On boundary overlaps in open XXZ spin chains

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based on joined work with C. Abetian and N. Kitanine (diagonal case)
$+$ special case of non-diag. b. c. with G. Niccoli

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The open XXZ spin chain with boundary fields

$$H_{h_+,h_-} = \sum_{m=1}^{L-1} \left[\sigma_m^x\sigma_{m+1}^x + \sigma_m^y\sigma_{m+1}^y + \Delta\, \sigma_m^z\sigma_{m+1}^z\right] + \sum_{a\in\{x,y,z\}} \left[h_+^a\sigma_1^a + h_-^a\sigma_L^a\right]$$

- space of states: $\mathcal{H} = \otimes_{n=1}^{L}\mathcal{H}_n$ with $\mathcal{H}_n \simeq \mathbb{C}^2$
- $\sigma_m^{x,y,z} \in \mathrm{End}(\mathcal{H}_n)$: local spin-1/2 operators (Pauli matrices) at site m
- anisotropy parameter $\Delta = \cosh\eta$

- simplest case : longitudinal boundary fields $h_\pm^{x,y} = 0$

  $h_\pm^z = \sinh\eta\, \coth\xi_\pm$

  $\longrightarrow$ solvable by (boundary version of) ABA

- more general case : boundary fields $h_\pm^{x,y,z}$ parametrised in terms of 6 boundary parameters $\xi_\pm, \kappa_\pm, \tau_\pm$, or alternatively $\varphi_\pm, \psi_\pm, \tau_\pm$:

$$h_\pm^x = 2\kappa_\pm\, \sinh\eta\, \frac{\cosh\tau_\pm}{\sinh\xi_\pm}, \quad h_\pm^y = 2i\kappa_\pm\, \sinh\eta\, \frac{\sinh\tau_\pm}{\sinh\xi_\pm}, \quad h_\pm^z = \sinh\eta\, \coth\xi_\pm$$

$$\sinh\varphi_\pm\, \cosh\psi_\pm = \frac{\sinh\xi_\pm}{2\kappa_\pm}, \quad \cosh\varphi_\pm\, \sinh\psi_\pm = \frac{\cosh\xi_\pm}{2\kappa_\pm}$$

$\longrightarrow$ more complicated (no direct ABA solution. . .)

Boundary quench

Quench: dynamics of a system after abrupt change of one parameter.

We change **one** boundary field $h_+ \longrightarrow \widetilde{h}_+$. Local change, but can drastically modify the ground state (globally).

Question : compute overlaps $\langle \Psi | \widetilde{\Psi} \rangle$ for $| \Psi \rangle$ eigenstate of $H = H_{h_+,h_-}$ and $| \widetilde{\Psi} \rangle$ eigenstate of $\widetilde{H} = H_{\widetilde{h}_+,h_-}$

The most basic overlap: normalised scalar product of ground states:

$$|\mathcal{F}|^2 = \frac{\langle \Psi | \widetilde{\Psi} \rangle \langle \widetilde{\Psi} | \Psi \rangle}{\langle \Psi | \Psi \rangle \langle \widetilde{\Psi} | \widetilde{\Psi} \rangle}$$

with $| \Psi \rangle$, $| \widetilde{\Psi} \rangle$: ground states before and after the change of field

Gives for example the dominant term for the Loschmidt echo (dynamics of the initial state after the change of the boundary magnetic field):

$$\mathcal{L}(t) = \left| \langle \Psi | e^{-i\widetilde{H}t} | \Psi \rangle \right|^2$$

Algebraic framework: the QISM for the open chain (1)

The open XXZ spin chain is solvable in the framework of the representation theory of the reflection algebra (or boundary Yang-Baxter algebra) [Cherednik 84,Sklyanin 88]

