Classifying Integrable Deformations in Quantum Spin Chains

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Outline

Motivation

- Integrable models have special and useful features, such as solvability;
- However, adding a generic perturbation to an integrable model usually spoils these properties.
- Is it possible to add perturbations to integrable models that preserve, at least to some extent, their integrability?

Motivation

- Significant progress has been made in the classification of integrable models using the Boost Operator Method;

  M. de Leeuw, C. Paletta, A. Pribytok, A. L. Retore & P. Ryan, 2010.11231

  L. Corcoran & M. de Leeuw, 2306.10423

- During this classification, it was possible to identify certain integrable deformations, for example, in AdS/CFT and XXX spin chains:

  M. de Leeuw, C. Paletta, A. Pribytok, A. L. Retore & P. Ryan, 2109.00017

  M. de Leeuw, A. Fontanella & J. M. Nieto García, 2506.13598

- However, one must perform the full classification to identify these deformations.

Motivation

- Several works have explored the classification of long-range deformations directly at the level of the charges or by applying the boost operator;

T. Bargheer, N. Beisert & F. Loebbert, 0902.0956

T. Gombor & B. Pozsgay, 2108.02053

M. de Leeuw & A. L. Retore, 2206.08390

F. M. Surace & O. Motrunich, 2302.12804

Motivation

Key Idea: Use the Boost Operator to obtain a systematic classification of nearest-neighbour (NN) deformations of integrable models.

Key Idea

- We deform an integrable model using some expansion parameter ε:

$$\boxed{\mathcal{H}_{\text{int}} \to \mathcal{H}_{\text{int}} + \varepsilon \mathcal{H}_1 + \cdots}$$

  and use the boost operator to classify $\mathcal{H}_i$ terms that preserve integrability up to a certain order.

- Next, we compute **indicators of integrability breaking** as **level spacing distributions**;
- Investigate the onset of quantum chaos.

The Boost Operator Method

- A quantum integrable model can be defined by the presence of a tower of commuting charges:

$$[\mathbb{Q}_i(z), \mathbb{Q}_j(z)] = 0, \quad i, j = 1, 2, \cdots ;$$

- One way to classify and obtain new integrable models is to use The Boost Operator Method:

$$\mathcal{B}(\mathcal{O}) = \sum_k k\ \mathcal{O}_{k,k+1} \quad \text{such that}$$

$$\mathbb{Q}_{m+1}(z) = [\mathcal{B}(\mathbb{Q}_2), \mathbb{Q}_m], m > 1$$

where $\mathbb{Q}_2$ is the total Hamiltonian.

The Boost Operator Method

- The next non-trivial charge is given by

$$\mathbb{Q}_3 = \sum_{i=1}^{L} [\mathcal{H}_{i,i+1}, \mathcal{H}_{i-1,i}] + \mathcal{G}_{i,i+1},$$

M de Leeuw & AL Retore, 2206.08390

for a periodic spin-chain with Nearest-Neighbour Hamiltonian density $\mathcal{H}_{j,j+1}$ and where $\mathcal{G}_{i,i+1}$ is a range-two operator.

- The necessary condition for integrability is given by

$$[\mathbb{Q}_2, \mathbb{Q}_3] = 0$$

and allows us to constrain the degrees of freedom of the Hamiltonian densities to classify integrable models.

Deformations

Let us consider a nearest-neighbour deformation in ε of an integrable Hamiltonian density:

$$\mathcal{H}_{ij}^{(0)} \to \mathcal{H}_{ij}(\varepsilon) = \mathcal{H}_{ij}^{(0)} + \varepsilon \mathcal{H}_{ij}^{(1)} + \varepsilon^2 \mathcal{H}_{ij}^{(2)} + \dots$$

Assuming periodic boundary conditions, the deformed charges using the Boost are given by

$$\mathbb{Q}_2(\varepsilon) = \sum_{i=1}^{L} \mathcal{H}_{i,i+1}(\varepsilon) = \mathbb{Q}_2^{(0)} + \varepsilon \mathbb{Q}_2^{(1)} + \dots,$$

$$\mathbb{Q}_3(\varepsilon) = \sum_{i=1}^{L} [\mathcal{H}_{i,i+1}(\varepsilon), \mathcal{H}_{i-1,i}(\varepsilon)] + \mathcal{G}_{i,i+1}(\varepsilon).$$

Boost Setup

- Then, we impose the necessary condition for integrability:

$$[\mathbb{Q}_2(\varepsilon), \mathbb{Q}_3(\varepsilon)] = 0,$$

order by order in ε.

