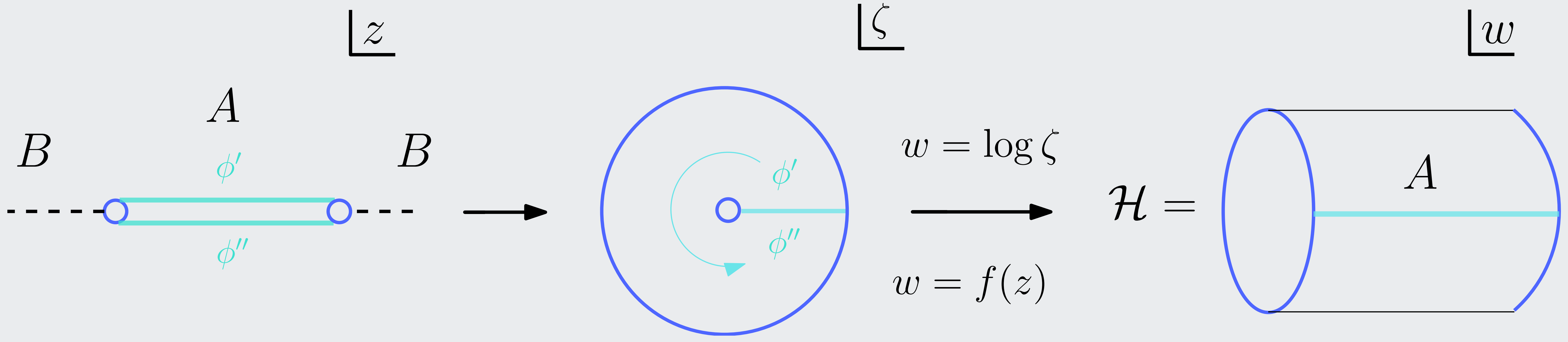


- The RDM has the form of a thermal state wrt to the original Hamiltonian, with a position-dependent inverse temperature $\frac{2\pi}{c} x_1$.

J. J. Bisognano and E. H. Wichmann, *J. Math. Phys.* **16**, *985–1007 (1975)*
J. J. Bisognano and E. H. Wichmann, *J. Math. Phys.* **17**, *303–321 (1976)*

MAPPING TO THE ANNULUS

- (1+1)D CFTs (<u>equilibrium case</u>)

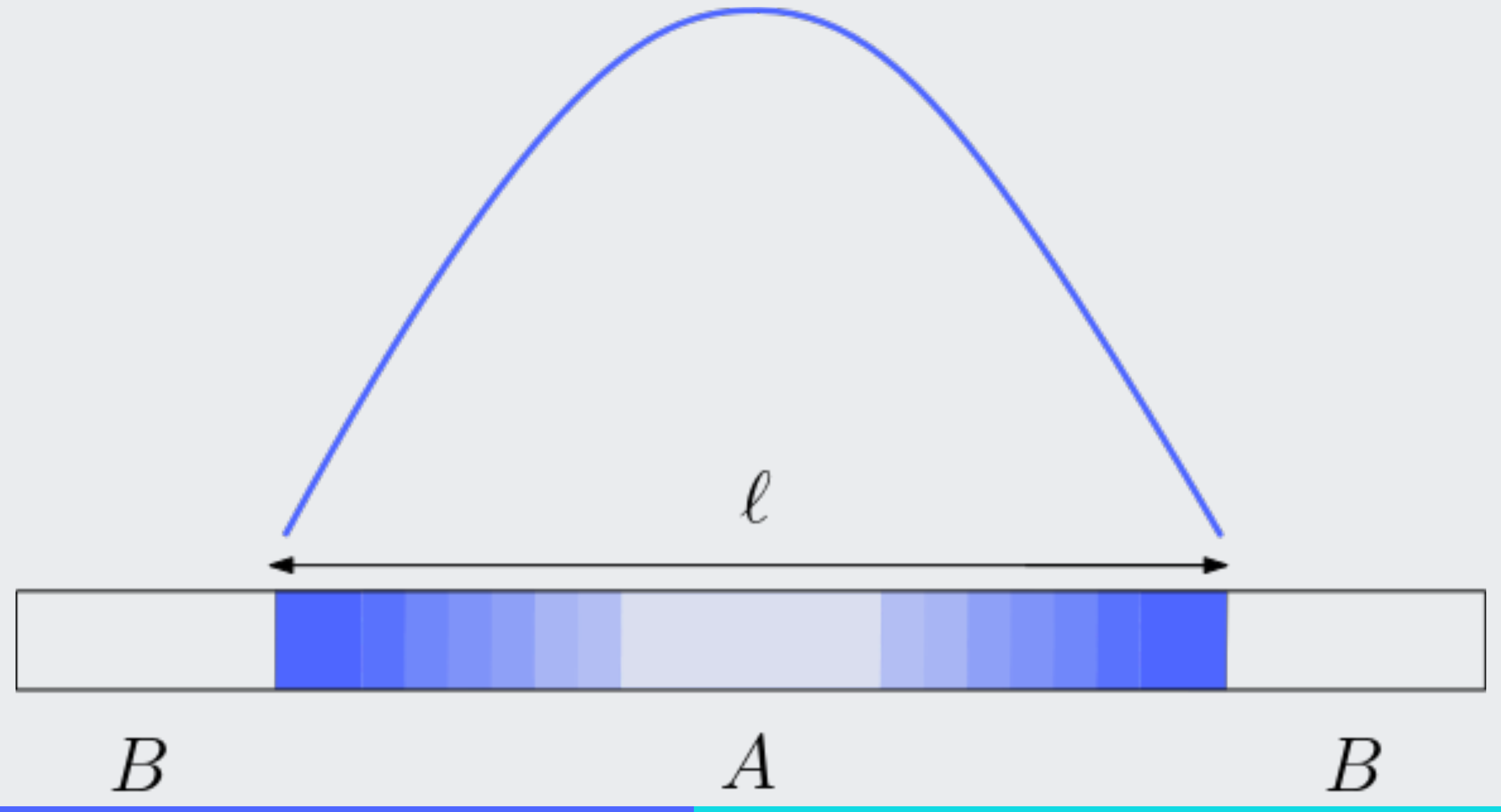


- The EH can be written as a local integral over the energy-momentum tensor

$$\mathscr{H} = \frac{2\pi}{c} \int_A \beta(x) T_{00}(x) \mathrm{d}x$$

- <u>*Example:*</u> finite interval $A = [0, \ell]$

$$\beta(x) \propto \frac{x}{\ell}\left(1 - \frac{x}{\ell}\right)$$



J. Cardy and E. Tonni, *J. Stat. Mech. 123103 (2016)*

MAPPING TO THE ANNULUS

- In the final strip geometry, the EH is given by the integral of the generator of translations in the imaginary time direction $v = \mathrm{Im} w$.

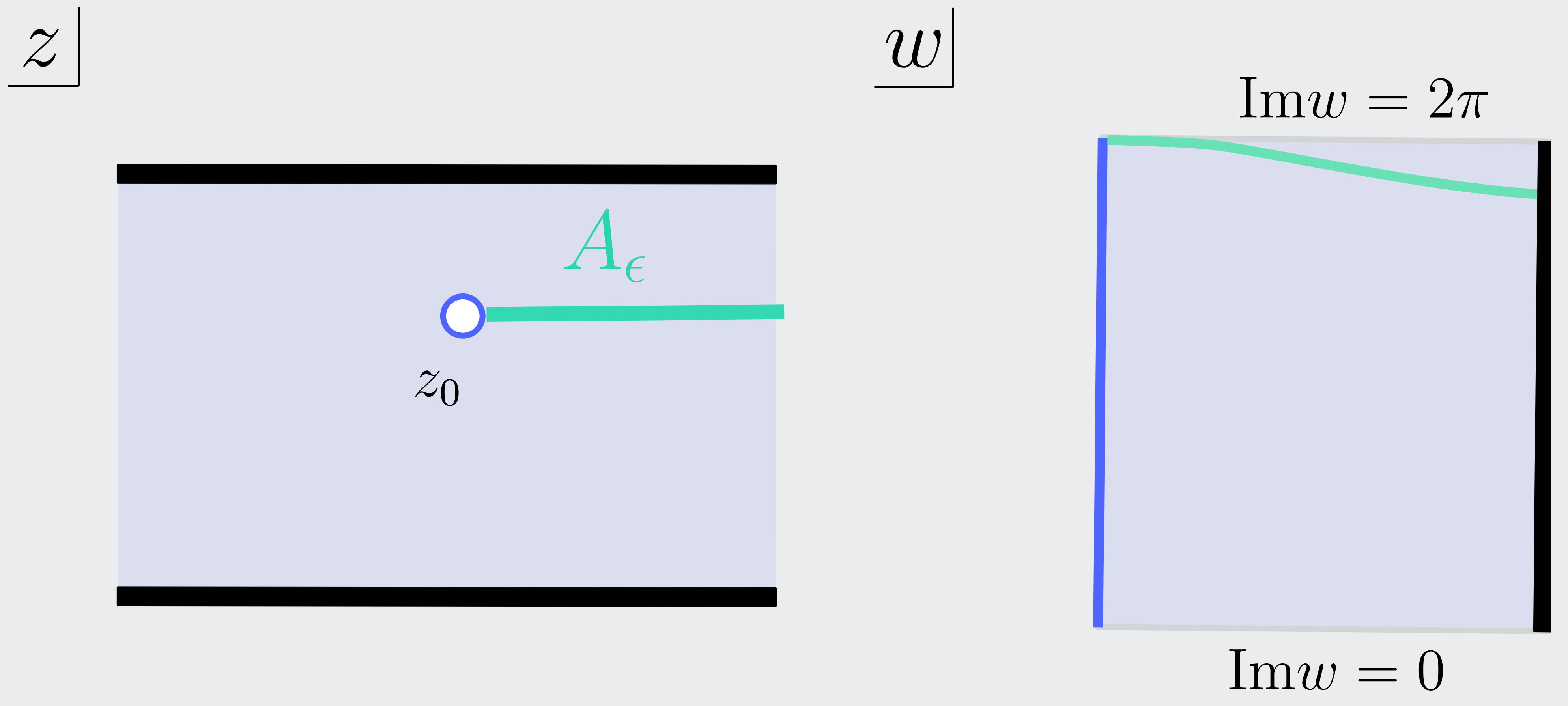
$$\mathscr{H} = -\int_{v=cnst} T_{vv} \mathrm{d}u = \int_{f(C)} T(w)\mathrm{d}w + \int_{\bar{f}(C)} \bar{T}(\bar{w})\mathrm{d}\bar{w}$$

Back to original Euclidean spacetime

$$\mathscr{H} = \int_C \frac{T(z)}{f'(z)}\mathrm{d}z + \int_{\bar{C}} \frac{\bar{T}(\bar{z})}{\bar{f}'(\bar{z})}\mathrm{d}\bar{z}$$

- (1+1)D CFTs <u>out of equilibrium:</u>

$$z \quad \rightarrow \quad w = u + \mathrm{i}v = f(z)$$

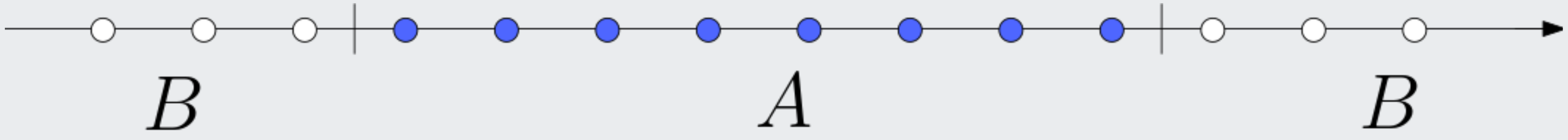


J. Cardy and E. Tonni, *J. Stat. Mech.* 123103 (2016)

EH ON THE LATTICE: FREE FERMIONS

- Free fermions hopping on a lattice: $$\hat{H} = -\frac{1}{2}\sum_n t_n (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n)$$
- Subsystem $A = [1, \cdots, L_A]$

$$\rho_A = \frac{\mathrm{e}^{-\mathscr{H}}}{Z}, \qquad \mathscr{H} = \sum_{i,j \in A}^{L_A} H_{ij} c_i^\dagger c_j = \sum_k^{L_A} \varepsilon_k \tilde{c}_k^\dagger \tilde{c}_k$$



I. Peschel and V. Eisler, J. Phys. A **42** 504003 (2009)

EH ON THE LATTICE: FREE FERMIONS

- Free fermions hopping on a lattice: $\hat{H} = -\frac{1}{2}\sum_n t_n(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n)$
- Subsystem $A = [1, \cdots, L_A]$

$$\rho_A = \frac{e^{-\mathcal{H}}}{Z}, \qquad \mathcal{H} = \sum_{i,j\in A}^{L_A} \boxed{H_{ij}} c_i^\dagger c_j = \sum_k^{L_A} \varepsilon_k \tilde{c}_k^\dagger \tilde{c}_k$$

- Correlation matrix restricted to $A = [1, \cdots, L_A]$: $(C_A)_{i,j} = \mathrm{Tr}[\rho_A c_i^\dagger c_j] = \langle c_i^\dagger c_j \rangle, \qquad i,j \in A$

$$\boxed{H} = \ln[(\mathbb{1} - C_A)/C_A]$$

$$\varepsilon_k = \ln[(1 - \zeta_k)/\zeta_k]$$ eigenvalues

I. Peschel and V. Eisler, *J. Phys. A* ***42*** *504003 (2009)*

BW ON THE LATTICE?

- The lattice EH cannot be obtained as a simple discretization of the continuum result.

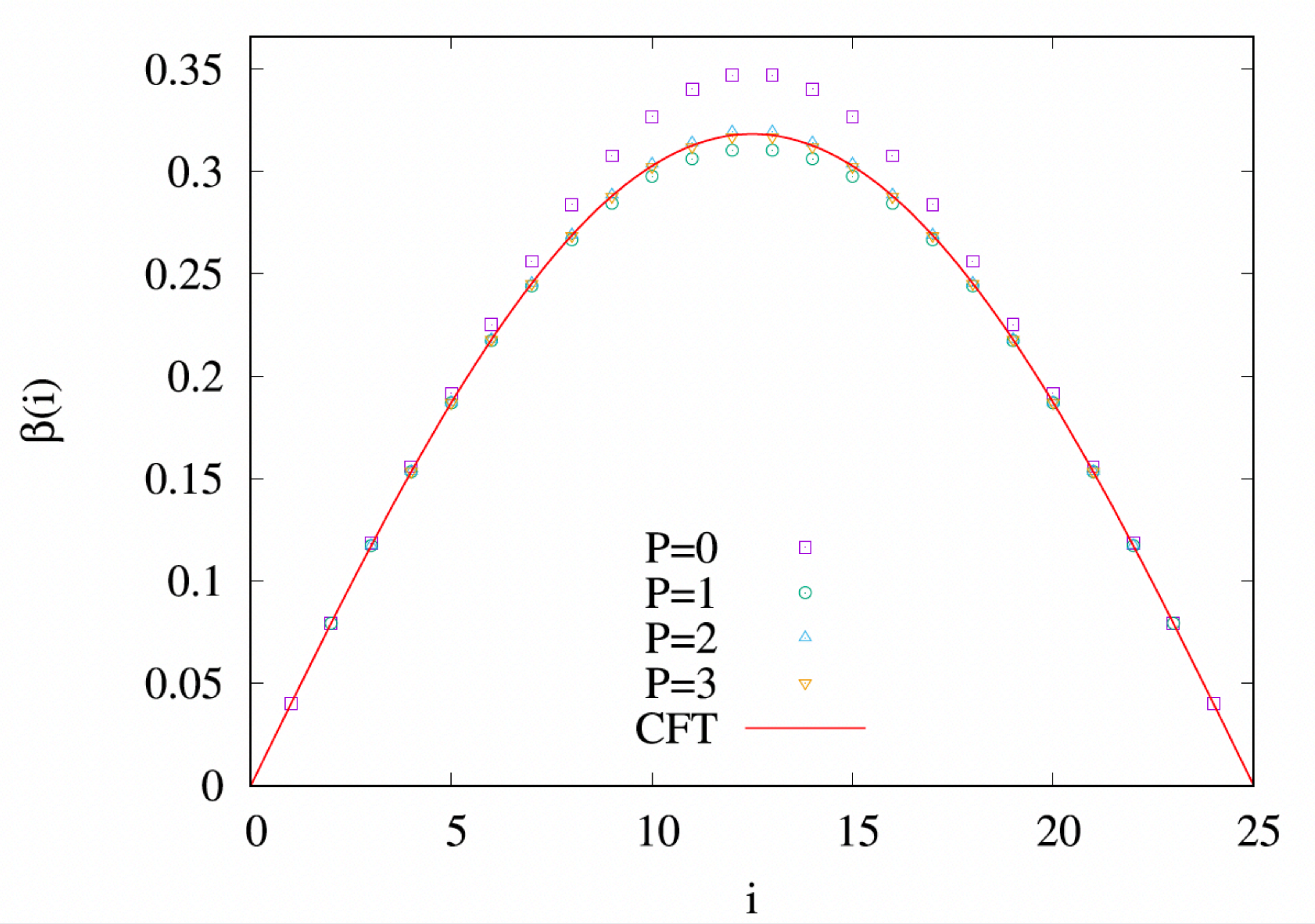
- Results:

  - The CFT result is recovered from the lattice EH including all the long range hopping terms in the continuum limit

V. Eisler, E. Tonni and I. Peschel, *J. Stat. Mech. 073V101 (2019)*

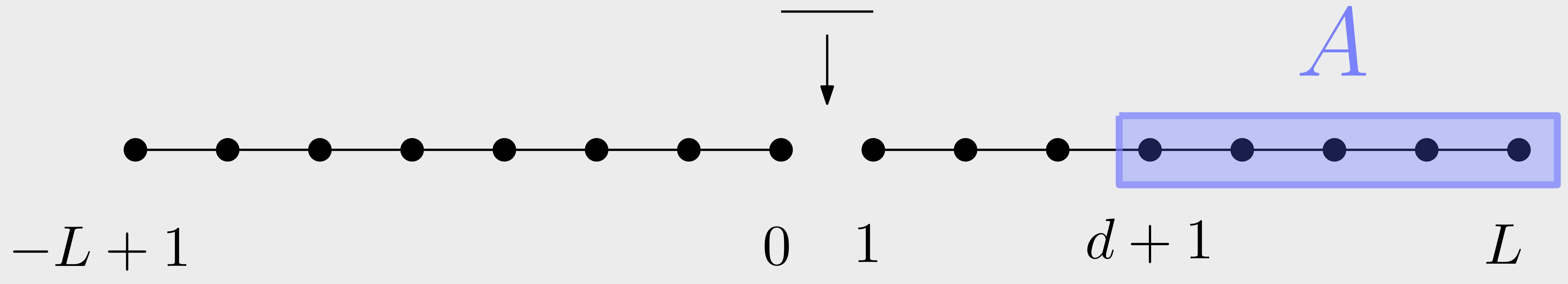
  - *Example:* finite interval $A = [1, \cdots, L]$ in a finite chain

$$\beta(x) = \frac{1}{\pi L} \sum_{p=0}^{P} (-1)^p (2p+1) H_{i-p,i+p+1}$$



EH AFTER A LOCAL QUENCH

THE MODEL



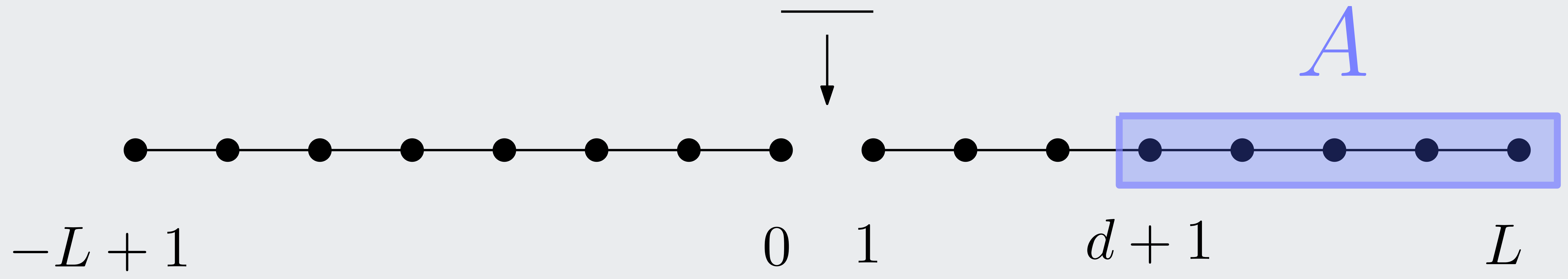
- Two half-chains are in their ground states at half filling, initially disconnected

$$\hat{H}_0 = \hat{H}_\ell + \hat{H}_r = -\frac{1}{2}\sum_{n=-L+1}^{-1}\left(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n\right) - \frac{1}{2}\sum_{n=1}^{L-1}\left(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n\right)$$

- $t=0$: two halves joined → The system evolves under the action of the homogeneous Hamiltonian

$$\hat{H} = -\frac{1}{2}\sum_{n=-L+1}^{L-1}\left(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n\right).$$

THE MODEL



- Two point correlation

$$C_{m,n}(t) = \langle \psi(t) | c_m^\dagger c_n | \psi(t) \rangle, \qquad | \psi(t) \rangle = \mathrm{e}^{-\mathrm{i}\hat{H}t} | \psi_0 \rangle_\ell \otimes | \psi_0 \rangle_r$$

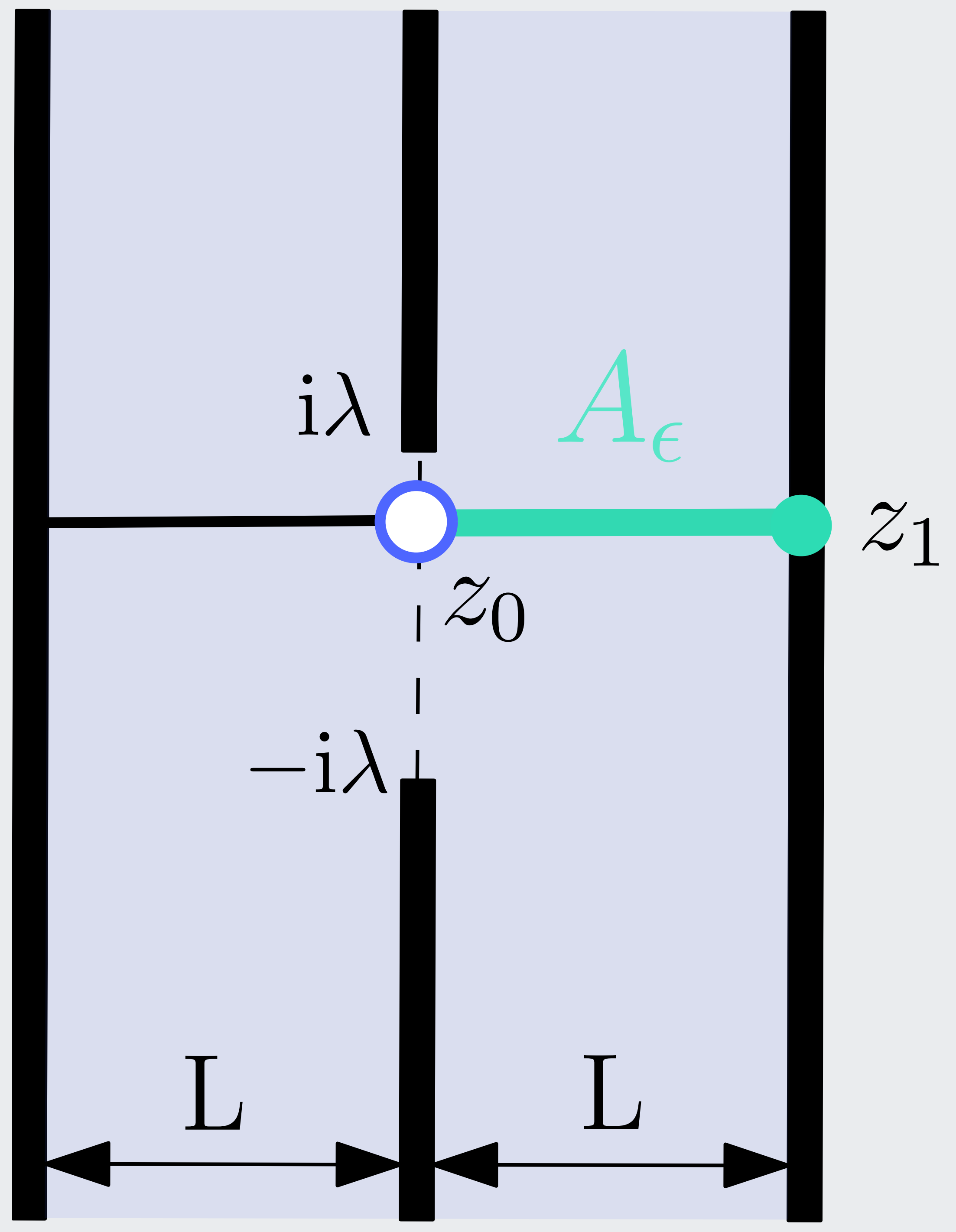
- The EH associated to a subsystem $A = [d+1, L]$ is

$$H(t) = \ln \left[\frac{1 - C_A(t)}{C_A(t)} \right].$$

$$\rho_A(t) = \frac{\mathrm{e}^{-\mathscr{H}(t)}}{\mathcal{Z}}, \qquad \mathscr{H}(t) = \sum_k \boxed{H_{i,j}(t)} c_i^\dagger c_j,$$

CFT FORMULATION

- Path-integral representation of the time-evolved density matrix $\rho(t)$

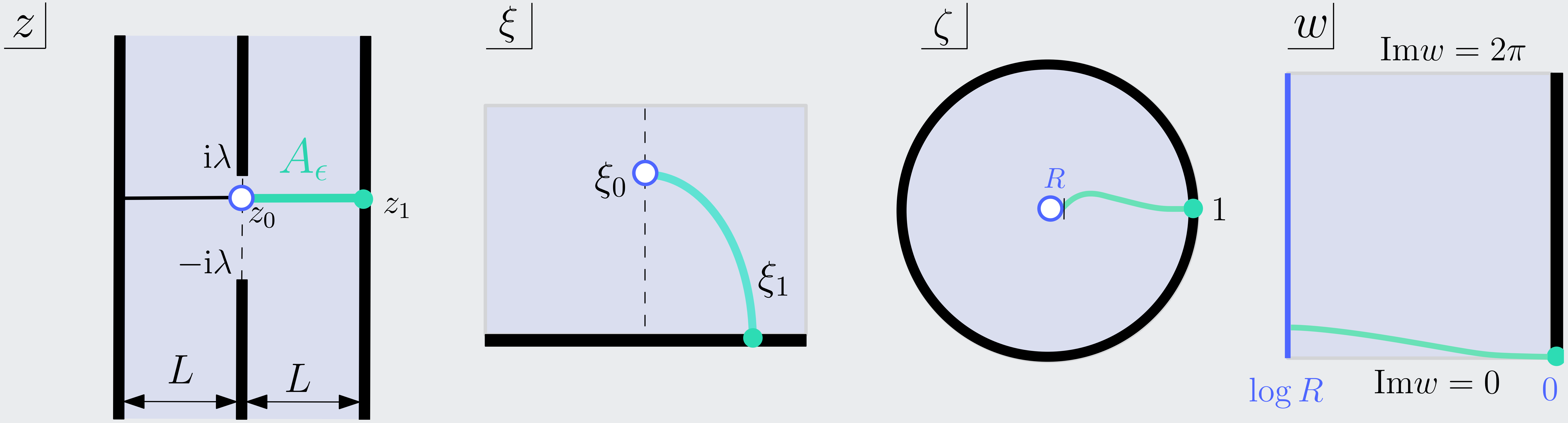


$$\langle\varphi''|\rho(t)|\varphi'\rangle = Z^{-1}\langle\varphi''|\mathrm{e}^{-\mathrm{i}t\hat{H}}\mathrm{e}^{-\lambda\hat{H}}\boxed{|\varphi_\ell\otimes\varphi_r\rangle}\langle\varphi_\ell\otimes\varphi_r|\mathrm{e}^{\mathrm{i}t\hat{H}}\mathrm{e}^{-\lambda\hat{H}}|\varphi'\rangle$$

Boundary states: $$|\varphi_\ell\otimes\varphi_r\rangle = \lim_{\beta\to\infty}\frac{\mathrm{e}^{-\beta(\hat{H}_\ell+\hat{H}_r)}\mathrm{e}^{\beta E_0}|s\rangle}{\langle\varphi_\ell\otimes\varphi_r|s\rangle}.$$

- Reduced density matrix:
  - Cut along $C = \{z = x + \mathrm{i}\tau, x_0 < x < L\}$
  - Regularisation disk around entangling point

CONFORMAL MAPPING TO THE ANNULUS



- Conformal mapping to the annulus

$$\xi(z) = \mathrm{i}\sqrt{\frac{\sin\left(\frac{\pi}{2L}(\mathrm{i}\lambda + z)\right)}{\sin\left(\frac{\pi}{2L}(\mathrm{i}\lambda - z)\right)}} \quad \rightarrow \quad \zeta(z) = \left(\frac{\xi_1 - \bar{\xi}_0}{\xi_1 - \xi_0}\right)\frac{\xi(z) - \xi_0}{\xi(z) - \bar{\xi}_0} \quad \rightarrow \quad w(z) = \log\left[\left(\frac{\xi_1 - \bar{\xi}_0}{\xi_1 - \xi_0}\right)\frac{\xi(z) - \xi_0}{\xi(z) - \bar{\xi}_0}\right]$$

CFT RESULT FOR THE EH

- The EH in the strip geometry is

$$\mathcal{H} = 2\pi \int_{w(C_\epsilon)} T(w)\mathrm{d}w + 2\pi \int_{\overline{w(C_\epsilon)}} \bar{T}(\bar{w})\mathrm{d}\bar{w}\,.$$

$$\beta(z) = \frac{2\pi}{w'(z)}, \quad \bar{\beta}(\bar{z}) = \frac{2\pi}{\overline{w'(z)}}$$

- The EH in the original double-pant geometry is

$$\mathcal{H} = \int_{C_\epsilon} \beta(z)T(z)\mathrm{d}z + \int_{\bar{C}_\epsilon} \bar{\beta}(\bar{z})\bar{T}(\bar{z})\mathrm{d}\bar{z},$$

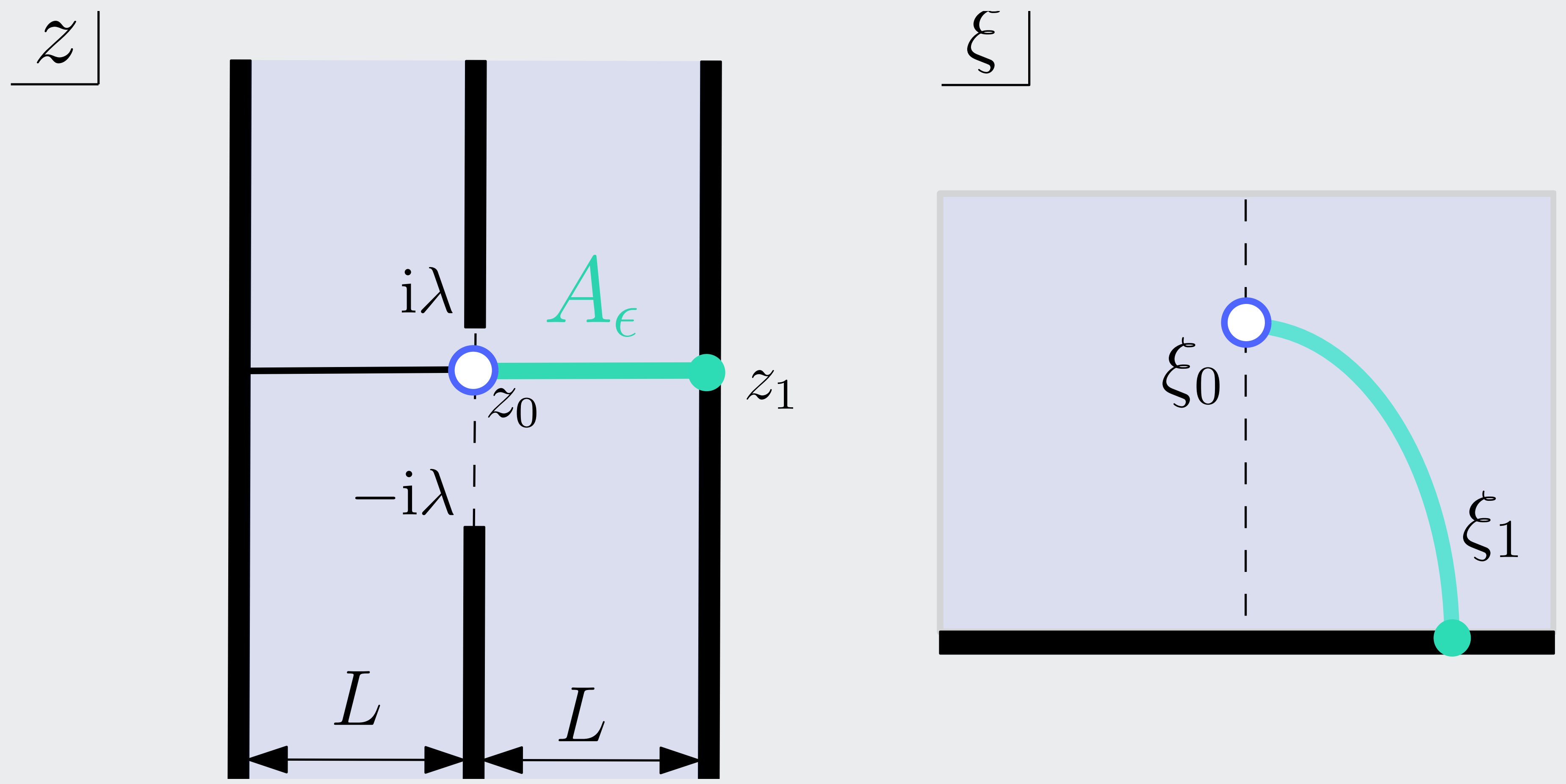
$$\beta(z) = 2\pi \frac{(\xi(z)-\xi_0)(\xi(z)-\bar{\xi}_0)}{(\xi_0-\bar{\xi}_0)\xi'(z)}\,.$$