- generators $\mathcal{U}_{ij}(\lambda)$, $1 \le i,j \le 2$ $\leftarrow$ elements of the boundary monodromy matrix $\mathcal{U}(\lambda)$
- relations given by the reflection equation:

$$R_{12}(\lambda-\mu)\,\mathcal{U}_1(\lambda)\,R_{12}(\lambda+\mu-\eta)\,\mathcal{U}_2(\mu) = \mathcal{U}_2(\mu)\,R_{12}(\lambda+\mu-\eta)\,\mathcal{U}_1(\lambda)\,R_{12}(\lambda-\mu)$$

where R is the 6-vertex R-matrix

$\hookrightarrow$ most general 2×2 trigonometric solution of the refl. eq [de Vega, Gonzalez-Ruiz; Ghoshal, Zamolodchikov 93] :

$$K(\lambda;\xi,\kappa,\tau) = \frac{1}{\sinh\xi}\begin{pmatrix} \sinh(\lambda-\frac{\eta}{2}+\xi) & \kappa\, e^{\tau}\sinh(2\lambda-\eta) \\ \kappa\, e^{-\tau}\sinh(2\lambda-\eta) & \sinh(\xi-\lambda+\frac{\eta}{2}) \end{pmatrix}$$

$\rightsquigarrow$ boundary matrices $K_+(\lambda)\equiv K(\lambda+\eta/2;\xi_+,\kappa_+,\tau_+)$ and $K_-(\lambda)\equiv K(\lambda-\eta/2;\xi_-,\kappa_-,\tau_-)$ describing left/right boundary fields:

$$h^x_\pm = 2\kappa_\pm \sinh\eta\,\frac{\cosh\tau_\pm}{\sinh\xi_\pm},\quad h^y_\pm = 2i\kappa_\pm \sinh\eta\,\frac{\sinh\tau_\pm}{\sinh\xi_\pm},\quad h^z_\pm = \sinh\eta\,\coth\xi_\pm$$

Algebraic framework: the QISM for the open chain (2)

The boundary monodromy matrix $\mathcal{U}(\lambda) \equiv \mathcal{U}_-(\lambda)$ solution of the reflection equation is explicitly constructed

- from the bulk monodromy matrix:

$$T(\lambda) = R_{01}(\lambda - \xi_1 - \eta/2) \dots R_{0L}(\lambda - \xi_L - \eta/2) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

- from the boundary K-matrix $K_-(\lambda)$ encoding the right boundary field h_-

as

$$\mathcal{U}_-(\lambda) = T(\lambda)\, K_-(\lambda)\, \hat{T}(\lambda) = \begin{pmatrix} \mathcal{A}_-(\lambda) & \mathcal{B}_-(\lambda) \\ \mathcal{C}_-(\lambda) & \mathcal{D}_-(\lambda) \end{pmatrix}$$

with $\hat{T}(\lambda) \propto \sigma^y\, T^t(-\lambda)\, \sigma^y$

Remark 1: $\mathcal{U}_-$ does not depend on h_+

Remark 2: One can alternatively use $\mathcal{U}_+$ constructed from the boundary matrix K_+ which does not depend on h_-

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$\rightsquigarrow$ **transfer matrix:**

$$\mathcal{T}(\lambda) = \operatorname{tr}\{K_+(\lambda)\mathcal{U}_-(\lambda)\} \quad \text{such that} \quad [\mathcal{T}(\lambda), \mathcal{T}(\mu)] = 0$$

$\rightsquigarrow$ **Hamiltonian** (in the homogeneous limit $\xi_1 = \ldots = \xi_L = 0$):

$$H_{h_+,h_-} = c_1 \frac{d}{d\lambda} \log \mathcal{T}(\lambda)\Big|_{\lambda=\eta/2} + c_2$$

Solution by ABA in the diagonal case

When both boundary matrices $K_\pm$ are diagonal ($\kappa_\pm = 0$, i.e. boundary fields along σ_1^z and σ_L^z only):

- the bulk reference state $|\,0\,\rangle = |\uparrow\uparrow\ldots\uparrow\rangle$ can still be used to construct the eigenstates as Bethe states in the ABA framework [Sklyanin 88] :

$$|\,\{\lambda\}\,\rangle = \prod_{k=1}^{M} \mathcal{B}_-(\lambda_k)\,|\,0\,\rangle \in \mathcal{H}, \qquad \langle\,\{\lambda\}\,| = \langle\,0\,|\prod_{k=1}^{M} \mathcal{C}_-(\lambda_k) \in \mathcal{H}^*$$

  are eigenstates of $\mathcal{T}(\mu)$ if $\{\lambda\}$ satisfy the system of Bethe equations

$$\mathfrak{a}_{\boldsymbol{\lambda}}(\lambda_j) = 1, \qquad j = 1, \ldots, M,$$

  in which $\mathfrak{a}_{\boldsymbol{\lambda}}$ is the exponential counting function:

$$\mathfrak{a}_{\boldsymbol{\lambda}}(u) = -\frac{\mathbf{A}(u)}{\mathbf{A}(-u)}\,\frac{Q_{\boldsymbol{\lambda}}(u-\eta)}{Q_{\boldsymbol{\lambda}}(u+\eta)} \quad \text{with} \quad Q_{\boldsymbol{\lambda}}(u) = \prod_{k=1}^{M} \sinh(u-\lambda_k)\sinh(u+\lambda_k)$$