- Since we assumed $\mathcal{H}^{(0)}$ is integrable, the lowest-order condition is trivially satisfied:

$$\left[\mathbb{Q}_2^{(0)}, \mathbb{Q}_3^{(0)}\right] = 0.$$

- The first-order condition is linear in $\mathcal{H}^{(1)}$ and $\mathcal{G}^{(1)}$:

$$\underbrace{\left[\mathbb{Q}_2^{(0)}, \mathbb{Q}_3^{(1)}\right]}_{\text{linear in } h_{ij}^{(1)}, g_{ij}^{(1)}} + \underbrace{\left[\mathbb{Q}_2^{(1)}, \mathbb{Q}_3^{(0)}\right]}_{\text{linear in } h_{ij}^{(1)}} = 0,$$

so we can solve it and find a closed model.

Boost Setup

- Substituting the first-order solutions into the second-order condition, we obtain:

$$\underbrace{\left[\mathbb{Q}_2^{(0)},\mathbb{Q}_3^{(2)}\right]}_{\text{quadratic in } h_{ij}^{(1)},\text{ linear in } h_{ij}^{(2)},\, g_{ij}^{(2)}} + \underbrace{\left[\mathbb{Q}_2^{(2)},\mathbb{Q}_3^{(0)}\right]}_{\text{linear in } h_{ij}^{(2)}} + \underbrace{\left[\mathbb{Q}_2^{(1)},\mathbb{Q}_3^{(1)}\right]}_{\text{quadratic in } h_{ij}^{(1)}} = 0.$$

  This leads to solutions for $\mathcal{H}^{(2)}$ and $\mathcal{G}^{(2)}$, but also to additional constraints for $\mathcal{H}^{(1)}$.
- This procedure can be systematically extended to higher orders.

Boost Setup

This allows us to classify the deformations into different categories:

- Integrability-Breaking;
- Integrable Deformations:
  - Straightforward Integrability;
  - High-Order in ε Integrability;
- **Quasi-Integrable Models**.

XXZ Spin Chain

As a concrete example, we consider the XXZ spin chain, whose Hamiltonian density is given by

$$\mathcal{H}^{(0)} = \mathcal{H}^{\mathrm{XXZ}} = \frac{\operatorname{csch}(\eta)}{2}\Big[\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \cosh(\eta)(1 + \sigma_z \otimes \sigma_z)\Big]$$

where σ_i are the Pauli matrices.

First Order - Integrable Deformations

Considering a general deformation, we solve the integrability condition at first order, obtaining the following integrable deformation (along with constraints for $g^{(1)}$).

- First-order integrable deformations:
  - 1 sum of the spins in the z direction;
  - 3 sums of mixed spin terms in all directions;
  - 1 difference of mixed spin terms in the xy direction.

$$\mathcal{H}^{(1)}_{\text{Int}} = \mathcal{H}^{(1)}_{\text{XYZ}} + \delta^{(1)}_{z,+} \left(\mathbb{1} \otimes \sigma_z + \sigma_z \otimes \mathbb{1}\right)$$
$$+h^{(1)}_{yz,+} \left(\sigma_y \otimes \sigma_z + \sigma_z \otimes \sigma_y\right) + h^{(1)}_{xz,+} \left(\sigma_x \otimes \sigma_z + \sigma_z \otimes \sigma_x\right)$$
$$+h^{(1)}_{xy,+} \left(\sigma_x \otimes \sigma_y + \sigma_y \otimes \sigma_x\right) + h^{(1)}_{xy,-} \left(\sigma_x \otimes \sigma_y - \sigma_y \otimes \sigma_x\right).$$

Second Order

Solving the 2nd-order integrability condition, we find two possible deformations that preserve integrability.

- **Model A - 4 free parameters**

$$\boxed{\begin{aligned}\mathcal{H}^{(1)}_{\text{ModA}} = \;& \delta^{(1)}_{z,+}\,(\mathbb{1}\otimes\sigma_z+\sigma_z\otimes\mathbb{1})+h^{(1)}_{xz,+}\,(\sigma_x\otimes\sigma_z+\sigma_z\otimes\sigma_x)\\ &+h^{(1)}_{yz,+}\,(\sigma_y\otimes\sigma_z+\sigma_z\otimes\sigma_y)+h^{(1)}_{xy,-}\,(\sigma_x\otimes\sigma_y-\sigma_y\otimes\sigma_x)\,.\end{aligned}}$$

along with a second-order deformation $\mathcal{H}^{(2)}$ that satisfies specific conditions.