- Analytical continuation $\tau \to \mathrm{i}t$

$$\mathcal{H} = \int_{A_\epsilon} \beta(x,t)T(x-t)\mathrm{d}x + \int_{A_\epsilon} \bar{\beta}(x,t)\bar{T}(x+t)\mathrm{d}x\,.$$

EH FOR THE HALF CHAIN

- Half-chain partition $x_0 = 0 \quad \rightarrow \quad z_0 = \mathrm{i}\tau$



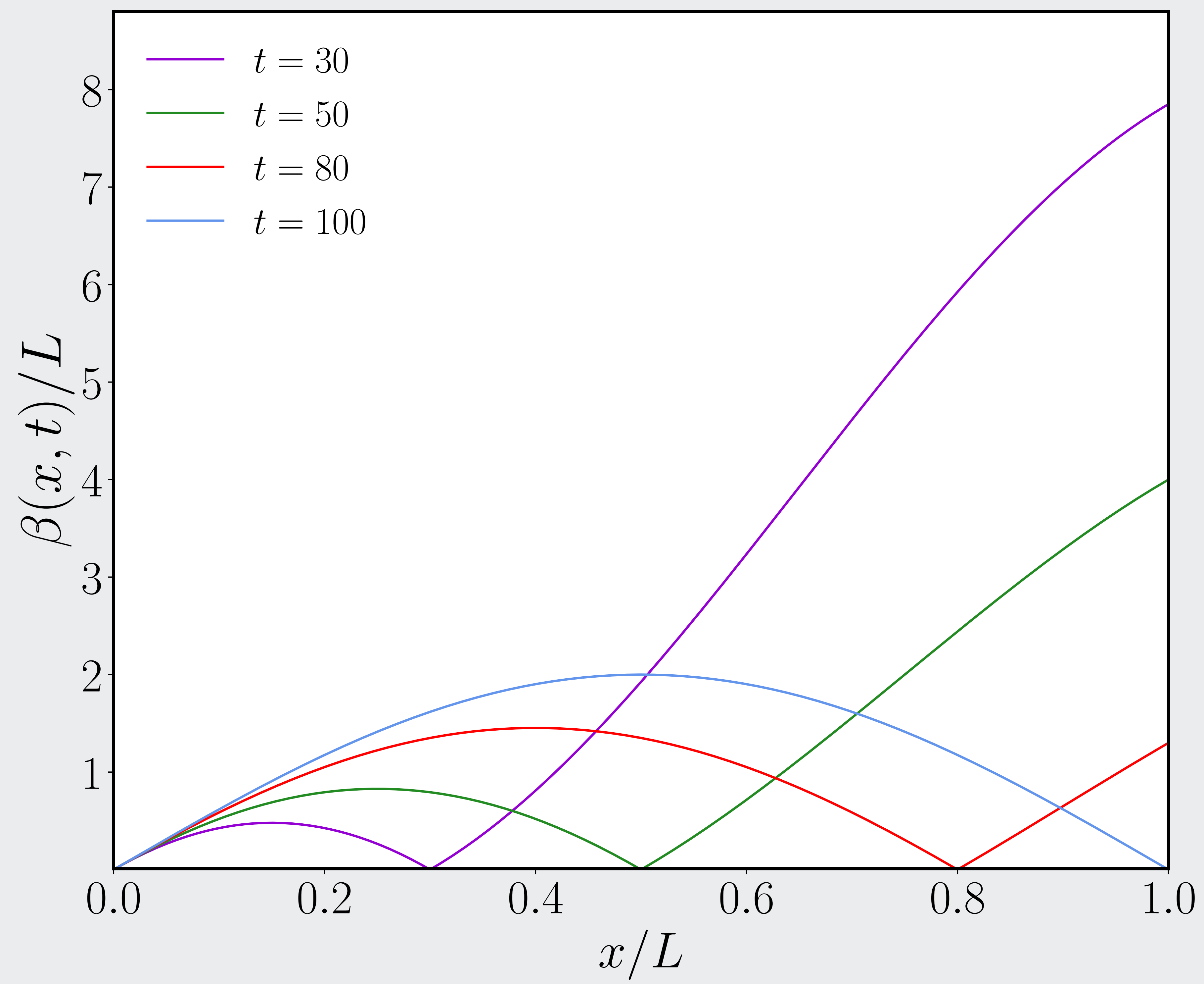
$$\xi_0 = \xi(\mathrm{i}\tau) = \mathrm{i}\sqrt{\frac{\sinh \frac{\pi}{2L}(\lambda+\tau)}{\sinh \frac{\pi}{2L}(\lambda-\tau)}},$$

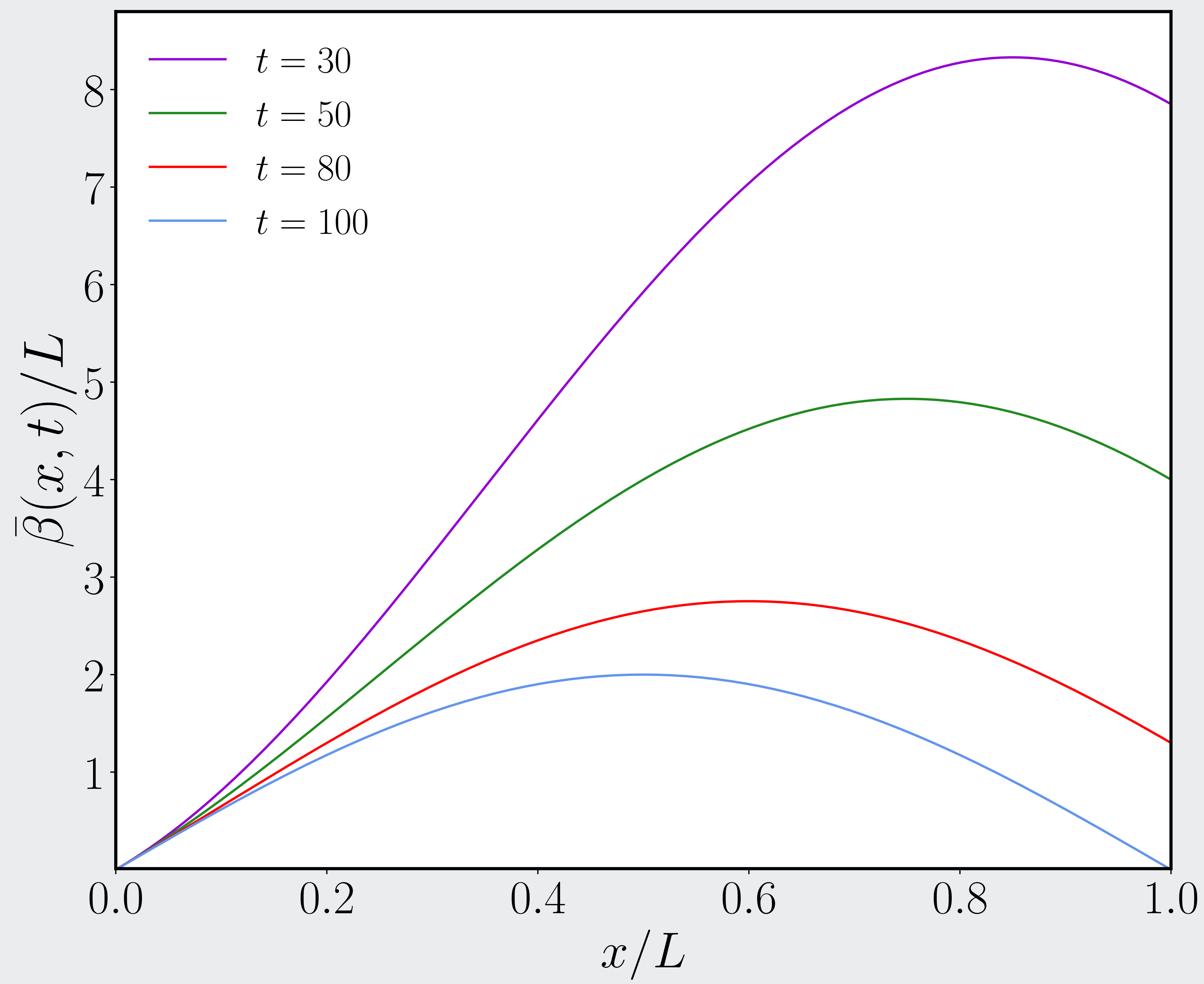
$$\beta(z) = 4L \sin\left[\frac{\pi}{2L}(z-\mathrm{i}\tau)\right]\sqrt{\frac{\cosh\frac{\pi\lambda}{L} - \cos\frac{\pi z}{L}}{\cosh\frac{\pi\lambda}{L} - \cosh\frac{\pi\tau}{L}}}$$

- After analytic continuation, and in the limit $t \gg \lambda$,

$$t < L: \qquad \beta(x,t) = 4L\left|\frac{\sin\frac{\pi x}{2L}\sin\frac{\pi}{2L}(x-t)}{\sin\frac{\pi t}{2L}}\right|, \qquad \bar{\beta}(x,t) = 4L\left|\frac{\sin\frac{\pi x}{2L}\sin\frac{\pi}{2L}(x+t)}{\sin\frac{\pi t}{2L}}\right|$$

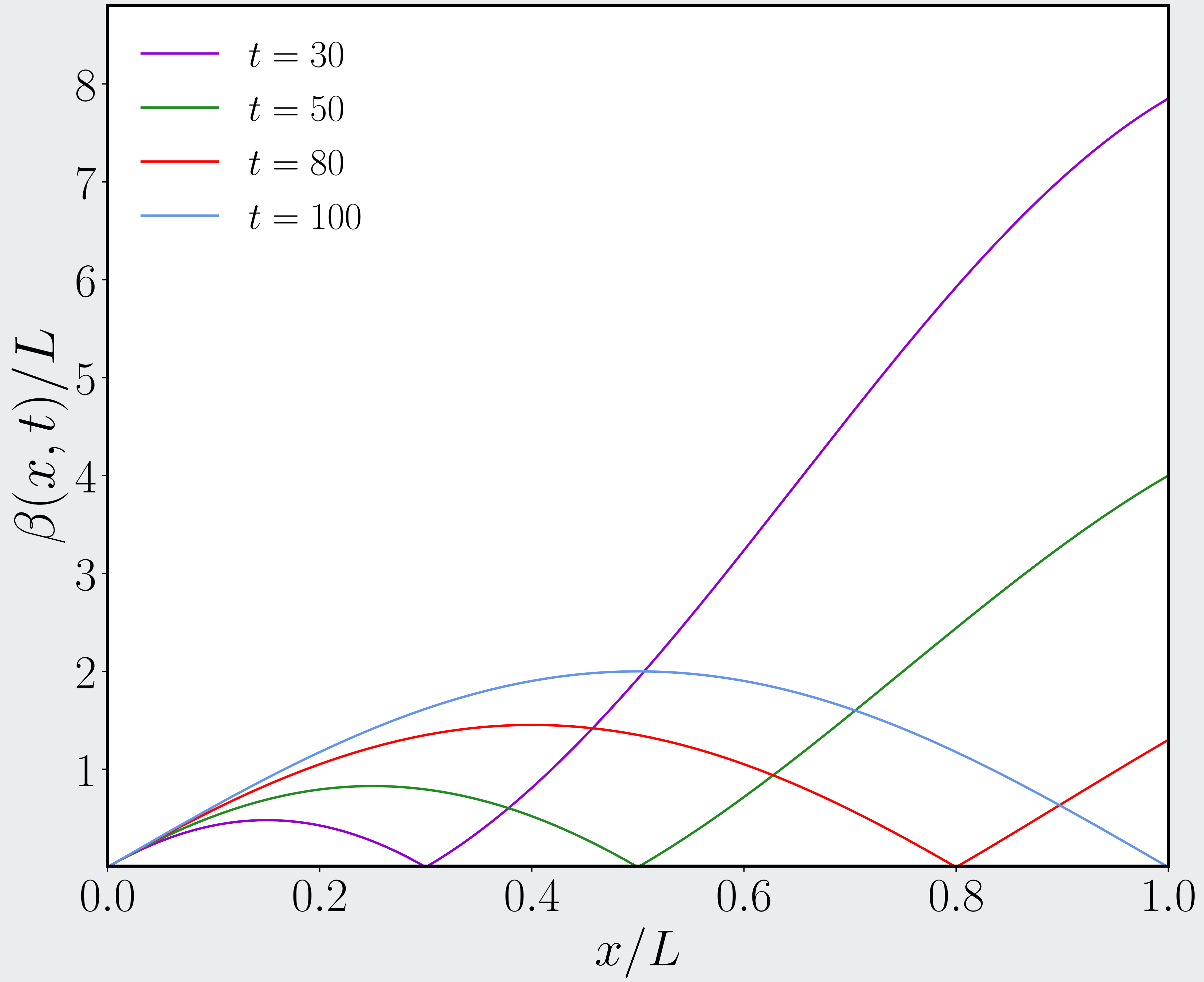
EH FOR THE HALF CHAIN

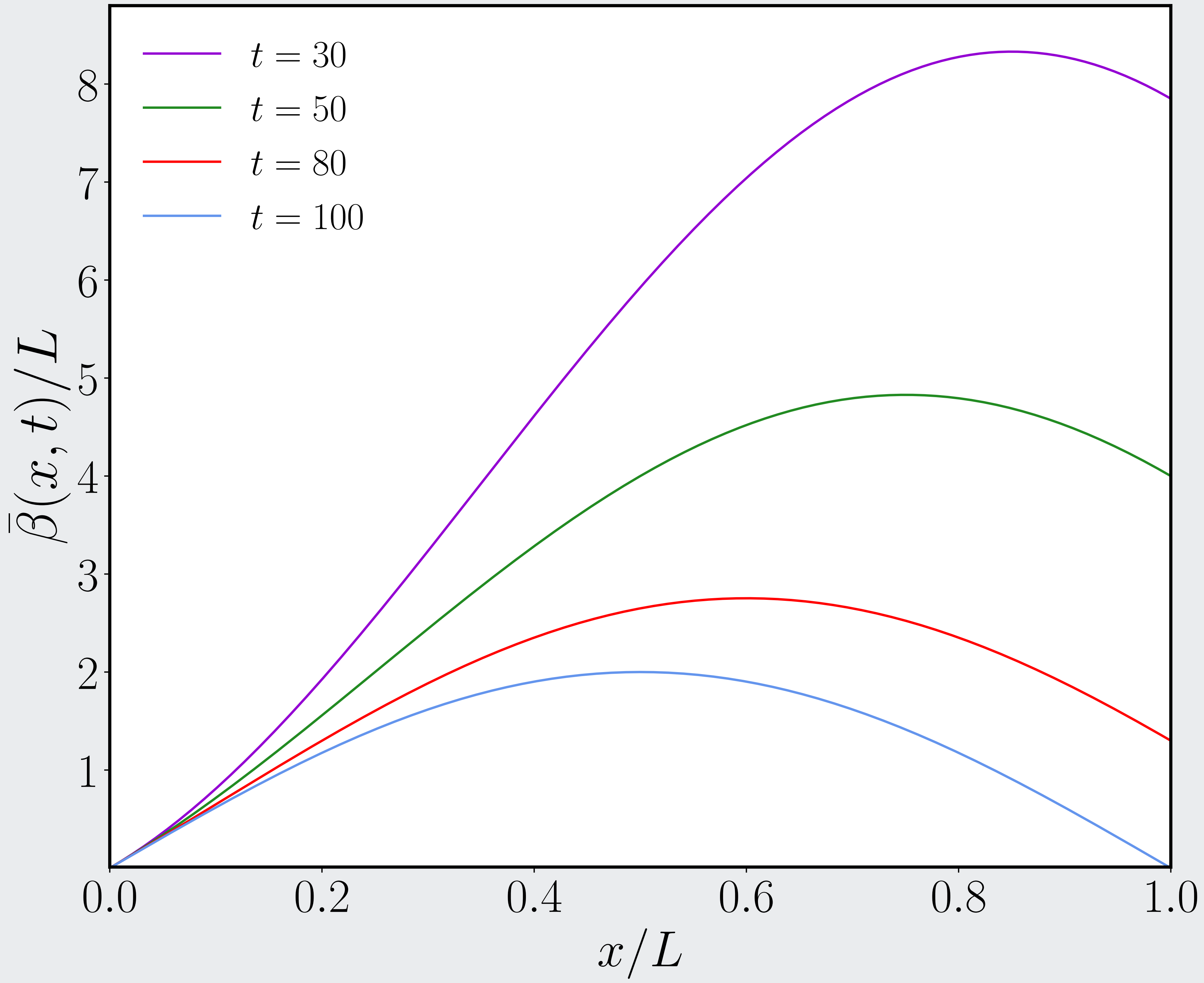




$$t < L: \qquad \beta(x,t) = 4L \left| \frac{\sin \frac{\pi x}{2L} \sin \frac{\pi}{2L}(x-t)}{\sin \frac{\pi t}{2L}} \right|, \qquad \bar{\beta}(x,t) = 4L \left| \frac{\sin \frac{\pi x}{2L} \sin \frac{\pi}{2L}(x+t)}{\sin \frac{\pi t}{2L}} \right|$$