  Corresponding eigenvalue:

$$\tau_{\boldsymbol{\lambda}}(u) = \mathbf{A}(u)\,\frac{Q_{\boldsymbol{\lambda}}(u-\eta)}{Q_{\boldsymbol{\lambda}}(u)} + \mathbf{A}(-u)\,\frac{Q_{\boldsymbol{\lambda}}(u+\eta)}{Q_{\boldsymbol{\lambda}}(u)}$$

- $\exists$ generalization of Slavnov's determinant representation for the scalar products of Bethe states $\langle\{\mu\}_{\text{off-shell}}|\{\lambda\}_{\text{on-shell}}\rangle$ [Tsuchiya 98; Wang 02]

Boundary quench and overlaps in the diagonal case

Consider the open XXZ chain with longitudinal boundary fields h_+, h_-, in the massive regime $\Delta = \cosh\zeta > 1$ ($\eta = -i\zeta$, $\zeta > 0$)

$\rightsquigarrow$ boundary quench: change the magnetic field at site 1

$$h_+ \longrightarrow \widetilde{h}_+$$

while keeping Δ and h_- fixed.

The change of h_+ leaves the boundary monodromy matrix $\mathcal{U}_-$ unchanged

$\rightsquigarrow$ the two sets of Bethe states before and after the quench are constructed from the same operators $\mathcal{B}_-$

$\rightsquigarrow$ one can use the boundary version of Slavnov's determinant representation for scalar products of Bethe states to compute the overlaps

Overlaps in the diagonal case (1)

Let $|\{\lambda\}\rangle$ be an eigenstate of H with Bethe roots satisfying $\mathfrak{a}_{\boldsymbol{\lambda}}(\lambda_j) = 1$
and $|\{\mu\}\rangle$ be an eigenstate of $\widetilde{H}$ with Bethe roots satisfying $\widetilde{\mathfrak{a}}_{\boldsymbol{\mu}}(\mu_j) = 1$

The normalized overlap can therefore be computed as a ratio of (boundary version of) Slavnov and Gaudin determinants:

$$S(\{\lambda\}|\{\mu\}) \equiv \frac{\langle\{\mu\}|\{\lambda\}\rangle\langle\{\lambda\}|\{\mu\}\rangle}{\langle\{\lambda\}|\{\lambda\}\rangle\langle\{\mu\}|\{\mu\}\rangle} = \prod_{j=1}^{M} \frac{Q_\lambda(\mu_j)Q_\mu(\lambda_j)}{Q_\lambda(\lambda_j)Q_\mu(\mu_j)} \cdot \frac{\det \mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{\mu}} \det \mathcal{M}_{\boldsymbol{\mu},\boldsymbol{\lambda}}}{\det \mathcal{N}_{\boldsymbol{\lambda}} \det \mathcal{N}_{\boldsymbol{\mu}}}$$

- Slavnov matrix:

$$\left[\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{\mu}}\right]_{jk} = \mathfrak{a}_{\boldsymbol{\lambda}}(\mu_k)\big[t(-\mu_k+\lambda_j) - t(-\mu_k-\lambda_j)\big] + t(\mu_k-\lambda_j) - t(\mu_k+\lambda_j)$$

- Gaudin matrix: $\left[\mathcal{N}_{\boldsymbol{\lambda}}\right]_{jk} = \mathfrak{a}'_\lambda(\lambda_j)\delta_{jk} - K(\lambda_j-\lambda_k) + K(\lambda_j+\lambda_k)$

with $\quad t(\lambda) = \frac{i\sinh\zeta}{\sin\lambda\,\sin(\lambda-i\zeta)} \qquad K(\lambda) = t(\lambda) + t(-\lambda).$

Goal: thermodynamic limit $L \to \infty$ of the overlaps in the massive antiferromagnetic regime [Abetian, Kitanine, VT: SciPost Phys 18, 026 (2025) + in preparation]