Second Order - Model A

- To extend the integrability of this model to higher orders, one must add $\mathcal{H}^{(3)}, \mathcal{H}^{(4)}, \cdots$.
- Another way to see this is to write Model A using some redefinitions and a basis transformation.
- Consider the integrable model composed of XXZ, an addition of the spin in the z direction, and a DMI term:

$$\mathcal{H}_{\mathrm{Int}} = \mathcal{H}_{XXZ} + \varepsilon\delta^{(1)}_{z,+}\ (\mathbb{1}\otimes\sigma_z + \sigma_z\otimes\mathbb{1}) + \varepsilon h^{(1)}_{xy,-}\ (\sigma_x\otimes\sigma_y - \sigma_y\otimes\sigma_x)$$

Alcaraz & Wreszinski (1990)

N. Beisert, L. Fiévet, M. de Leeuw & F. Loebbert, 1308.1584

Second Order - Model A

- Now, we use the transformation given by

$$U = e^{i\varepsilon \tanh\left(\frac{\eta}{2}\right)\left(h_{yz,+}(\mathbb{1}\otimes\sigma_x+\sigma_x\otimes\mathbb{1})-h_{xz,+}(\mathbb{1}\otimes\sigma_y+\sigma_y\otimes\mathbb{1})+\mathcal{O}(\varepsilon^2)\right)}$$

to rewrite

$$\mathcal{H}_{\text{ModA}}(\varepsilon) = U(\varepsilon)\mathcal{H}_{\text{Int}}\, U(\varepsilon)^{-1}.$$

So Model A can be extended to be integrable at all orders by solving recursively for the coefficients in the transformation.

Second Order - Model B

Model B - 6 free parameters

Higher-order integrability conditions repeat this pattern. Hence, the second integrable deformation is given by:

$$\boxed{\mathcal{H}_{\mathrm{ModB}} = \sum_{A,B=x,y,z} h^{(1)}_{AB,+}\left(\sigma_A \otimes \sigma_B + \sigma_B \otimes \sigma_A\right)}$$

where $h_{AB}(\epsilon)$ is an arbitrary power series in $\varepsilon \to$ rotate XYZ.

Quasi-Integrable Models

We define quasi-integrable models as those that are integrable only up to a finite order in ε. Combining information from the 1^{st} and 2^{nd} orders, we can construct a quasi-integrable model at first order:

$$H_{\text{QInt}} = H_{\text{XXZ}} + \epsilon\alpha \sum_{i=1}^{L} (\sigma_{x,i}\sigma_{x,i+1} - \sigma_{y,i}\sigma_{y,i+1}) + \epsilon\beta \sum_{i=1}^{L} (\sigma_{z,i} + \sigma_{z,i+1}) + \epsilon\gamma \sum_{i=1}^{L} (\sigma_{x,i}\sigma_{y,i+1} - \sigma_{y,i}\sigma_{x,i+1}).$$

This is a deformation of the XXZ spin chain including a total spin operator, an XYZ term, and a DMI term.

Camley & Livesey, Surf. Sci. Rep. 78 (2023)

Quasi-Integrable Models

- Each of the deformations separately preserves integrability, but together they are only integrable up to 1st order.
- This happens due to the fact that, for H_{XYZ}, rotations about the z-axis are no longer a symmetry.
- The term proportional to γ is called the anti-symmetric or Dzyaloshinskii–Moriya interaction (DMI) term. It results in a spiral structure that has been associated with magnetic skyrmions.

Camley & Livesey, Surf. Sci. Rep. 78 (2023)

R-matrix

- We verify that these deformations originate from a solution to the Yang-Baxter equation up to first order in ε, such that:

$$R = R_{XXZ} + \varepsilon \sinh(u-v) R^{(1)},$$

$$R^{(1)} = \begin{pmatrix} -2\beta\frac{\sinh(2v-\eta)}{\sinh(\eta)} & 0 & 0 & 2\alpha\frac{\sinh(u-v+\eta)}{\sinh(\eta)} \\ 0 & -2\beta\frac{\sinh(2u)+\sinh(2v)}{\sinh(\eta)} - 2i\gamma & 0 & 0 \\ 0 & 0 & 2i\gamma & 0 \\ 2\alpha\frac{\sinh(u-v+\eta)}{\sinh(\eta)} & 0 & 0 & -2\beta\frac{\sinh(2u+\eta)}{\sinh(\eta)} \end{pmatrix}.$$

- It is of non-difference form and satisfies braiding unitarity up to 1^{st} order in ε.