EH FOR THE HALF CHAIN

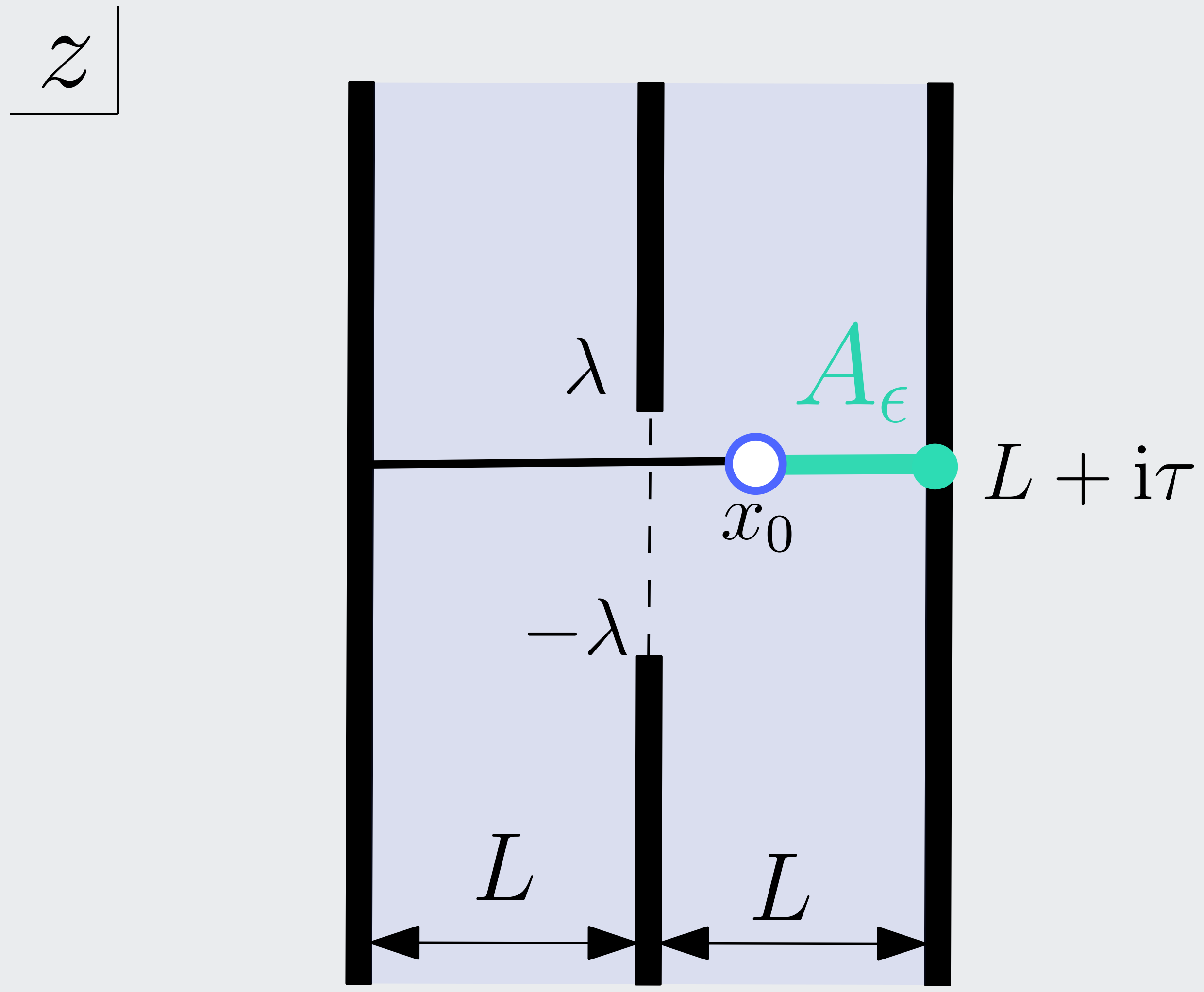


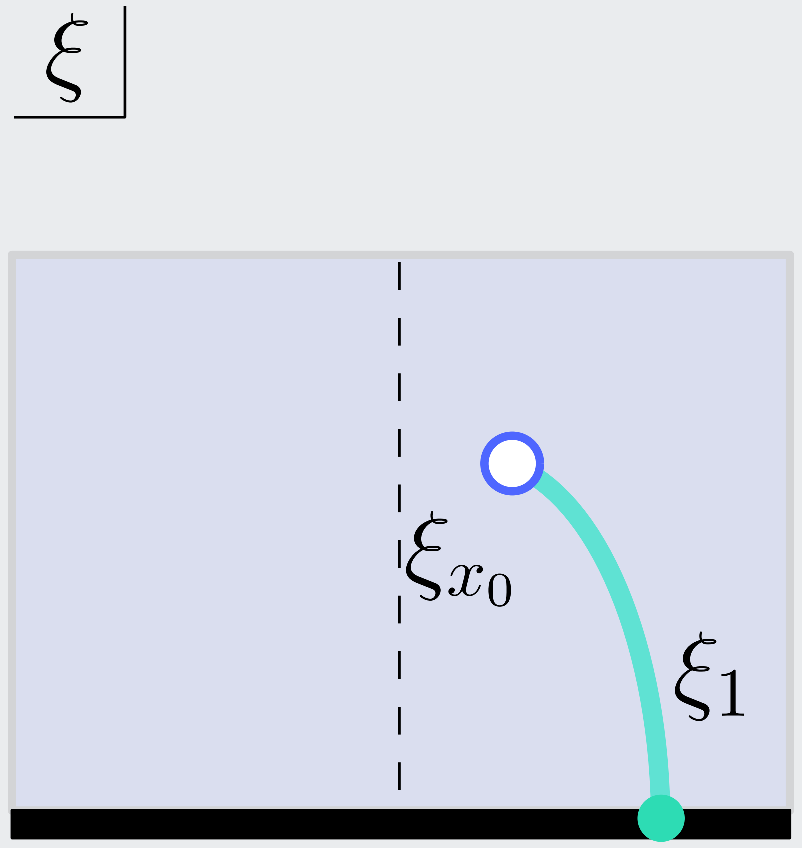


- After reflection at $t = L$, we have the symmetry property

$$L < t < 2L: \qquad \bar{\beta}(x,t) = \beta(x, 2L - t), \qquad \beta(x,t) = \bar{\beta}(x, 2L - t)$$

DECENTERED INTERVAL

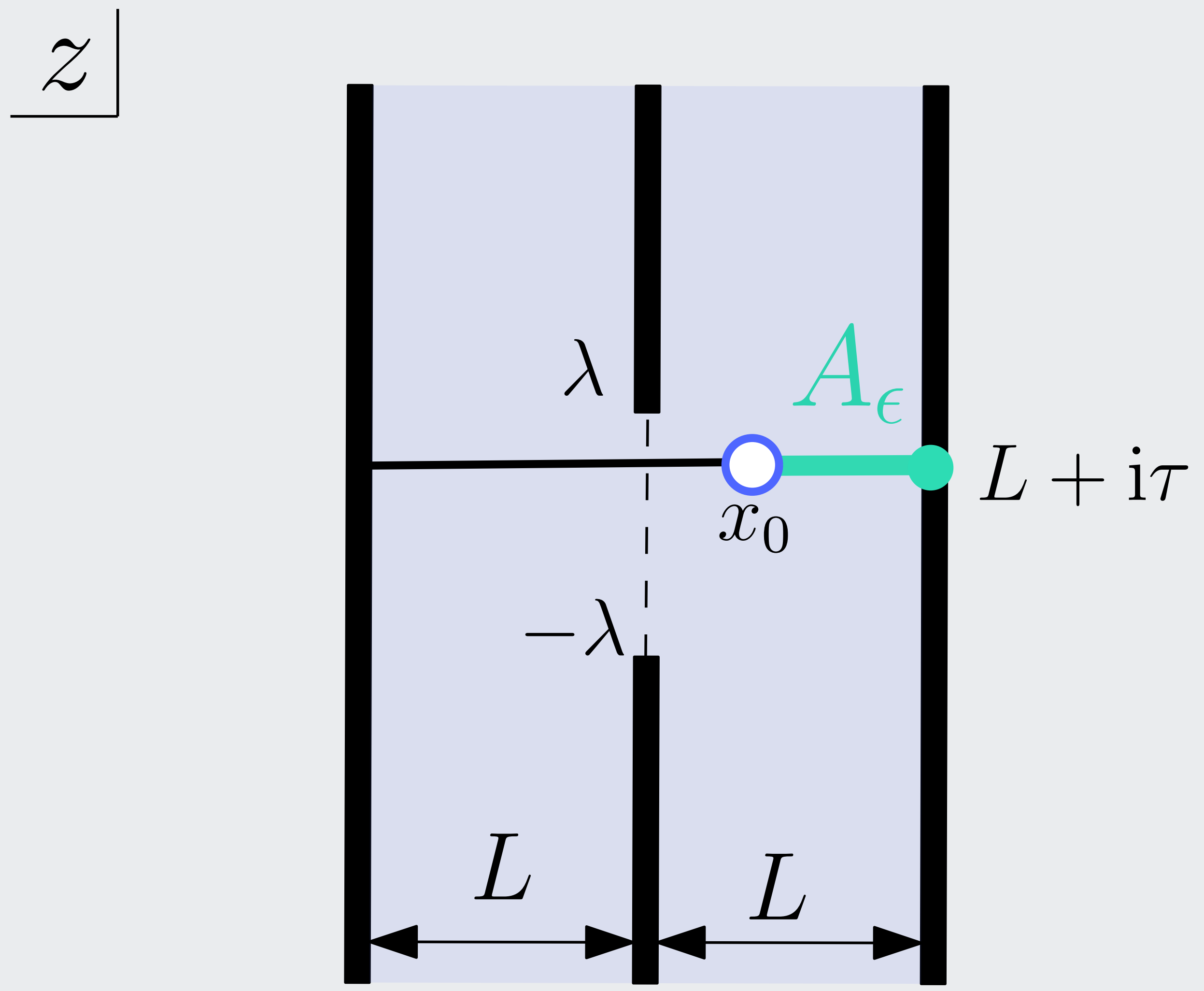


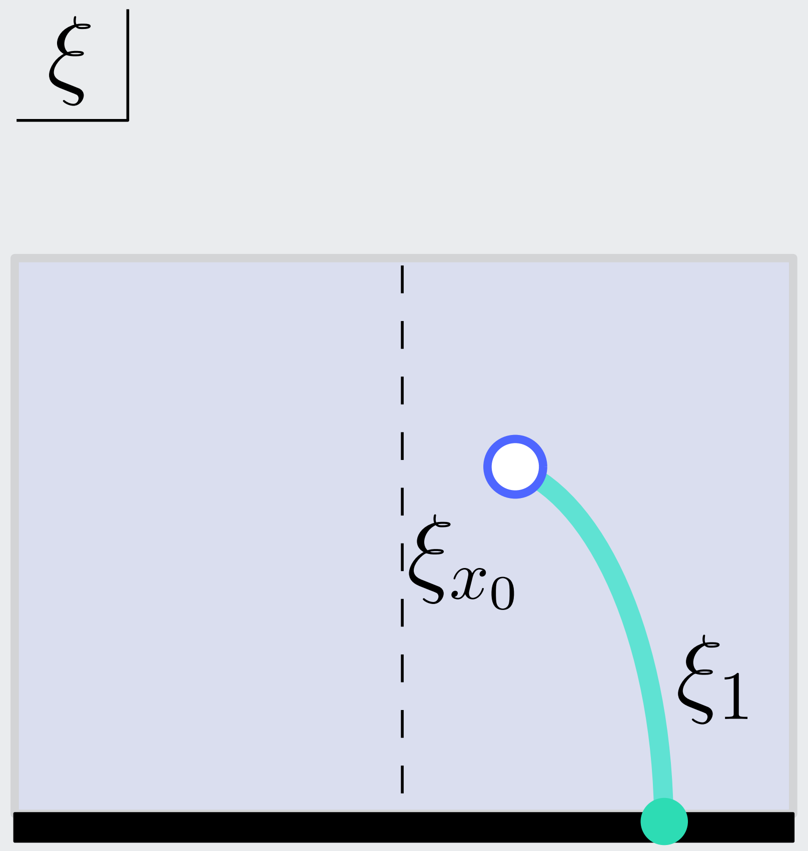


- The boundary point x_0 does not coincide with the quench location $x = 0$.

$$\beta(x,t) = \begin{cases} 2L\dfrac{\cos\frac{\pi x_0}{L} - \cos\frac{\pi x}{L}}{\sin\frac{\pi x_0}{L}} \\ 4L\dfrac{\sin\frac{\pi}{2L}(x-x_0)\sin\frac{\pi}{2L}(t-x)}{\sin\frac{\pi}{2L}(t-x_0)} \\ 4L\dfrac{\sin\frac{\pi}{2L}(x+x_0)\sin\frac{\pi}{2L}(x-t)}{\sin\frac{\pi}{2L}(t+x_0)} \end{cases} \quad \bar\beta(x,t) = \begin{cases} 2L\dfrac{\cos\frac{\pi x_0}{L} - \cos\frac{\pi x}{L}}{\sin\frac{\pi x_0}{L}} & t < x_0 < x \\ 4L\dfrac{\sin\frac{\pi}{2L}(x-x_0)\sin\frac{\pi}{2L}(x+t)}{\sin\frac{\pi}{2L}(t+x_0)} & x_0 < x < t \\ 4L\dfrac{\sin\frac{\pi}{2L}(x-x_0)\sin\frac{\pi}{2L}(x+t)}{\sin\frac{\pi}{2L}(t+x_0)} & x_0 < t < x \end{cases}$$

DECENTERED INTERVAL





- The boundary point x_0 does not coincide with the quench location $x = 0$.