- ground state – ground state overlaps
- excited states with holes

Overlaps in the diagonal case (2)

$\longrightarrow$ system of linear equations for the ratio of the Slavnov and Gaudin determinants $F_\lambda = \mathcal{N}_{\boldsymbol{\lambda}}^{-1}\mathcal{M}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ (cf [Kitanine, Kulkarni 19]):

$$i\mathfrak{a}'_\lambda(\lambda_j)\,[F_\lambda]_{j,k} - \sum_{a=1}^{M}\left[K(\lambda_j-\lambda_a) - K(\lambda_j+\lambda_a)\right][F_\lambda]_{a,k}$$
$$= \mathfrak{a}_\lambda(\mu_k)\,[t(\lambda_j-\mu_k) - t(-\lambda_j-\mu_k)] - t(\mu_k-\lambda_j) - t(\mu_k+\lambda_j)$$

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$\longrightarrow$ contour integral equation for a meromorphic function:

$$G_\lambda(\lambda;\mu_k) - \frac{1}{2\pi}\oint_\Gamma d\nu\, K(\lambda-\nu)\,\frac{G_\lambda(\nu;\mu_k)}{\mathfrak{a}_\lambda(\nu)-1} = r.h.s.(\lambda,\mu_k)$$

with $G_\lambda(\lambda_j,\mu_k) = i\mathfrak{a}'_\lambda(\lambda_j)\,[F_\lambda]_{j,k}$

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with $G_\lambda(\lambda_j,\mu_k) = i\mathfrak{a}'_\lambda(\lambda_j)\,[F_\lambda]_{j,k}$

$\longrightarrow$ Integral equation in the thermodynamic limit

$$G_\lambda(\lambda;\mu) + \frac{1}{2\pi i}\int_{-\pi/2+i0}^{\pi/2+i0} d\nu\, K(\lambda-\nu)\, G_\lambda(\nu;\mu) = r.h.s.(\lambda,\mu).$$

cf. Lieb equation for the density of Bethe roots!

$\longrightarrow$ Solution: elliptic Cauchy determinant (if no holes, $q = e^{-\zeta}$)

$$\boxed{[F_\lambda]_{j,k} = \frac{\mathfrak{a}_\lambda(\mu_k)-1}{\mathfrak{a}'_\lambda(\lambda_j)}\cdot\frac{(q^2,q^2)_\infty}{(-q^2,q^2)_\infty}\left(\frac{\vartheta_2(\lambda_j-\mu_k,q)}{\vartheta_1(\lambda_j-\mu_k,q)} + \frac{\vartheta_2(\mu_k+\lambda_j,q)}{\vartheta_1(\mu_k+\lambda_j,q)}\right)}$$

Ground state – ground state overlap

$\longrightarrow$ Product form for the overlap (up to exponentially small correction in L):

$$\boxed{S(\{\lambda\}|\{\mu\}) = \prod_{j=1}^{M} \frac{\chi(\lambda_j)}{\chi(\mu_j)} \prod_{j=1}^{M}\prod_{k=1}^{M} \frac{\varphi(\lambda_j-\lambda_k)\varphi(\mu_j-\mu_k)\varphi(\lambda_j+\lambda_k)\varphi(\mu_j+\mu_k)}{\varphi^2(\lambda_j-\mu_k)\varphi^2(\lambda_j+\mu_k)}}$$

in which $\qquad \chi(u) = \dfrac{\widetilde{\tau}_\mu(u)}{\tau_\lambda(u)}, \qquad \varphi(\lambda) = \dfrac{\vartheta_1(\lambda, q)}{\sin\lambda}.$

It remains to specify the two states and evaluate the products in the thermodynamic limit.