Non-difference Form

- One could propose to continue the expansion to higher orders, i.e.,

$$R = R^{(0)} + \varepsilon R^{(1)} + \varepsilon^2 R^{(2)},$$

  and try to solve it to second order.
- We obtain equations that do not involve the functions $R_{ij}^{(2)}$, but depend only on the entries of the Hamiltonian. In this way, we recover sub-cases of either Model A or Model B.

Spectral Statistics - Do these models exhibit different physics?

- Level Spacing: the difference between "unfolded" energy eigenvalues:

$$s_k = \epsilon_{k+1} - \epsilon_k$$

- Their probability distributions reveal information about the underlying physics of the model:

$$P_P(s) = e^{-s}, \qquad P_{GOE}(s) = \frac{1}{2}\pi s e^{-\frac{\pi}{4}s^2}$$

Integrable Chaotic

- How does H_{QInt} fit into this scenario? Compare H_{QInt} with an integrability-breaking model: H_{dXYZ}.

Onset of Chaos

- Do these two systems transition to chaos in different ways?
- To answer this, we introduce the Brody distribution:

$$P_{\text{Brody}}(s) = (\omega_B + 1)\Gamma\left(\frac{\omega_B + 2}{\omega_B + 1}\right)^{\omega_B+1} s^{\omega_B} e^{-\Gamma\left(\frac{\omega_B+2}{\omega_B+1}\right)^{\omega_B+1} s^{\omega_B+1}}$$

which recovers the Poisson distribution for $\omega_B = 0$ and the GOE Wigner-Dyson distribution for $\omega_B = 1$.

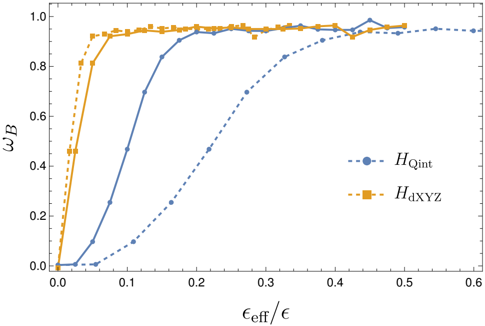

Figure 1: Brody parameter ω_B as a function of deformation strength for $L = 21$ in H_{dXYZ} with $J_x = 1.63$, $J_y = 0.73$, $J_z = 1.18$ and H_{QInt} with $J_z = 0.68$, $\alpha = 0.82$, $\beta = 1.43$, $\gamma = 1$. Dashed lines are for effective coupling.

Volume Dependence

- To analyze how the transition scales with the spin chain size L, we define the critical coupling ϵ_c such that:

$$\epsilon_c \implies \text{the distribution is midway between integrable and chaotic, i.e., } \omega_B = \frac{1}{2}.$$

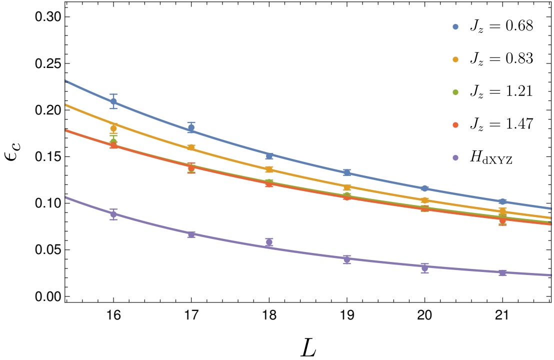

Figure 2: Scaling of critical coupling ϵ_c as a function of L for the dXYZ model (with $J_x = 1.63$, $J_y = 0.73$, $J_z = 1.18$) and the QInt model ($\alpha = 0.82$, $\beta = 1.43$, $\gamma = 1$) for different values of J_z.