Equilibrium result

E. Tonni, J. Rodriguez-Laguna, and G. Sierra, *J. Stat. Mech. (2018) 043105*

$$\beta(x,t) = \begin{cases} \boxed{2L \dfrac{\cos \frac{\pi x_0}{L} - \cos \frac{\pi x}{L}}{\sin \frac{\pi x_0}{L}}} \\ 4L \dfrac{\sin \frac{\pi}{2L}(x - x_0) \sin \frac{\pi}{2L}(t - x)}{\sin \frac{\pi}{2L}(t - x_0)} \\ 4L \dfrac{\sin \frac{\pi}{2L}(x + x_0) \sin \frac{\pi}{2L}(x - t)}{\sin \frac{\pi}{2L}(t + x_0)} \end{cases} \qquad \bar{\beta}(x,t) = \begin{cases} \boxed{2L \dfrac{\cos \frac{\pi x_0}{L} - \cos \frac{\pi x}{L}}{\sin \frac{\pi x_0}{L}}} & t < x_0 < x \\ 4L \dfrac{\sin \frac{\pi}{2L}(x - x_0) \sin \frac{\pi}{2L}(x + t)}{\sin \frac{\pi}{2L}(t + x_0)} & x_0 < x < t \\ 4L \dfrac{\sin \frac{\pi}{2L}(x - x_0) \sin \frac{\pi}{2L}(x + t)}{\sin \frac{\pi}{2L}(t + x_0)} & x_0 < t < x \end{cases}$$

EH ON THE LATTICE: CONTINUUM LIMIT

- We write the lattice EH as

$$\mathcal{H} = \sum_j t_0(j) c_j^\dagger c_j + \sum_j \sum_{r\geq 1} \left[t_r(j+r/2)\left(c_j^\dagger c_{j+r} + c_{j+r}^\dagger c_j\right) + \mathrm{i} s_r(j+r/2)\left(c_j^\dagger c_{j+r} - c_{j+r}^\dagger c_j\right)\right].$$

where

$$t_r(j+r/2) = \mathrm{Re}\, H_{j,j+r}(t), \qquad s_r(j+r/2) = \mathrm{Im}\, H_{j,j+r}(t),$$

- Continuum limit: $aj \to x$

Fermi points: $\pm a q_F$

$$c_j \quad \to \quad \sqrt{a}\left(\mathrm{e}^{\mathrm{i}\boxed{q_F}x}\psi(z) + \mathrm{e}^{-\mathrm{i}\boxed{q_F}x}\bar{\psi}(\bar{z})\right),$$

$$\begin{cases} \psi(z), \bar{\psi}(\bar{z}) & \text{R/L movers} \\ z = x - t, & \bar{z} = x + t \end{cases}$$

EH ON THE LATTICE: CONTINUUM LIMIT

- Continuum limit of the EH

$$\mathscr{H} = \int_A \mathrm{d}x \left[\beta_0(x,t)T_{00}(z,\bar{z}) - \beta_1(x,t)T_{01}(z,\bar{z}) + \mu_0(x,t)\rho(z,\bar{z}) - \mu_1(x,t)j(z,\bar{z})\right],$$

  - density and current of the $U(1)$ charge

$$\rho(z,\bar{z}) = \psi^\dagger(z)\psi(z) + \bar{\psi}^\dagger(\bar{z})\bar{\psi}(\bar{z}), \qquad j(z,\bar{z}) = \psi^\dagger(z)\psi(z) - \bar{\psi}^\dagger(\bar{z})\bar{\psi}(\bar{z})\,.$$

  - weight functions

$$\beta_0(x,t) = -\,2a\sum_{r=1}^{\infty} r\sin(rq_F a)t_r(x) \qquad \mu_0(x,t) = t_0 + 2\sum_{r=1}^{\infty}\cos(rq_F a)t_r(x),$$

$$\beta_1(x,t) = 2a\sum_{r=1}^{\infty} r\cos(rq_F a)s_r(x) \qquad \mu_1(x,t) = 2a\sum_{r=1}^{\infty}\sin(rq_F a)s_r(x)\,.$$

RESULTS AT HALF FILLING

- Local quench at half-filling: $q_F a = \pi/2$
- The diagonals of $H(t)$ are diagonals are either purely real or imaginary,

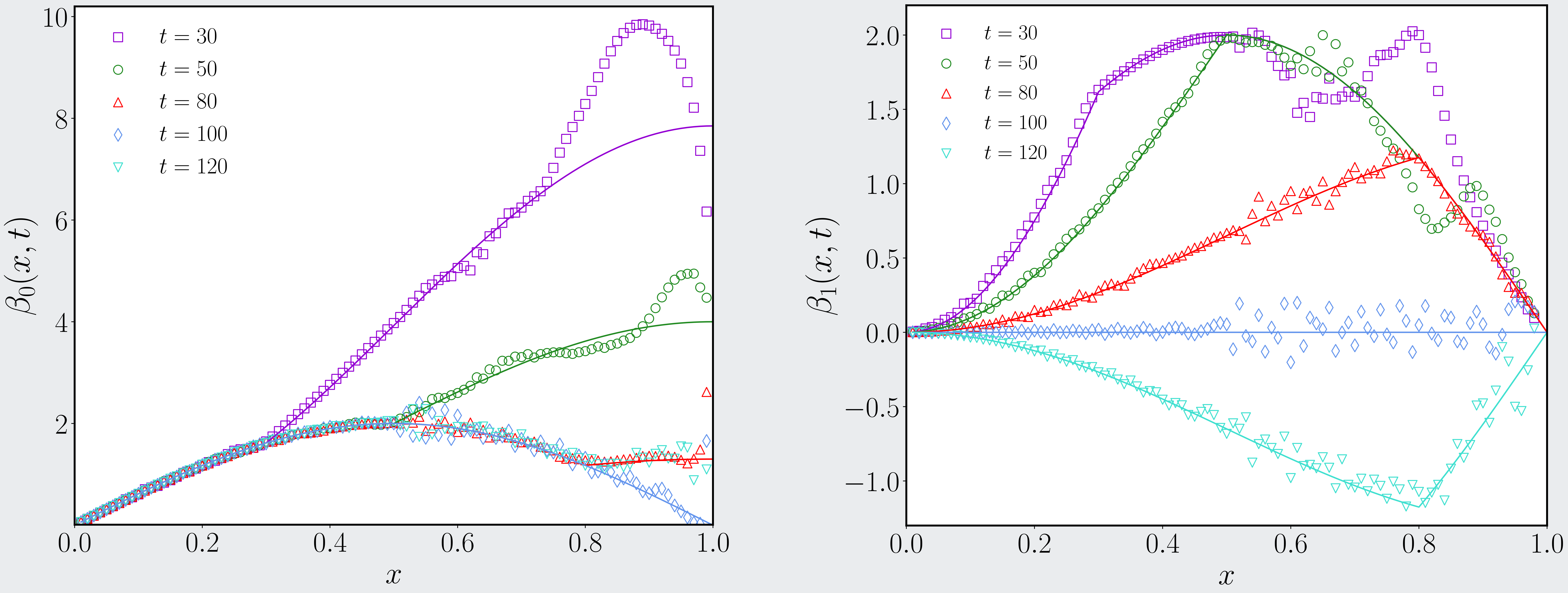
$$t_{2p}(x) \equiv 0, \quad s_{2p+1}(x) \equiv 0 \longrightarrow \mu_0(x,t) \equiv 0, \quad \mu_1(x,t) \equiv 0,$$

- Nonvanishing weights

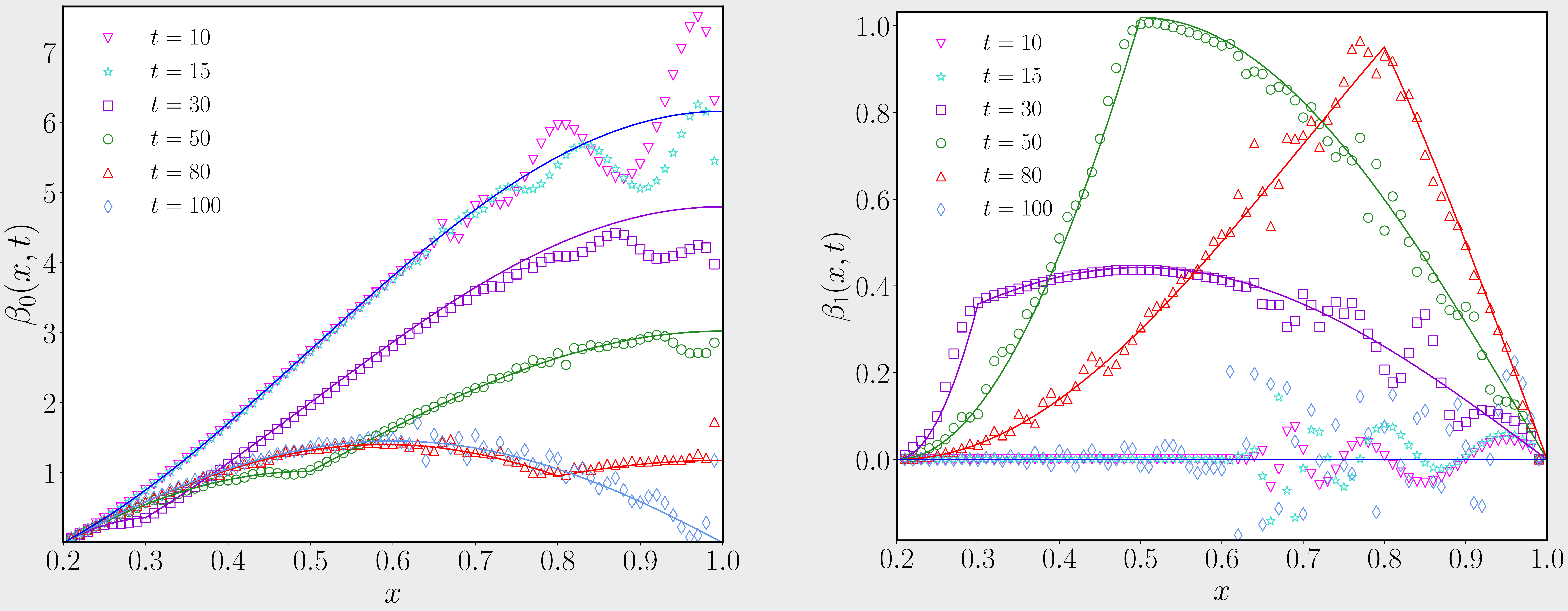
$$\beta_0(x,t) = -2a \sum_{p=0}^{\infty} (2p+1)(-1)^p t_{2p+1}(x), \qquad \beta_1(x,t) = 2a \sum_{p=1}^{\infty} (2p)(-1)^p s_{2p}(x)\,.$$

- Comparison with CFT expressions $\beta_0(x,t) = \dfrac{\beta(x,t) + \bar{\beta}(x,t)}{2}, \qquad \beta_1(x,t) = \dfrac{\bar{\beta}(x,t) - \beta(x,t)}{2}\,.$

CENTERED INTERVAL



DECENTERED INTERVAL



CONCLUSIONS

- Main results:
  - Derivation of the CFT expression of the EH after a local quench between two finite subsystems
  - The CFT prediction for the EH after a local quench can be recovered via proper continuum limit procedure from the lattice EH.
- Further investigation:
  - Local approximation of lattice EH (commuting operators)
  - Different conformal boundary conditions

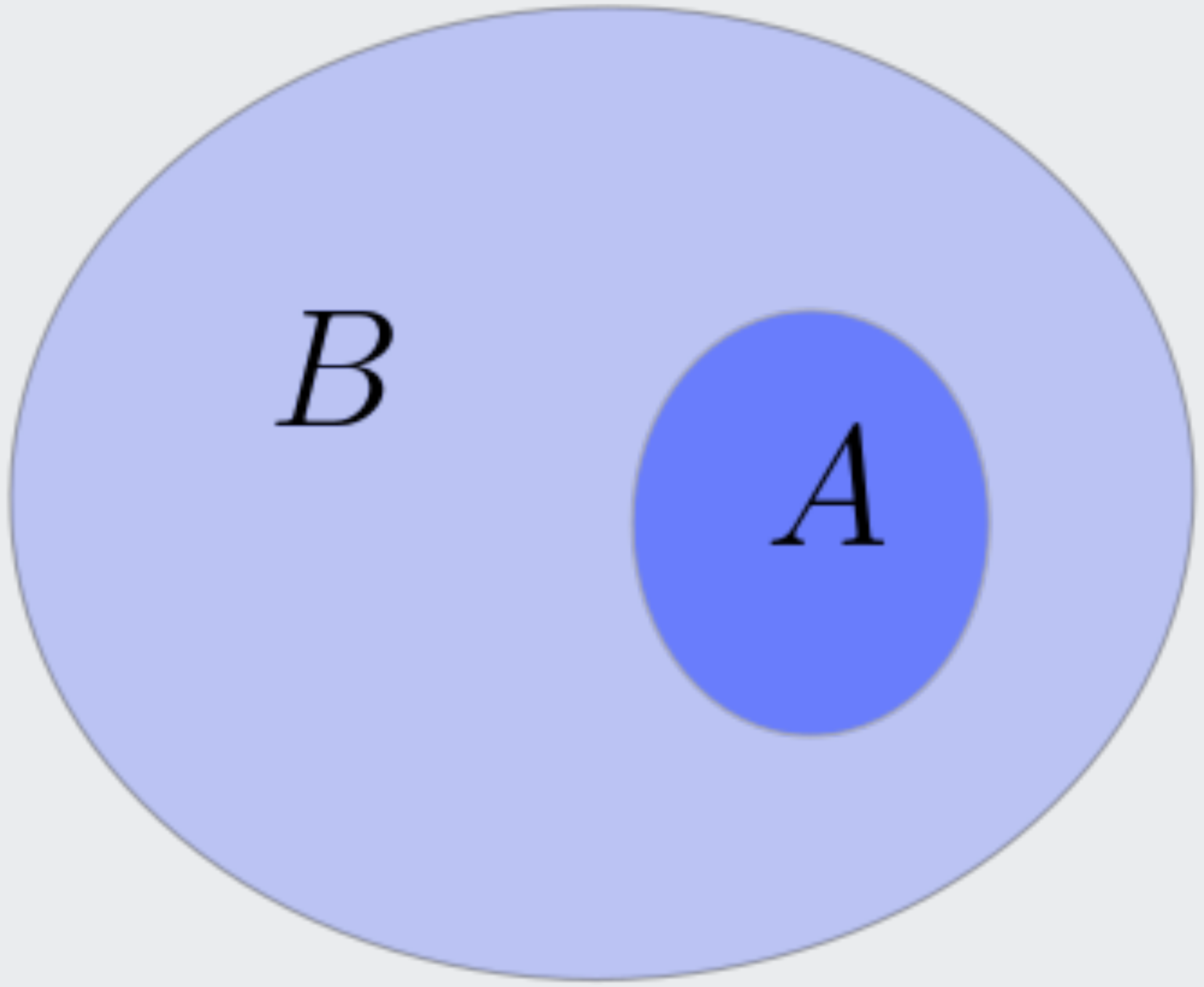
THANK YOU!

CONCLUSIONS

- EH of gradient chain and Fermi gas in an harmonic trap:
  - Effective curved-space CFT description with a space-dependent Fermi velocity, which yields the EH in a Bisognano-Wichmann form for both cases
  - The BW structure is inherited by operators that commute exactly with the EH
  - The low-lying entanglement spectra can be well approximated by rescaling the spectra of the commuting operators by a local v_F
- EH and orthogonal polynomials:
  - The T matrix is a deformation of the physical Hamiltonian
  - The deformation factor in the T matrix is determined by the dual spectrum

ENTANGLEMENT: DEFINITIONS

- Let us consider a bipartite quantum system $S = A \cup B$ in a pure state $|\Psi\rangle \in \mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B$

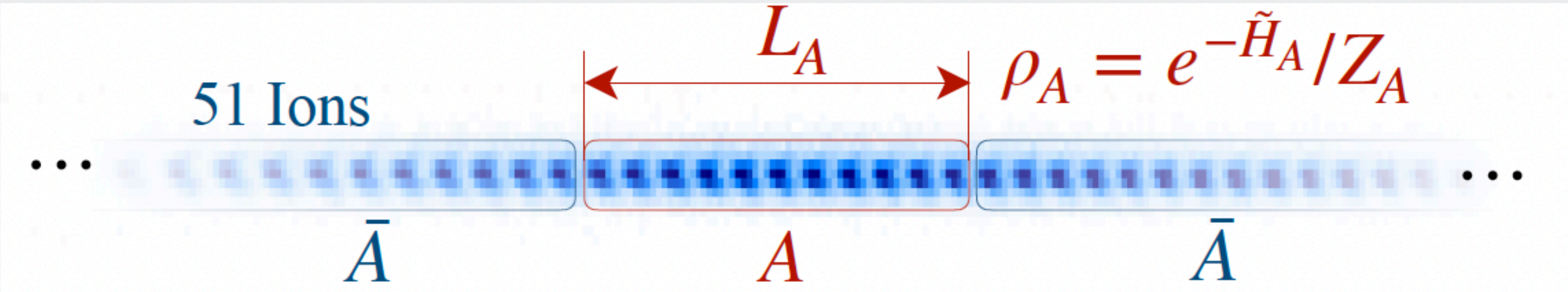


- A pure state $|\Psi\rangle \in \mathscr{H}_{AB}$ is said to be separable if and only if it can be written as Kronecker product of pure states $|\psi\rangle_A \in \mathscr{H}_A$ and $|\phi\rangle_B \in \mathscr{H}_B$, that is

$$|\Psi\rangle = |\psi\rangle_A \otimes |\phi\rangle_B$$

otherwise it is said to be entangled.