In the massive antiferromagnetic regime: different sub-regimes according to the value of $h_+ = -\sinh\zeta\,\coth\xi_+$ (first site) and $h_- = -\sinh\zeta\,\coth\xi_-$ (last site) with respect to two critical fields

$$h_{cr}^{(1)} = \Delta - 1, \qquad h_{cr}^{(2)} = \Delta - 1.$$

$\rightsquigarrow$ the ground-state Bethe-roots may be composed in relevant cases of

- only real roots
- real roots + one complex boundary root of the form:
  $\lambda_{\mathrm{BR}}^{\sigma} = -i(\zeta/2 + \xi_\sigma) + O(L^{-\infty}) \quad \sigma = \pm$

Ground state – ground state overlap: final result

Depending on the respective configurations of $h_+ = -\sinh\zeta\,\coth\xi_+$ (pre-quench field), $\widetilde{h}_+ = -\sinh\zeta\,\coth\widetilde{\xi}_+$ (post-quench field) and h_-:

either $\boxed{S(\{\lambda\},\{\mu\}) = O(L^{-\infty})}$ or

$$\boxed{S(\{\lambda\},\{\mu\}) = \frac{(p^{2\epsilon}q^2;q^4,q^4)_\infty\,(\widetilde{p}^{2\epsilon}q^2;q^4,q^4)_\infty\,((p\widetilde{p})^\epsilon q^4;q^4,q^4)^2_\infty}{(p^{2\epsilon}q^4;q^4,q^4)_\infty\,(\widetilde{p}^{2\epsilon}q^4;q^4,q^4)_\infty\,((p\widetilde{p})^\epsilon q^2;q^4,q^4)^2_\infty} + O(L^{-\infty})}$$

where $p = e^{-2\xi_+}$, $\widetilde{p} = e^{-2\widetilde{\xi}_+}$, $q = e^{-\zeta}$, and $\epsilon = \pm$.
Here $(x;q_1,q_2)_\infty$ denotes the double q-Pochhammer symbol defined as

$$(x;q_1,q_2)_\infty = \prod_{n_1,n_2=0}^{\infty}(1 - xq_1^{n_1}q_2^{n_2}).$$

- ⇝ good agreement with numerics (exact diagonalisation with small L)
- ⇝ can also be obtained from q-vertex operators (result of R. Weston)

Overlap of excited states with holes

Consider excited states with a finite number of holes in the real-root distribution, excluding additional complex roots for simplicity.

For states with hole rapidities $\{\lambda_{h_1}, \ldots, \lambda_{h_n}\}$ and $\{\mu_{h_1}, \ldots, \mu_{h_m}\}$,

$$|\mathcal{F}(\{\lambda_h\}, \{\mu_h\})|^2 = \frac{\langle \Psi(\{\lambda_h\}) | \widetilde{\Psi}(\{\mu_h\}) \rangle \langle \widetilde{\Psi}(\{\mu_h\}) | \Psi(\{\lambda_h\}) \rangle}{\langle \Psi(\{\lambda_h\}) | \Psi(\{\lambda_h\}) \rangle \langle \widetilde{\Psi}(\{\mu_h\}) | \widetilde{\Psi}(\{\mu_h\}) \rangle}.$$

Final result: (product form) $\times$ (finite-size determinants (size $=$ number of holes))

$$\boxed{|\mathcal{F}|^2 = \underbrace{S_0(\{\lambda_h\}|\{\mu_h\})}_{\text{similar product form as before}} \cdot \frac{\det_n(\delta_{jk} - \mathcal{I}(\lambda_{h_j}, \lambda_{h_k}))\, \det_m(\delta_{jk} - \widetilde{\mathcal{I}}(\mu_{h_j}, \mu_{h_k}))}{\det_n(\delta_{jk} + \rho_h(\lambda_{h_j}, \lambda_{h_k}))\, \det_m(\delta_{jk} + \widetilde{\rho}_h(\mu_{h_j}, \mu_{h_k}))}}$$

The non-diagonal case

Description of the spectrum:

- It is possible to generalize usual Bethe ansatz equations to the open XXZ chain with non-longitudinal boundary fields with one constraint on the boundary parameters $\varphi_\pm, \psi_\pm, \tau_\pm$ [Nepomechie 03] :

$$\cosh(\tau_+ - \tau_-) \\ = \epsilon_+\epsilon_- \cosh(\epsilon_+(\varphi_+ - \psi_+) + \epsilon_-(\varphi_- + \psi_-) + (L - 2M - 1)\eta)$$

  with $M \in \mathbb{N}$ (numbers of Bethe roots), $\epsilon_\pm \in \{+, -\}$

  $\rightsquigarrow$ incomplete in general (except for $M = L$)

  $\rightsquigarrow$ the solutions for $(M, \epsilon_+, \epsilon_-)$ together with the solutions for $(M' = L - M - 1, -\epsilon_+, -\epsilon_-)$ should produce the complete spectrum [Nepomechie, Ravanini 03]

- most general boundaries ?