For the QInt model with $J_z = 0.68$, fitting to a power-law scaling:

$$\epsilon_c = aL^{-b}$$

we find that

$$a_{\text{QInt}} = (3.1 \pm 0.5) \times 10^2 \,, \quad b_{\text{QInt}} = 2.64 \pm 0.06.$$

- Strong integrability breaking: $\epsilon_c \propto L^{-3}$;

  R. Modak & S. Mukerjee, 1310.1443

  R. Modak, S. Mukerjee & S. Ramaswamy, 1309.1865

- Weak integrability breaking: $\epsilon_c \propto L^{-2}$;

  D. Szász-Schagrin, B. Pozsgay & G. Takács, 2103.06308

  T. McLoughlin & A. Spiering, 2202.12075

Hubbard Model

- Let us consider the standard Hubbard model

$$\mathcal{H}^{\text{Hub}} = c^{\dagger}_{j,\downarrow}\, c_{j+1,\downarrow} + c^{\dagger}_{j+1,\downarrow}\, c_{j,\downarrow} + c^{\dagger}_{j,\uparrow}\, c_{j+1,\uparrow} + c^{\dagger}_{j+1,\uparrow}\, c_{j,\uparrow} \\ - \gamma(1-2n_{j,\uparrow})\,(1-2n_{j,\downarrow}) + \gamma\mathbb{1}$$

where γ is the coupling constant of the model, $c^{\dagger}_{j,\uparrow/\downarrow}$, $c_{j,\uparrow/\downarrow}$ are the creation and annihilation operators

$$\{c_{j,\alpha}, c_{k,\beta}\} = \left\{c^{\dagger}_{j,\alpha}, c^{\dagger}_{k,\beta}\right\} = 0, \\ \left\{c_{j,\alpha}, c^{\dagger}_{k,\beta}\right\} = \delta_{\alpha,\beta}\delta_{j,k},$$

where j, k refer to the local Hilbert spaces, $\alpha, \beta = \uparrow, \downarrow$ and n is the local number operator $n_{j,\alpha} = c^{\dagger}_{j,\alpha} c_{j,\alpha}$.

- Using a proper Jordan-Wigner transformation, one can map it onto a spin chain model defined in a local $\mathbb{C}^2 \otimes \mathbb{C}^2$ Hilbert space where

$$\sigma^i = \sigma^i \otimes \mathbb{1} \quad \text{and} \quad \tau^i = \mathbb{1} \otimes \sigma^i$$

such that we have

$$\mathcal{H}^{\text{Hub}} = \gamma\left(\mathbb{1} - \sigma_j^z \tau_j^z\right) + \sigma_{j+1}^- \sigma_j^+ + \sigma_j^- \sigma_{j+1}^+ + \tau_{j+1}^- \tau_j^+ + \tau_j^- \tau_{j+1}^+.$$

- Then, we use the same machinery to determine possible integrable deformations;
- Higher Complexity: a full deformation contains 256 degrees of freedom;
- Ansatz: requiring the deformation to preserve the fermion number reduces the parameter space to 128 d.o.f.;

First-Order Integrable Deformations

At first order, we identify a 16-parameter non-trivial deformation, split into three distinct physical contributions:

$$\mathcal{H}^{(1)}_{\text{int}} = \mathcal{H}^{(1)}_{\alpha} + \mathcal{H}^{(1)}_{\beta} + \mathcal{H}^{(1)}_{\text{Q}}$$

- $\mathcal{H}^{(1)}_{\alpha}$ (Number & Jump Interactions – 6 parameters): Governs standard hopping and local interactions. It acts as an inherently integrable block.
- $\mathcal{H}^{(1)}_{\beta}$ (Creation/Annihilation & Flip-Jump – 4 parameters): Describes the simultaneous creation or annihilation of particle pairs, alongside spin-flip hopping dynamics. This component is also inherently integrable.

Second-Order Constraints & Full Integrability

- $\mathcal{H}_{\mathrm{Q}}^{(1)}$ (Weighted-Jump Interactions – 6 parameters): This term contains density-dependent hopping terms. Unconstrained, this term only provides a quasi-integrable deformation.
- Full Integrability:
  - Pushing the analysis to second order imposes constraints exclusively on $\mathcal{H}_{\mathrm{Q}}^{(1)}$.
  - These constraints reduce its free parameters to 4 and branch the system into 3 distinct, fully integrable models (Models A, B, and C).
- Summary: While $\mathcal{H}_{\alpha}^{(1)}$ and $\mathcal{H}_{\beta}^{(1)}$ are integrable deformations by themselves, $\mathcal{H}_{\mathrm{Q}}^{(1)}$ requires higher-order constraints to select which exact integrable model is realized.