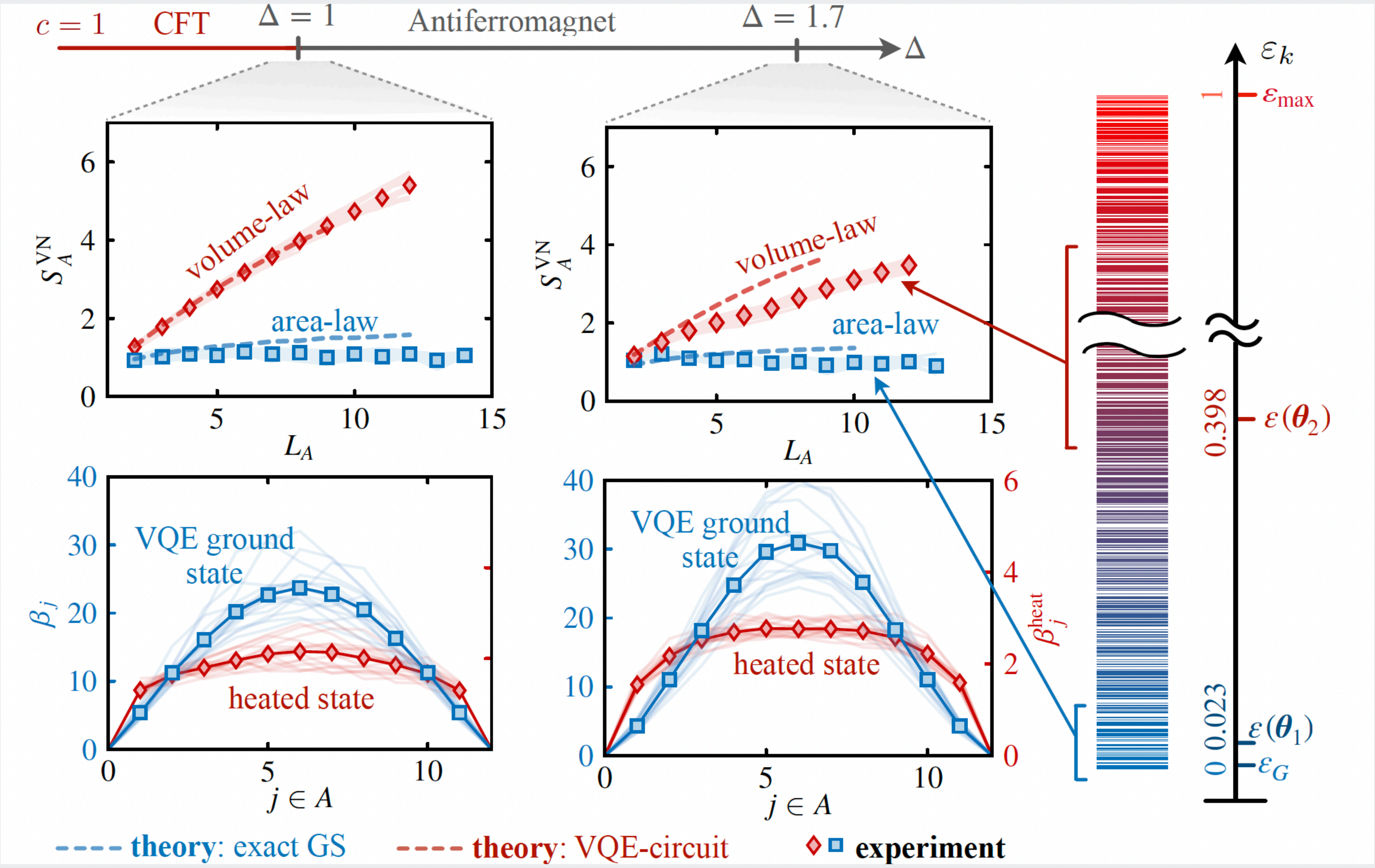
EXPERIMENTAL MOTIVATIONS



- One-dimensional XXZ Heisenberg chain prepared on a 51-ion programmable quantum simulator

$$\hat{H} = \sum_j H_j \quad \rightarrow \quad \mathscr{H} = \sum_j \beta_j H_j$$

- The assumption of a local form of the EH has enabled the recent successful reconstruction of the EH in quantum-simulator experiments.

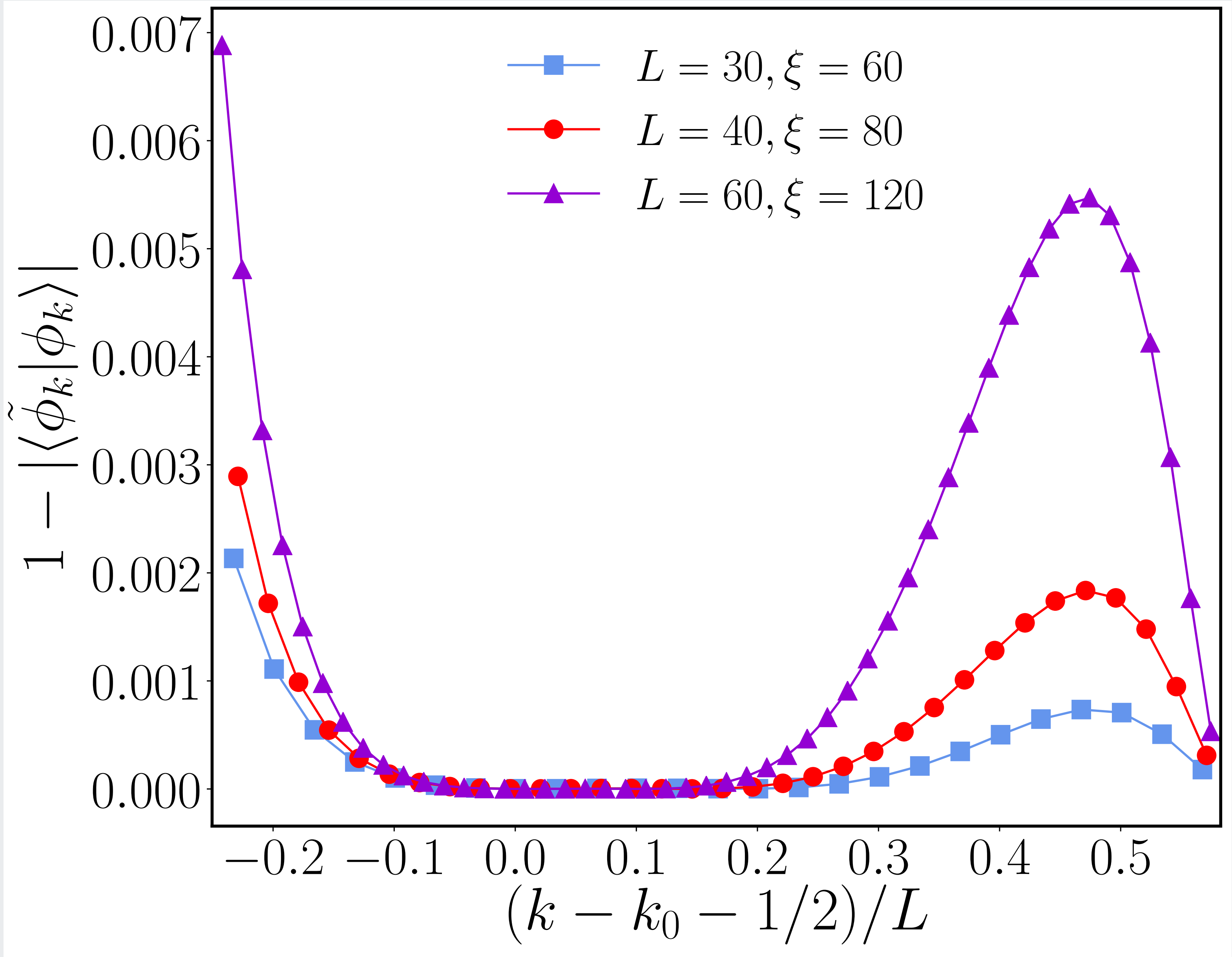


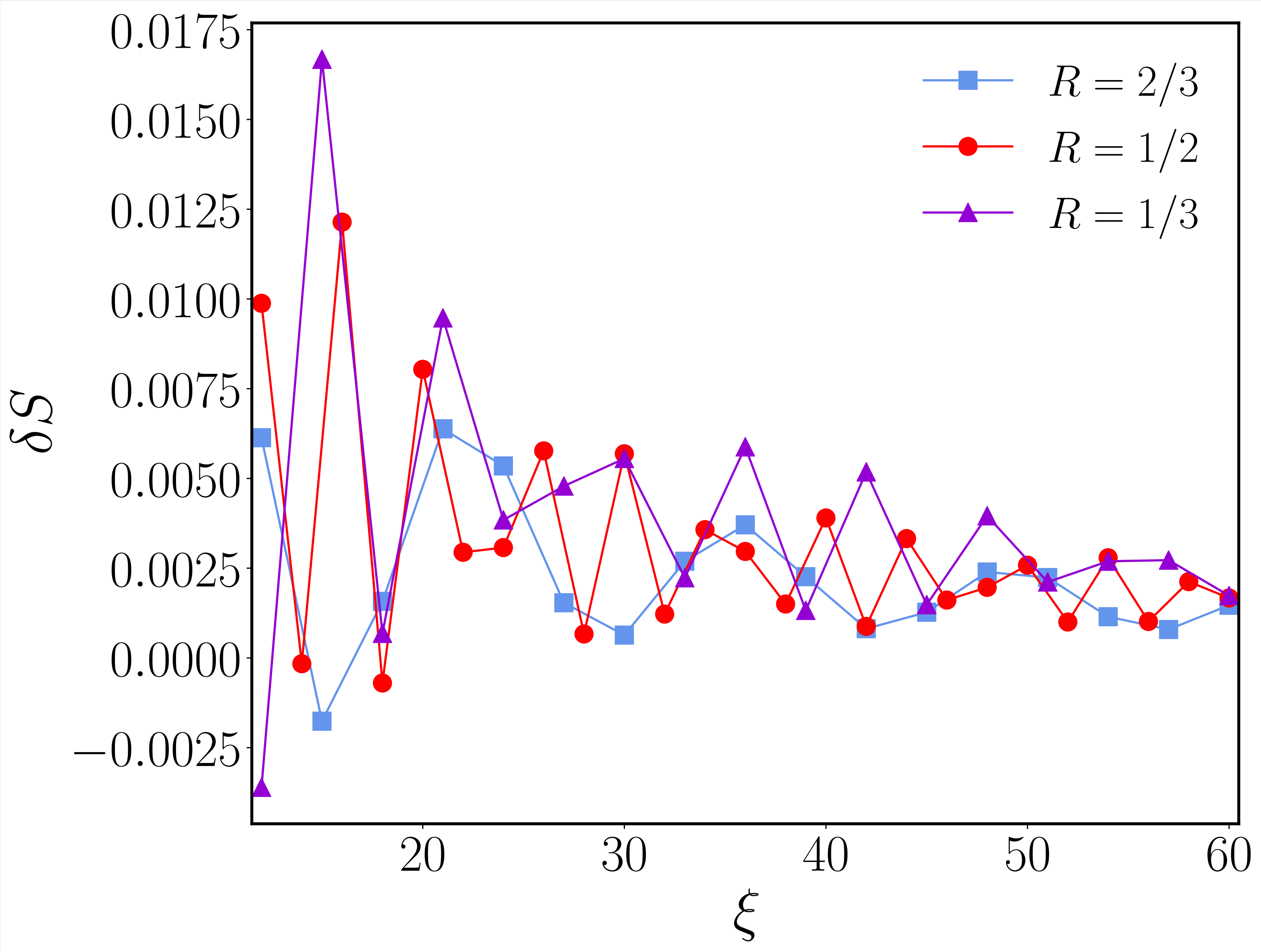
M. K. Joshi et al., *Nat. 624, 539 (2023)*.

EH FOR THE GRADIENT CHAIN: $L < \xi$

- Almost commuting tridiagonal matrix obtained from discretisation of CFT ansatz.

$$t_i = \tilde{\beta}(i) \qquad d_i = -\frac{2}{\xi}\left(i - \frac{1}{2}\right)\tilde{\beta}(i - 1/2)\,.$$





EH FOR THE HARMONIC TRAP

$$\mathscr{H} = \boxed{\frac{2\pi}{v_F(x_0)}} \int_A (x - x_0) H(x)$$

- Differential operator $[\mathscr{K}, \hat{D}] = 0$:

$$\hat{D} = \frac{\mathrm{d}}{\mathrm{d}x}(x - x_0)\frac{\mathrm{d}}{\mathrm{d}x} - x^2(x - x_0) + 2N(x - x_0)$$

Single-particle hamiltonian:

$$\boxed{\hat{H}} = -\boxed{\frac{1}{2}}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \boxed{\frac{1}{2}}x^2 + \mu$$

- BW Ansatz for the EH:

$$\mathscr{H}_{BW} = -\frac{\pi}{v_F(x_0)}\hat{D}$$

$$\boxed{\varepsilon_k = -\frac{\pi}{v_F(x_0)}\chi_k}$$

χ_k eigenvalues of $\hat{D}$

EH FOR THE HARMONIC TRAP

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$$\boxed{\hat{H}} = -\boxed{\frac{1}{2}}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \boxed{\frac{1}{2}}x^2 + \mu$$

- The operator $\hat{D}$ has the same spectrum $\{\chi_k\}$ of the tridiagonal matrix M

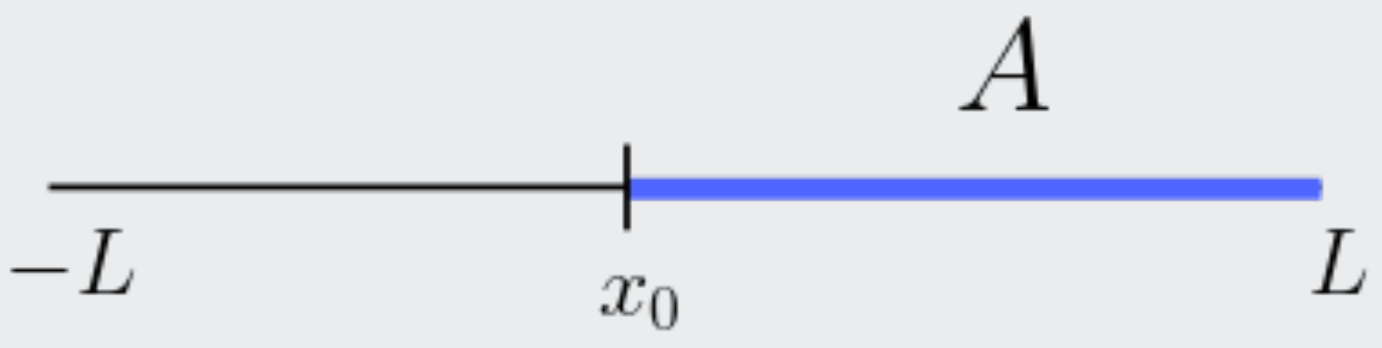
$$M_{m,n} = 2(N - n)\sqrt{\frac{n}{2}}\delta_{m,n-1} + 2(N - m)\sqrt{\frac{m}{2}}\delta_{n,m-1} - 2\left(N - n - \frac{1}{2}\right)x_0\delta_{m,n}$$

A. Grunbaum, *J. Math. Anal. Appl. 95 no. 2, 491–500 (1983)*

EH FOR THE GRADIENT CHAIN

- Fermi velocity: $$v_F(x) = \left.\frac{\mathrm{d}\omega_q(x)}{\mathrm{d}q}\right|_{q_F} = \sqrt{1-\left(\frac{x}{\xi}\right)^2}$$

  Local Density Approx



- Weak gradient regime $L < \xi$:

$$\tilde{\beta}(x) = \frac{2}{\pi}\xi \arcsin\left(\frac{L}{\xi}\right) \frac{\sin\left(\frac{\pi}{2}\frac{\arcsin(x/\xi)}{\arcsin(L/\xi)}\right) - \sin\left(\frac{\pi}{2}\frac{\arcsin(x_0/\xi)}{\arcsin(L/\xi)}\right)}{\cos\left(\frac{\pi}{2}\frac{\arcsin(x_0/\xi)}{\arcsin(L/\xi)}\right)}$$

- Infinite system $L \to \infty$:

$$\tilde{\beta}(x) = \frac{x - x_0}{\sqrt{1-\left(\frac{x_0}{\xi}\right)^2}} = \frac{x-x_0}{v_F(x_0)}$$