  $\exists$ description in terms of inhomogeneity parameters/discrete T-Q equations (for inhomogeneous models) but no known description in terms of usual Bethe equations

  Alternative proposals: Bethe equations with an additional term (Off-diagonal Bethe Ansatz...) [Cao et al 13...] or use transfer matrix roots [Qiao et al 21...]

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The non-diagonal case

Construction of the transfer matrix eigenstates ?

- Under the constraint, construction of some Bethe states by means of a Vertex-IRF transformation [Fan et al. 96; Cao et al 03; Yang, Zhang 07; Filali, Kitanine 11] but problems in the ABA construction of "compatible" sets of Bethe states in $\mathcal{H}$ and $\mathcal{H}^*$

  $\rightsquigarrow$ scalar products and correlation functions could not be computed

- Alternative methods of construction for general boundaries:
  - Modified Bethe Ansatz [Belliard et al 13...]
  - Separation of Variables [Frahm et al 10, Niccoli 12, Faldella et al 13...]
    In particular : connexion to generalized Bethe Ansatz (states and T-Q/Bethe equations) under the constraint
    + computation of the scalar products

Solution by SoV in the general case

Goal: identify a basis $\{|\,\mathbf{h}\,\rangle\}_{\mathbf{h}\in\{0,1\}^L}$ of $\mathcal{H}$ and $\{\langle\,\mathbf{h}\,|\}_{\mathbf{h}\in\{0,1\}^L}$ of $\mathcal{H}^*$, with

$\langle\,\mathbf{h}\,|\,\mathbf{k}\,\rangle \propto \frac{\delta_{\mathbf{h},\mathbf{k}}}{V_{\mathbf{h}}(\boldsymbol{\xi})}$

which "separates the variables" for the transfer matrix spectral problem:

$$\mathcal{T}(\lambda)\,|\,\Psi_\tau\,\rangle = \tau(\lambda)\,|\,\Psi_\tau\,\rangle \quad \text{with} \quad |\,\Psi_\tau\,\rangle = \sum_{\mathbf{h}\in\{0,1\}^L} \psi_\tau(\mathbf{h})\,|\,\mathbf{h}\,\rangle,$$

is solved by $\quad \psi_\tau(\mathbf{h}) = \prod_{n=1}^{L} Q_\tau(\xi_n^{(h_n)}) \cdot V_{\mathbf{h}}(\boldsymbol{\xi})$

where Q_τ and τ are solution of a discrete version of Baxter's T-Q equation:

$$\tau(x)\,Q_\tau(x) = \mathbf{A}(x)\,Q(x+\eta) + \mathbf{A}(-x)\,Q_\tau(x-\eta), \quad x \in \cup_{n=1}^{L}\{\xi_n^{(0)}, \xi_n^{(1)}\}$$

- $\rightarrow$ can be constructed by a generalisation of Sklyanin's method [Sklyanin 85,90] , see [Niccoli 12...] , using Baxter's vertex-IRF transformation (or by some new more general approach [Maillet, Niccoli 19])
- $\rightarrow$ works only on an inhomogeneous deformation of the model:

  $$T(\lambda) \longrightarrow T(\lambda; \xi_1, \ldots, \xi_L)$$

  such that the shifted inhomogeneity parameters $\xi_n^{(h_n)} = \xi_n + \eta/2 - h_n\eta$, $1 \le n \le L$, $h_n \in \{0,1\}$, are all pairwise distincts
- $\rightarrow$ completeness/works for any K-matrices (not both proportional to identity)

Solution by Sklyanin's SoV approach: more details

1. **simplify the expression of $\mathcal{T}(\lambda) = \text{tr}\{K^{+}(\lambda)\mathcal{U}_{-}(\lambda)\}$** : use (a trigonometric version of) Baxter's Vertex-IRF transformation to diagonalize K^{+}

$$R_{12}(\lambda-\mu)\, S_1(\lambda|\alpha,\beta)\, S_2(\mu|\alpha,\beta+\sigma_1^z) = S_2(\mu|\alpha,\beta)\, S_1(\lambda|\alpha,\beta+\sigma_2^z)\, R_{12}^{\mathrm{SOS}}(\lambda-\mu|\beta)$$

with $\quad S(\lambda|\alpha,\beta) = \begin{pmatrix} e^{\lambda-\eta(\beta+\alpha)} & e^{\lambda+\eta(\beta-\alpha)} \\ 1 & 1 \end{pmatrix}$