Summary

- The Boost Operator Method provides a systematic, order-by-order classification of nearest-neighbour deformations in integrable models.
- Applied to the XXZ spin chain, it distinguishes exactly integrable deformations from Quasi-Integrable models (where integrability holds up to a finite order).
- Extended to the Hubbard model: we systematically identified a 16-parameter deformation, classifying its constraints into fully and quasi-integrable regimes.
- Spectral statistics reveal the onset of quantum chaos in these models scales as $\epsilon_c \propto L^{-2.64}$, placing it between standard weak and strong integrability breaking.

Outlook & Future Directions

- Phenomenology of XXZ Deformations: Investigate the physical implications of quasi-integrability, focusing on the DMI term and its connection to magnetic skyrmions.
- Full Hubbard Classification: Systematically extend our analysis to the complete Hubbard deformation space, encompassing all 256 degrees of freedom.
- Twisted Hubbard Models: Explore deformations associated with "Twisted Hubbard Model" that is associated to AdS/CFT and open quantum systems (Lindbladian dynamics).
- Drinfeld Twists: Compare our algebraic deformation results with the exact deformations generated by Drinfeld twists.

Thank you!

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FAPESP

Part II

Appendix

Considering a general deformation, we solve the integrability condition at first order, obtaining two distinct cases (along with constraints for $g^{(1)}$).

- ⋆ Integrability-breaking deformations:
  - 2 sums of spins in the x, y directions;
  - 2 differences of mixed spin terms in the yz, xz directions;

$$\boxed{\begin{aligned}\mathcal{H}^{(1)}_{\mathsf{BInt}} = {} & \delta^{(i)}_{x,+}\ (\mathbb{1}\otimes\sigma_x+\sigma_x\otimes\mathbb{1})+\delta^{(i)}_{y,+}\ (\mathbb{1}\otimes\sigma_y+\sigma_y\otimes\mathbb{1})\\ & +h^{(i)}_{xz,-}\ (\sigma_x\otimes\sigma_z-\sigma_z\otimes\sigma_x)+h^{(i)}_{yz,-}\ (\sigma_y\otimes\sigma_z-\sigma_z\otimes\sigma_y)\,.\end{aligned}}$$

GOE × Poisson

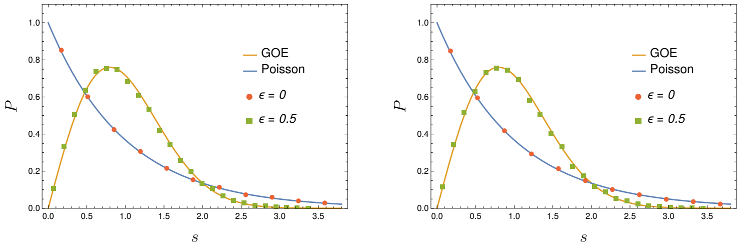

Figure 3: Level-spacing distributions for integrable ($\epsilon = 0.0$) and chaotic ($\epsilon = 0.5$) regimes in an $L = 21$ chain for (Left) H_{QInt} with $J_z = 0.68$, $\alpha = 0.82$, $\beta = 1.43$, $\gamma = 1$. (Right) H_{dXYZ} with $J_x = 1.63$, $J_y = 0.73$, $J_z = 1.18$.

R-matrices

- Integrable deformations: one can find a solution to any order in ε, already classified as a regular 4×4 R-matrix;
- Integrability-breaking deformations: it is not possible to find an R-matrix, as expected;
- Quasi-Integrable Models: R-matrix that respects the YBE up to a certain order in ε.

Non-Difference Form

- $\beta \neq 0 \rightarrow$ non-trivial $\mathcal{G} \rightarrow R$-matrix is of non-difference form:

$$R(u, v) \quad \text{cannot be written as} \quad R(u - v).$$

- The coefficients of the Hamiltonian generically depend on the spectral parameter. In this case, it is convenient to construct the Lax matrix instead.

Non-Difference Form

- We propose a correction to the original Lax:

$$\mathcal{L}_{QInt}(u) = \mathcal{L}_{XXZ}(u) + \varepsilon\mathcal{P}\left(\sinh(u)\mathcal{H}^{(1)} - \Lambda(u)\right),$$

where $\mathcal{P}$ is the permutation operator and $\Lambda(u)$ is the correction term that we need to determine, such that $\Lambda(0) = \Lambda'(0) = 0$.