EH AND ORTHOGONAL POLYNOMIALS

FREE FERMIONS AND ORTHOGONAL POLYNOMIALS

N. Crampé, R. Nepomechie and L. Vinet, *J. Stat. Mech. 093101 (2019)*

- Hamiltonian:

$$\widehat{H} = \sum_{n=0}^{N-2} J_n(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) - \sum_{n=0}^{N-1} B_n c_n^\dagger c_n,$$

- The single-particle wavefunctions $\phi_n(\epsilon_k)$ satisfy
  - Three-term recurrence relation

Single-particle spectrum

$$\boxed{\epsilon_k} R_n(\epsilon_k) = A_n R_{n+1}(\epsilon_k) - (A_n + C_n) R_n(\epsilon_k) + C_{n-1} R_{n-1}(\epsilon_k)$$

  - Three-term difference relation

Dual spectrum

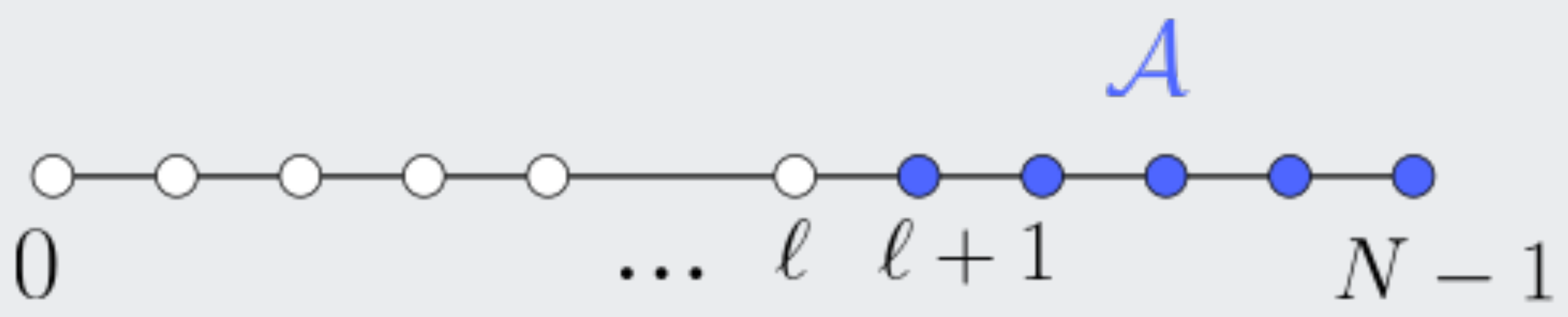
$$\boxed{\chi_n} R_n(\epsilon_k) = \bar{A}_k R_n(\epsilon_{k+1}) - (\bar{A}_k + \bar{C}_k) R_n(\epsilon_k) + \bar{C}_k R_n(\epsilon_{k-1})$$

Where

$$J_n = \sqrt{A_n C_{n+1}}, \quad B_n = A_n + C_n, \quad \phi_n(\epsilon_k) = \sqrt{W_k} \sqrt{\frac{A_{n-1} \cdots A_0}{C_n \cdots C_1}} R_n(\epsilon_k)$$

COMMUTING OPERATOR

N. Crampé, R. Nepomechie and L. Vinet, *J. Stat. Mech. 093101 (2019)*

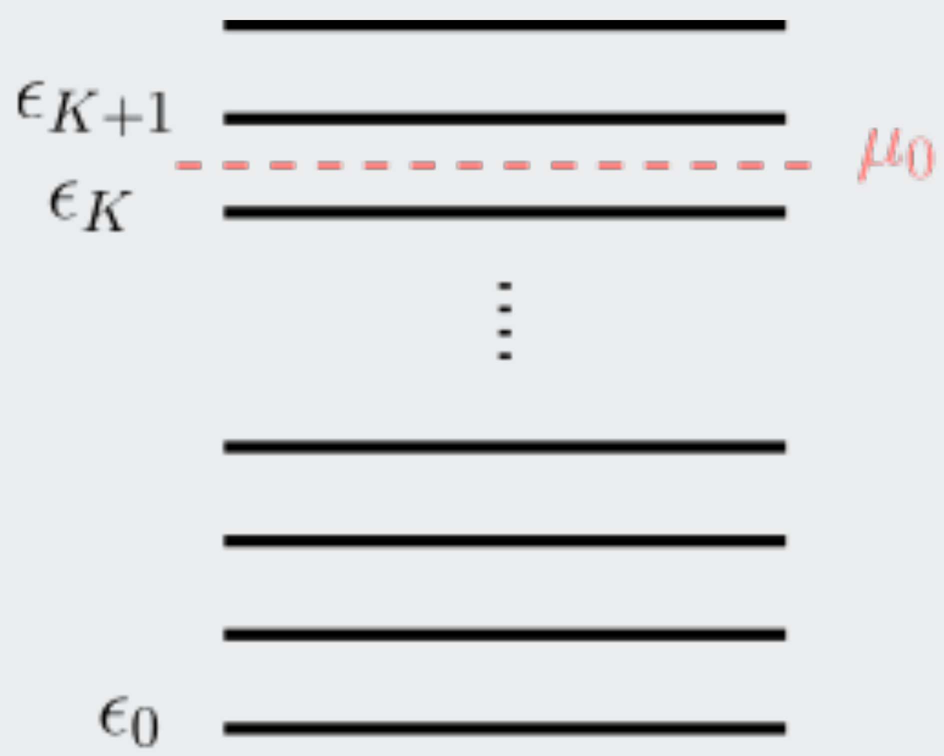


- Let us consider a subsystem $\mathscr{A} = \{\ell + 1, \cdots, N-1\}$
- Ground state composed of K filled levels $k \in \{0, 1, \ldots, K\}$
- Reduced correlation matrix: $[C_{\mathscr{A}}]_{ij} = \sum_{\epsilon_k < 0} \phi_i(\epsilon_k)\phi_j(\epsilon_k), \quad i, j \in \mathscr{A}$
- The tridiagonal operator T can be computed from the bispectral pair H and X:

$$T = \frac{1}{2}\{H - \boxed{\frac{\epsilon_{K+1} + \epsilon_K}{2}}, X - \frac{\chi_{\ell+1} + \chi_\ell}{2}\}$$

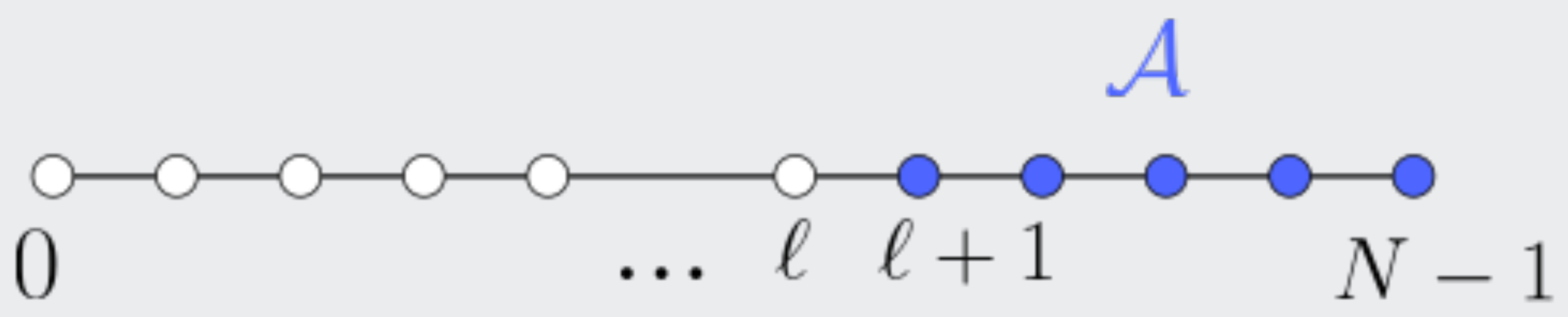
Chemical potential

$$\mu_0 = \frac{\epsilon_K + \epsilon_{K+1}}{2}$$



COMMUTING OPERATOR

N. Crampé, R. Nepomechie and L. Vinet, *J. Stat. Mech.* 093101 (2019)



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- The tridiagonal operator T can be computed from the bispectral pair H and X:

$$T = \begin{pmatrix} d_0 & t_0 & & & & \\ t_0 & d_1 & t_1 & & & \\ & t_1 & d_2 & t_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & d_{N-2} & t_{N-2} \\ & & & & t_{N-2} & d_{N-1} \end{pmatrix}$$

$$d_i = -\left(\chi_i - \frac{\chi_\ell + \chi_{\ell+1}}{2}\right)(B_i + \mu_0)$$

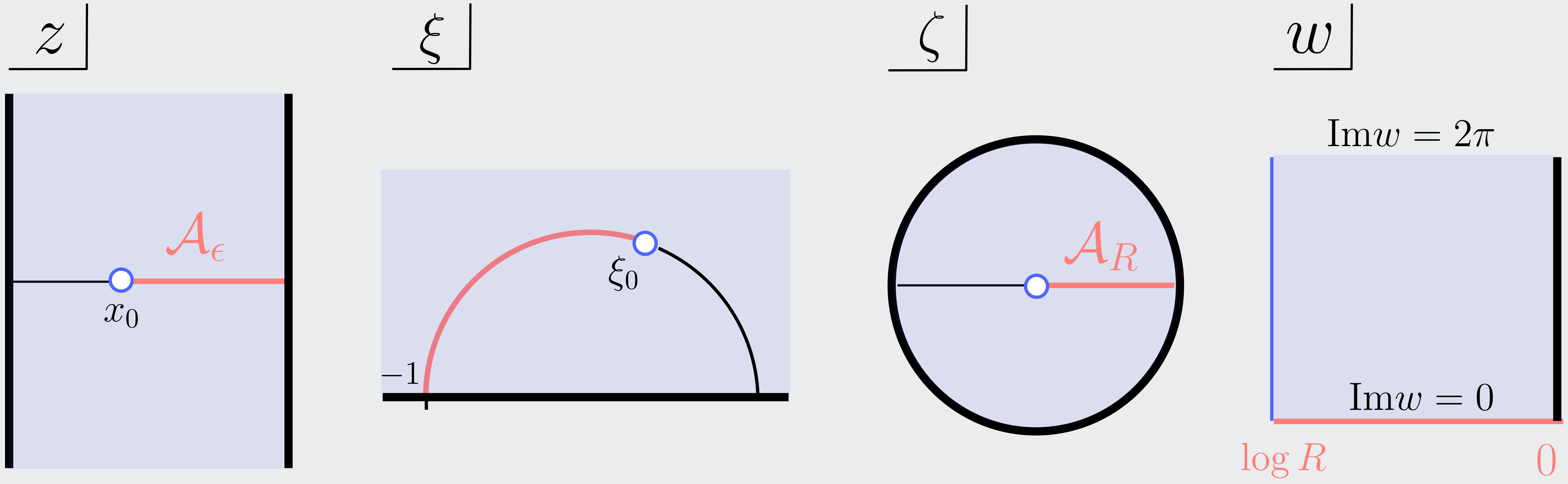
$$t_i = \left(\frac{\chi_i + \chi_{i+1}}{2} - \frac{\chi_\ell + \chi_{\ell+1}}{2}\right) J_i$$

EH IN CURVED SPACE CFT

- Chain of size L, bipartition $\mathscr{A} \in [x_0, L]$
  - Homogeneous system

Conformal mapping
to the annulus:

$$w = f(z) = \ln \left[\frac{\sin \frac{\pi(z - x_0)}{2L}}{\sin \frac{\pi(z + x_0)}{2L}} \right]$$

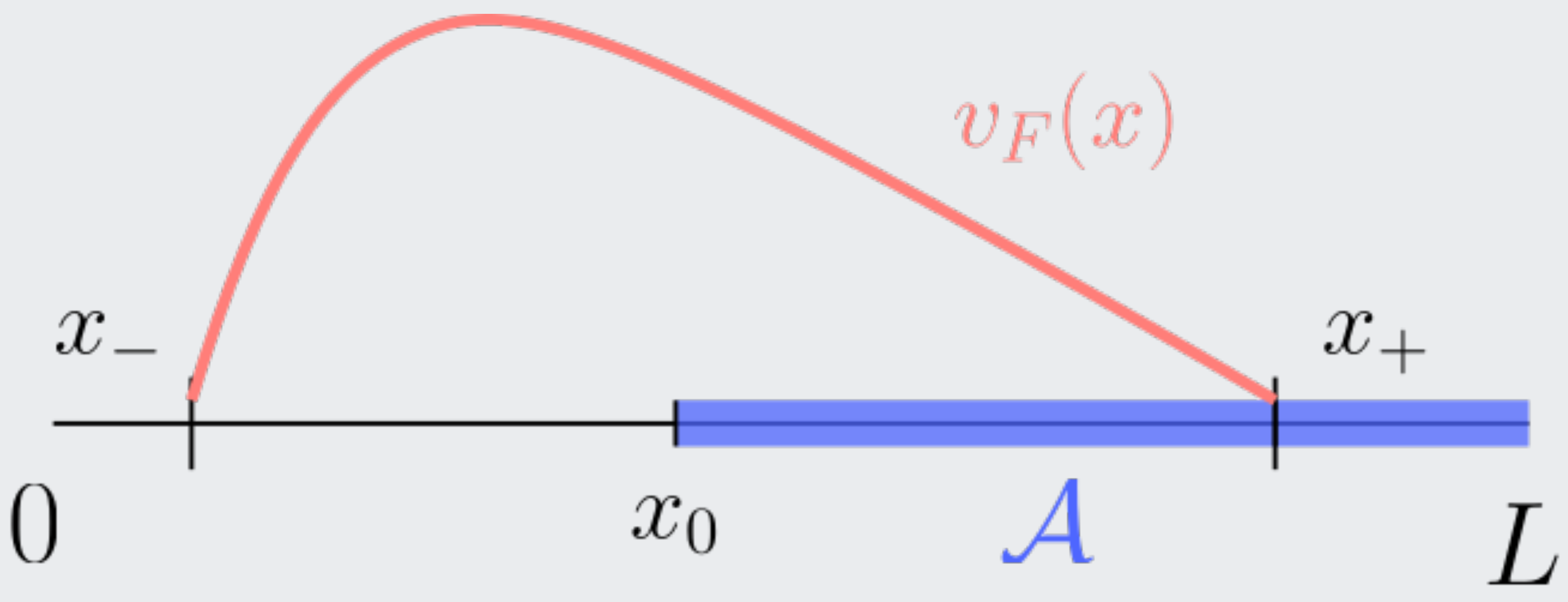


$$\frac{1}{f'(x)} = \beta(x) = \frac{L}{\pi} \frac{\cos\left(\frac{\pi x_0}{L}\right) - \cos\left(\frac{\pi x}{L}\right)}{\sin\left(\frac{\pi x_0}{L}\right)}$$

EH IN CURVED SPACE CFT

- Chain of size L, bipartition $\mathscr{A} \in [x_0, L]$
  - In-homogeneous system

    Non-symmetric v_F:



$$\begin{cases} \tilde{x}(x) = \displaystyle\int_{x_-}^{x} \frac{\mathrm{d}u}{v_F(u)} \\ \tilde{L} = \tilde{x}(x_+) = \displaystyle\int_{x_-}^{x_+} \frac{\mathrm{d}u}{v_F(u)} \end{cases}$$

$$\mathscr{H} = 2\pi \int_{x_-}^{x_+} \boxed{\tilde{\beta}(x)}\, v_F(x) T_{00}(x)\, \mathrm{d}x$$

$$\boxed{\tilde{\beta}(x)} = \frac{\tilde{L}}{\pi} \frac{\cos\left(\frac{\pi \tilde{x}_0}{\tilde{L}}\right) - \cos\left(\frac{\pi \tilde{x}(x)}{\tilde{L}}\right)}{\sin\left(\frac{\pi \tilde{x}_0}{\tilde{L}}\right)}$$

KRAWTCHOUCK CHAIN

- Couplings: $J_n = \sqrt{(n+1)(N-1-n)p(1-p)}, \qquad B_n = -(N-1)p - n(1-2p)$

- Spectrum and dual spectrum: $\epsilon_k = k, \qquad \chi_i = i,$

- Tridiagonal commuting matrix:

$$t_i = (i-\ell)\boxed{J_i,} \qquad d_i = \boxed{-}(i - 1/2 - \ell)\boxed{(B_i + \mu_0)}$$

- *The T matrix is a linear deformation of the physical Hamiltonian*

KRAWTCHOUCK CHAIN

- Continuum limit:

$$\omega_q(x) = 2J(x)\cos q - \mu(x) \quad \Rightarrow \quad v_F(x) = \left|\frac{\mathrm{d}\omega_q(x)}{\mathrm{d}q}\right|_{q_F(x)} = \sqrt{4J^2(x) - \mu^2(x)}$$