$\rightsquigarrow$ gauged transformed boundary monodromy matrix:

$$\begin{aligned} \mathcal{U}_{-}(\lambda|\alpha,\beta) &= S^{-1}(\eta/2-\lambda|\alpha,\beta)\,\mathcal{U}_{-}(\lambda)\, S(\lambda-\eta/2|\alpha,\beta) \\ &= \begin{pmatrix} \mathcal{A}_{-}(\lambda|\alpha,\beta) & \mathcal{B}_{-}(\lambda|\alpha,\beta) \\ \mathcal{C}_{-}(\lambda|\alpha,\beta) & \mathcal{D}_{-}(\lambda|\alpha,\beta) \end{pmatrix} \qquad \begin{cases} \beta : \text{dynamical parameter} \\ \alpha : \text{arbitrary shift} \end{cases} \end{aligned}$$

$\rightsquigarrow$ fix α, β in terms of the '+'-boundary parameters $\tau_+, \varphi_+, \psi_+$ (up to some signs/periodicity) such that

$$\mathcal{T}(\lambda) = \bar{\mathsf{a}}_{+}(\lambda)\,\mathcal{A}_{-}(\lambda|\alpha,\beta-1) + \bar{\mathsf{d}}_{+}(\lambda)\,\mathcal{D}_{-}(\lambda|\alpha,\beta+1)$$

2 **construct a SoV basis which pseudo-diagonalises $\mathcal{B}(\lambda|\alpha,\beta)$:**

$$|\,\mathbf{h},\alpha,\beta+1\,\rangle_{\mathrm{Sk}} \propto \prod_{j=1}^{L} \mathcal{D}_{-}(\xi_j+\eta/2|\alpha,\beta+1)^{h_j}\, S_{1\ldots L}(\{\xi\}|\alpha,\beta)\,|\,\underline{0}\,\rangle$$

$$_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,| \propto \langle\,0\,|\,S_{1\ldots N}(\{\xi\}|\alpha,\beta)^{-1}\prod_{j=1}^{L}\mathcal{A}_{-}(\eta/2-\xi_j|\alpha,\beta-1)^{1-h_j}$$

for $\quad \mathbf{h}\equiv(h_1,\ldots,h_L)\in\{0,1\}^L$,

$$\langle\,0\,| = \otimes_{n=1}^{L}(1,0)_n, \quad |\,\underline{0}\,\rangle = \otimes_{n=1}^{L}\begin{pmatrix}0\\1\end{pmatrix}_n$$

$$S_{1\ldots L}(\{\xi\}|\alpha,\beta) = \prod_{n=1\to L} S_n\left(-\xi_n\,\middle|\,\alpha,\beta+\sum_{j=1}^{n-1}\sigma_j^z\right)$$

such that

$$\mathcal{B}_{-}(\lambda|\alpha,\beta-1)\,|\,\mathbf{h},\alpha,\beta-1\,\rangle_{\mathrm{Sk}} = \mathrm{b}_R(\lambda|\alpha,\beta)\,a_{\mathbf{h}}(\lambda)\,a_{\mathbf{h}}(-\lambda)\,|\,\mathbf{h},\alpha,\beta+1\,\rangle_{\mathrm{Sk}},$$

$$_{\mathrm{Sk}}\langle\,\alpha,\beta+1,\mathbf{h}\,|\,\mathcal{B}_{-}(\lambda|\alpha,\beta+1) = \mathrm{b}_L(\lambda|\alpha,\beta)\,a_{\mathbf{h}}(\lambda)\,a_{\mathbf{h}}(-\lambda)\,{}_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,|,$$

where $a_{\mathbf{h}}(\lambda)=\prod_{n=1}^{L}\sinh(\lambda-\xi_n^{(h_n)})$ with $\xi_n^{(h_n)}=\xi_n+\eta/2-h_n\eta$