Non-Difference Form

From this Lax operator, we can compute the corresponding transfer matrix

$$tr(u) = Tr_a\left(\mathcal{L}_{an} \cdots \mathcal{L}_{a1}\right),$$

and demand that it commutes with the Hamiltonian:

$$[\mathbb{Q}_2, tr(u)] = 0.$$

This is enough to fix $\Lambda(u)$.

Non-Difference Form

Following this procedure, we are led to the solution

$$\Lambda = \sinh(u) \begin{pmatrix} 0 & 0 & 0 & 2\alpha\left(1-\frac{\sinh(u+\eta)}{\sinh(\eta)}\right) \\ 0 & 2\beta\frac{\sinh(2u)}{\sinh(u)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\alpha\left(1-\frac{\sinh(u+\eta)}{\sinh(\eta)}\right) & 0 & 0 & 2\beta\frac{\sinh(u)}{\sinh(\eta)}\cosh(\eta+u) \end{pmatrix}.$$

Non-Difference Form

Next, we substitute this result into the RLL equation:

$$R_{aa'}(u,v)\mathcal{L}_a(u)\mathcal{L}_{a'}(v) = \mathcal{L}_{a'}(v)\mathcal{L}_a(u)R_{aa'}(u,v),$$

to find an R-matrix up to order ε that satisfies it:

$$R = R_{XXZ} + \varepsilon \sinh(u-v)R^{(1)}.$$

Non-Difference Form

The result is given by

$$R^{(1)} = \begin{pmatrix} -2\beta\frac{\sinh(2v-\eta)}{\sinh(\eta)} & 0 & 0 & 2\alpha\frac{\sinh(u-v+\eta)}{\sinh(\eta)} \\ 0 & -2\beta\frac{\sinh(2u)+\sinh(2v)}{\sinh(\eta)} - 2i\gamma & 0 & 0 \\ 0 & 0 & 2i\gamma & 0 \\ 2\alpha\frac{\sinh(u-v+\eta)}{\sinh(\eta)} & 0 & 0 & -2\beta\frac{\sinh(2u+\eta)}{\sinh(\eta)} \end{pmatrix}.$$

- ⋆ Non-difference form, satisfies the YBE, and has braiding unitarity up to 1[st] order in ε.

Non-difference Form

- One could propose to continue the expansion to higher orders, i.e.,

$$R = R^{(0)} + \varepsilon R^{(1)} + \varepsilon^2 R^{(2)},$$

  and try to solve it to second order.

- We obtain equations that do not involve the functions $R_{ij}^{(2)}$ but only depend on the entries of the Hamiltonian. In this way, we recover either sub-cases of Model A or Model B.

- At first order, we find the following 16-parameter non-trivial integrable deformation for the Hubbard model:

$$\mathcal{H}^{(1)}_{\text{int}} = \mathcal{H}^{(1)}_{\alpha} + \mathcal{H}^{(1)}_{\beta} + \mathcal{H}^{(1)}_{\text{Q}},$$

where:

- $\mathcal{H}^{(1)}_{\alpha}$: Number operator and jump interactions - 6 parameters

$$\begin{aligned}\mathcal{H}^{(1)}_{\alpha} = \; & 2h_1 \left(1 - (n_{j+1,\downarrow} + n_{j,\downarrow})\right) + h_2 \left(1 - (n_{j+1,\uparrow} + n_{j,\uparrow})\right) \\ & + h_3 \left[(1 - 2n_{j,\uparrow})\,(1 - 2n_{j,\downarrow}) + (1 - 2n_{j+1,\uparrow})\,(1 - 2n_{j+1,\downarrow})\right] \\ & + (h_6 - h_4)\, c^{\dagger}_{j,\uparrow}\, c_{j+1,\uparrow} + (h_4 + h_6)\, c^{\dagger}_{j+1,\uparrow}\, c_{j,\uparrow} \\ & + (h_6 - h_5)\, c^{\dagger}_{j,\downarrow}\, c_{j+1,\downarrow} + (h_5 + h_6)\, c^{\dagger}_{j+1,\downarrow}\, c_{j,\downarrow}\end{aligned}$$