- CFT result from BW-theorem:

$$\tilde{\beta}(x) = \frac{x - x_0}{v_F(x_0)}$$

$\Rightarrow$ ○ CFT prediction for the EH has the BW form analogous to that of the gradient chain:

$$\mathcal{H} = \boxed{\frac{2\pi}{v_F(x_0)}} \int_A (x - x_0) H(x)$$

KRAWTCHOUCK CHAIN

- Continuum limit:

$$\omega_q(x) = 2J(x)\cos q - \mu(x) \quad \Rightarrow \quad v_F(x) = \left|\frac{\mathrm{d}\omega_q(x)}{\mathrm{d}q}\right|_{q_F(x)} = \sqrt{4J^2(x) - \mu^2(x)}$$

- CFT result from BW-theorem:

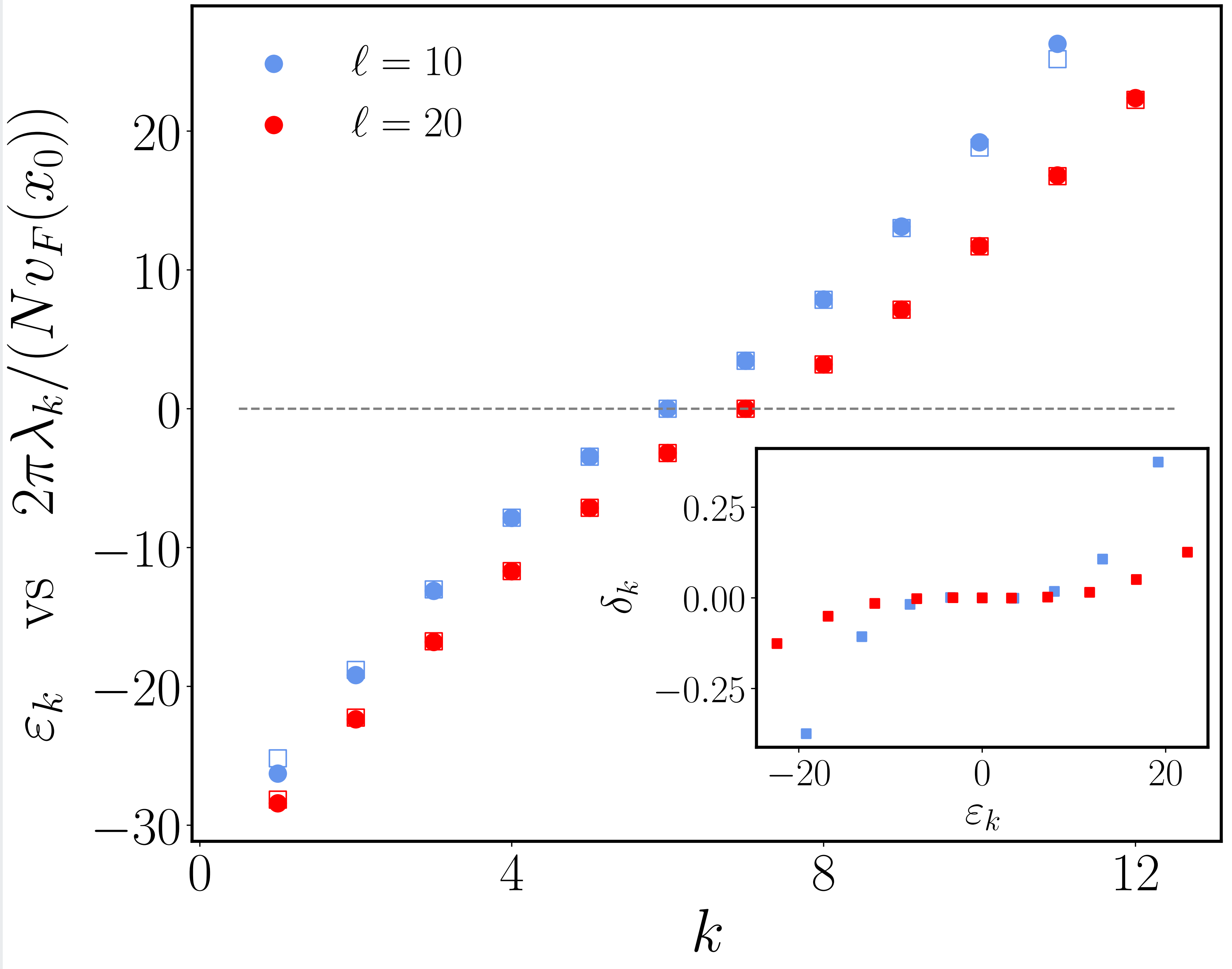
$$\tilde{\beta}(x) = \frac{x - x_0}{v_F(x_0)}$$

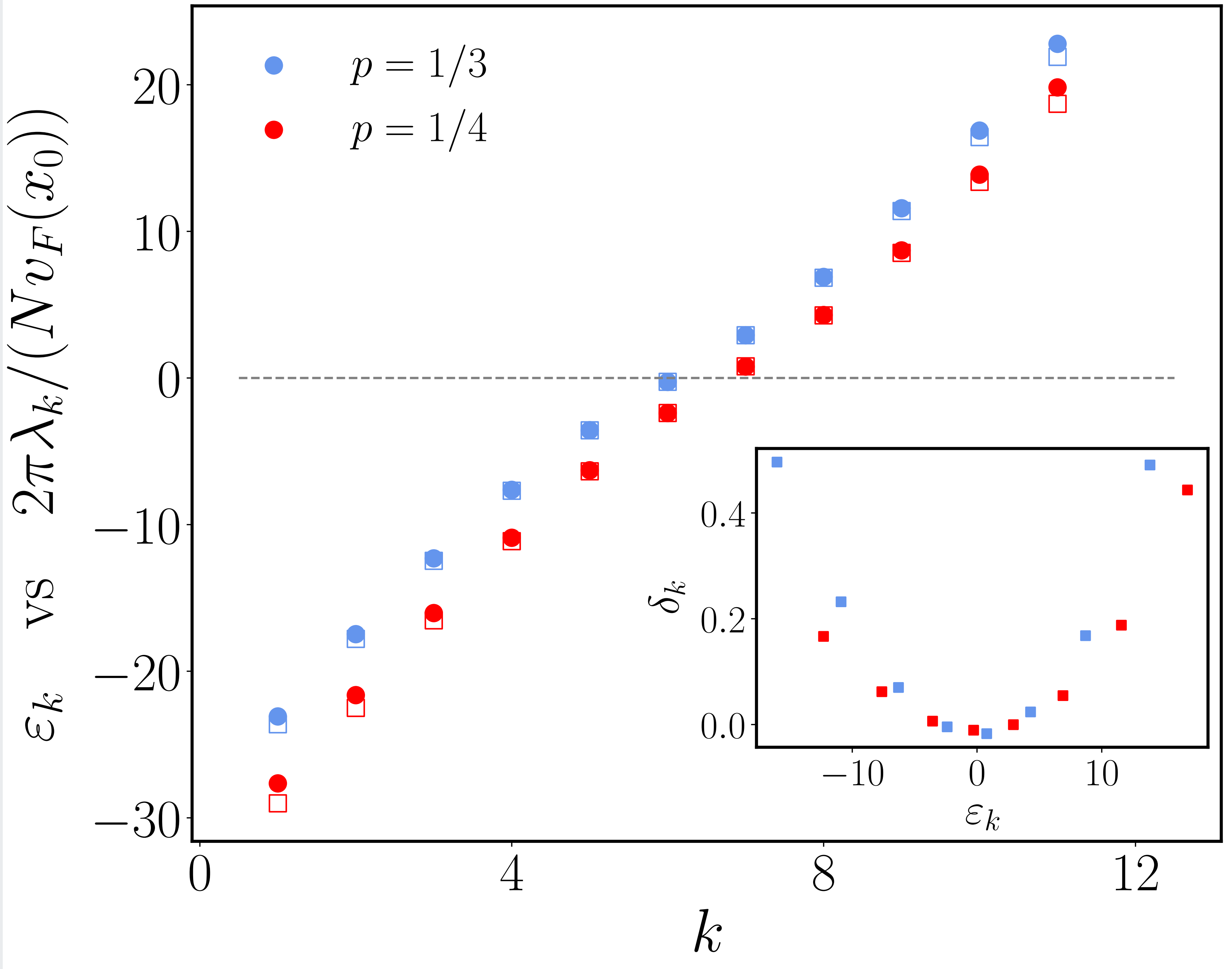
$\Rightarrow$ ∘ The lattice EH can be approximated as:

$$h \simeq \boxed{\frac{2\pi}{N v_F(x_0)}} T_{\mathscr{A}}$$

KRAWTCHOUCK CHAIN

- Comparison between the spectra:





HAHN CHAIN

- Couplings: $J_n = \sqrt{A_n C_{n+1}}, \quad B_n = A_n + C_n$

$$A_n = -\frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-1-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \qquad C_n = -\frac{n(n+\alpha+\beta+N)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}$$

- Spectrum and dual spectrum: $\epsilon_k = k, \qquad \boxed{\chi_i = i(i+\alpha+\beta+1)}$

- Tridiagonal commuting matrix:

$$t_i = \boxed{(i-\ell)(i+\ell+\alpha+\beta+2)}\boxed{J_i}$$

$$d_i = \boxed{-}\boxed{[(i-1/2-\ell)(i-1/2+\ell+\alpha+\beta+2)-1/4]}\boxed{(B_i+\mu_0)}$$

- *The deformation factor in the T matrix is determined by the dual spectrum.*

HAHN CHAIN

- CFT result from BW-theorem:

$$\tilde{\beta}(x) = \frac{(x - x_0)(x + x_0 + x_1 + x_2)}{(2x_0 + x_1 + x_2)v_F(x_0)}$$

where $x_1 = \alpha/N$ and $x_2 = \beta/N$.

- CFT prediction for the EH:

$$\mathscr{H} = \boxed{\frac{2\pi}{(2x_0 + x_1 + x_2)v_F(x_0)}} \int_A (x - x_0)(x + x_0 + x_1 + x_2)H(x)$$

HAHN CHAIN

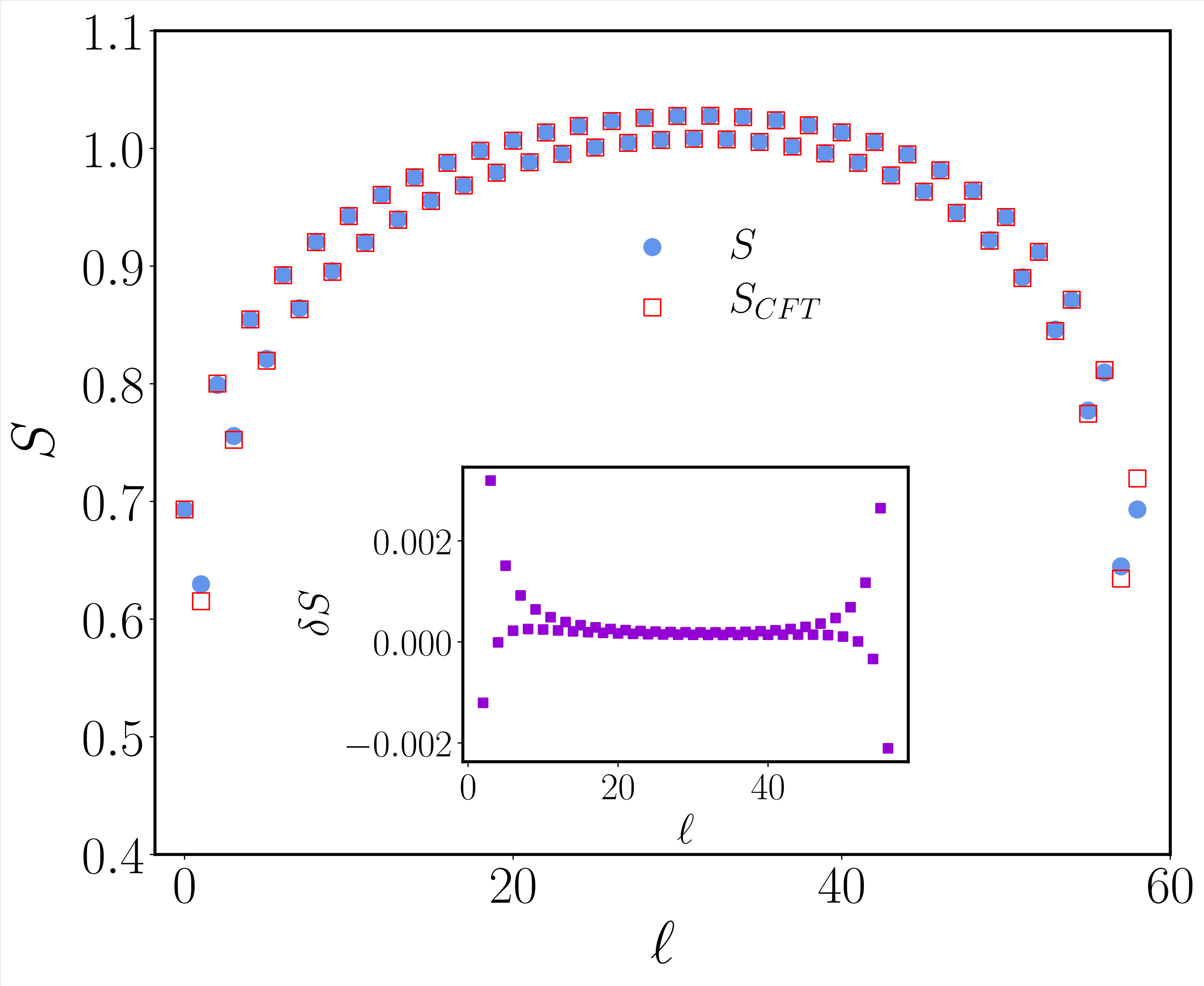
- CFT result from BW-theorem:

$$\tilde{\beta}(x) = \frac{(x - x_0)(x + x_0 + x_1 + x_2)}{(2x_0 + x_1 + x_2)v_F(x_0)}$$

where $x_1 = \alpha/N$ and $x_2 = \beta/N$.

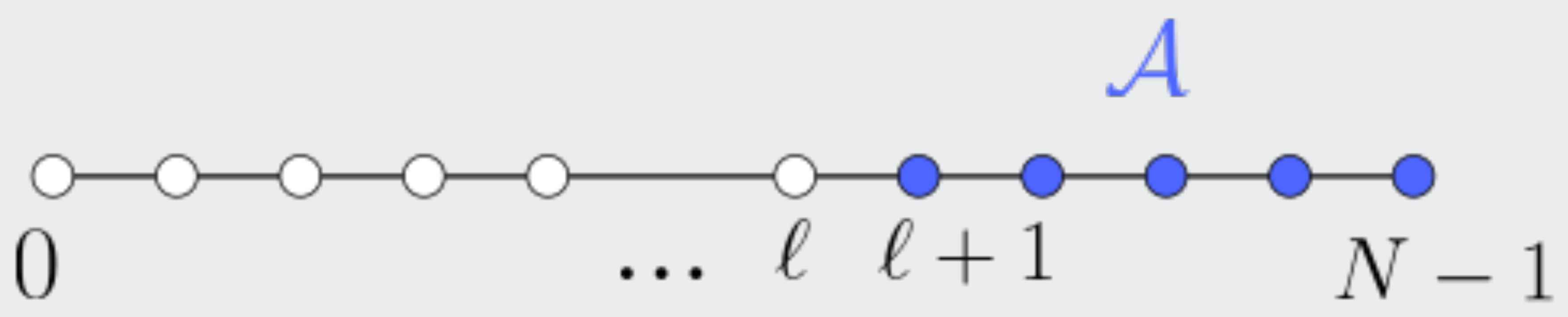
- CFT prediction for the EH:

$$\varepsilon_k \simeq \boxed{\frac{2\pi}{(2x_0 + x_1 + x_2)v_F(x_0)}}\frac{\lambda_k}{N^2}$$



COMMUTING OPERATOR

N. Crampé, R. Nepomechie and L. Vinet, *J. Stat. Mech. 093101 (2019)*



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The restriction $T_{\mathscr{A}}$ of T to subsystem $\mathscr{A}$ commutes with $C_{\mathscr{A}}$ and the EH h

$$[T_{\mathscr{A}}, C_{\mathscr{A}}] = [T_{\bar{\mathscr{A}}}, C_{\bar{\mathscr{A}}}] = 0$$

$$[T_{\mathscr{A}}, h] = 0$$