- $\mathcal{H}_{\beta}^{(1)}$: Double creation/annihilation and flip-jump interactions - 4 parameters

$$
\begin{aligned}
\mathcal{H}_{\beta}^{(1)} = \; & 2h_7 \left[c_{j,\downarrow}\, c_{j+1,\uparrow} \left(1 - (n_{j+1,\downarrow} + n_{j,\uparrow})\right) - c_{j,\uparrow}\, c_{j+1,\downarrow} \left(1 - (n_{j+1,\uparrow} + n_{j,\downarrow})\right) \right] \\
& + 2h_8 \left[c^{\dagger}_{j,\uparrow}\, c^{\dagger}_{j+1,\downarrow} \left(1 - (n_{j+1,\uparrow} + n_{j,\downarrow})\right) - c^{\dagger}_{j,\downarrow}\, c^{\dagger}_{j+1,\uparrow} \left(1 - (n_{j+1,\downarrow} + n_{j,\uparrow})\right) \right] \\
& + 2h_9 \left[c^{\dagger}_{j+1,\uparrow}\, c_{j,\downarrow} \left(n_{j,\uparrow} - n_{j+1,\downarrow}\right) + c^{\dagger}_{j,\uparrow}\, c_{j+1,\downarrow} \left(n_{j,\downarrow} - n_{j+1,\uparrow}\right) \right] \\
& + 2h_{10} \left[c^{\dagger}_{j,\downarrow}\, c_{j+1,\uparrow} \left(n_{j+1,\downarrow} - n_{j,\uparrow}\right) + c^{\dagger}_{j+1,\downarrow}\, c_{j,\uparrow} \left(n_{j+1,\uparrow} - n_{j,\downarrow}\right) \right] .
\end{aligned}
$$

* Weighted-jump interactions - 6 parameters:

$$\begin{aligned}
\mathcal{H}_{\mathrm{Q}}^{(1)} = {} & 2h_{11}\left(c_{j,\downarrow}^{\dagger}\, c_{j+1,\downarrow} - c_{j+1,\downarrow}^{\dagger}\, c_{j,\downarrow}\right)\left(n_{j+1,\uparrow} - n_{j,\uparrow}\right) \\
& + h_{12}\left(c_{j+1,\uparrow}^{\dagger}\, c_{j,\uparrow}\,(1-2n_{j+1,\downarrow}) - c_{j,\uparrow}^{\dagger}\, c_{j+1,\uparrow}\,(1-2n_{j,\downarrow})\right) \\
& + h_{13}\left(c_{j+1,\downarrow}^{\dagger}\, c_{j,\downarrow}(1-2n_{j+1,\uparrow}) - c_{j,\downarrow}^{\dagger}\, c_{j+1,\downarrow}(1-2n_{j,\uparrow})\right) \\
& + 2h_{14}\left(c_{j,\uparrow}^{\dagger}\, c_{j+1,\uparrow} - c_{j+1,\uparrow}^{\dagger}\, c_{j,\uparrow}\right)\left(n_{j+1,\downarrow} - n_{j,\downarrow}\right) \\
& + h_{15}\left(c_{j+1,\uparrow}^{\dagger}\, c_{j,\uparrow}\,(1-2n_{j,\downarrow}) - c_{j,\uparrow}^{\dagger}\, c_{j+1,\uparrow}\,(1-2n_{j+1,\downarrow})\right) \\
& + h_{16}\left(c_{j+1,\downarrow}^{\dagger}\, c_{j,\downarrow}(1-2n_{j,\uparrow}) - c_{j,\downarrow}^{\dagger}\, c_{j+1,\downarrow}(1-2n_{j+1,\uparrow})\right)
\end{aligned}$$

Going to second order, we obtain new constraints ***only*** for $\mathcal{H}_{\mathrm{Q}}^{(1)}$, yielding 3 distinct models that are fully integrable by adding higher-order deformations:

- Model A:

$$h_{16} = h_{13}, \quad h_{15} = h_{12};$$

- Model B:

$$h_{16} = h_{13}, \quad h_{14} = h_{11};$$

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- Model C:

$$h_{14} = h_{11}, \quad h_{15} = h_{12};$$

- $\mathcal{H}_{\alpha}^{(1)}, \mathcal{H}_{\beta}^{(1)}$: Integrable deformations;
- $\mathcal{H}_{Q}^{(1)}$ ***without*** additional constraints (6 parameters): Quasi-Integrable deformation;
- $\mathcal{H}_{Q}^{(1)}$ ***with*** additional constraints (4 parameters): 3 different types of integrable deformations.