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$$|\,\mathbf{h},\alpha,\beta+1\,\rangle_{\mathrm{Sk}} \text{ and } {}_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,| \text{ for } \mathbf{h}\equiv(h_1,\ldots,h_L)\in\{0,1\}^L$$

such that

$$\mathcal{B}_-(\lambda|\alpha,\beta-1)\,|\,\mathbf{h},\alpha,\beta-1\,\rangle_{\mathrm{Sk}} = \mathsf{b}_R(\lambda|\alpha,\beta)\,a_{\mathbf{h}}(\lambda)\,a_{\mathbf{h}}(-\lambda)\,|\,\mathbf{h},\alpha,\beta+1\,\rangle_{\mathrm{Sk}},$$
$$ {}_{\mathrm{Sk}}\langle\,\alpha,\beta+1,\mathbf{h}\,|\,\mathcal{B}_-(\lambda|\alpha,\beta+1) = \mathsf{b}_L(\lambda|\alpha,\beta)\,a_{\mathbf{h}}(\lambda)\,a_{\mathbf{h}}(-\lambda)\,{}_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,|,$$

where $a_{\mathbf{h}}(\lambda)=\prod_{n=1}^{L}\sinh(\lambda-\xi_n^{(h_n)})$ with $\xi_n^{(h_n)}=\xi_n+\eta/2-h_n\eta$

+ orthogonality conditions:

$${}_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,|\,\mathbf{k},\alpha,\beta+1\,\rangle_{\mathrm{Sk}} \propto \delta_{\mathbf{h},\mathbf{k}}\,\frac{e^{2\sum_{j=1}^{L}h_j\xi_j}}{V_{\mathbf{h}}(\boldsymbol{\xi})}$$

with $V_{\mathbf{h}}(\boldsymbol{\xi})=V(\xi_1^{(h_1)},\ldots,\xi_L^{(h_L)})=\det_N\left[\sinh^{2(j-1)}(\xi_i^{(h_i)})\right]$

Remarks: This construction

→ works only on an inhomogeneous deformation of the model:

$$T(\lambda)\longrightarrow T(\lambda;\xi_1,\ldots,\xi_L)$$

such that $\xi_i\neq\xi_k\pm\eta \mod i\pi$ if $i\neq k$

→ needs $[K^-(\lambda|\alpha,\beta)]_{12}\neq 0$

Spectrum and eigenstates by SoV (general case)

- For α, β fixed in terms of the '+'-boundary parameters $\tau_+, \varphi_+, \psi_+$, eigenstates are special cases of separate states:

$$\mathcal{T}(\lambda) \,|\, \Psi_\tau \,\rangle = \tau(\lambda) \,|\, \Psi_\tau \,\rangle \quad \text{and} \quad \langle\, \Psi_\tau \,|\, \mathcal{T}(\lambda) = \tau(\lambda) \,\langle\, \Psi_\tau \,|$$

with

$$|\, \Psi_\tau \,\rangle = \sum_{\mathbf{h}\in\{0,1\}^L} \prod_{n=1}^{L} Q(\xi_n^{(h_n)}) \; V_{\mathbf{h}}(\boldsymbol{\xi}) \;|\, \mathbf{h}, \alpha, \beta+1 \,\rangle_{\mathrm{Sk}}$$

$$\langle\, \Psi_\tau \,| = \sum_{\mathbf{h}\in\{0,1\}^L} \prod_{n=1}^{L} [f(\xi_n)^{h_n} Q(\xi_n^{(h_n)})] \; V_{1-\mathbf{h}}(\boldsymbol{\xi}) \;{}_{\mathrm{Sk}}\langle\, \alpha, \beta-1, \mathbf{h} \,|$$

where Q and τ are solution of a discrete T-Q equation:

$$\tau(x)\, Q(x) = \mathbf{A}(x)\, Q(x+\eta) + \mathbf{A}(-x)\, Q(x-\eta), \quad x \in \cup_{n=1}^{L} \{\xi_n^{(0)}, \xi_n^{(1)}\}$$

Remark: **A** and f are given by the model and depend on the boundary parameters

Spectrum and eigenstates by SoV (general case)

- For α, β fixed in terms of the '+'-boundary parameters $\tau_+, \varphi_+, \psi_+$, eigenstates are special cases of separate states:

$$\mathcal{T}(\lambda)\,|\,\Psi_\tau\,\rangle = \tau(\lambda)\,|\,\Psi_\tau\,\rangle \quad \text{and} \quad \langle\,\Psi_\tau\,|\,\mathcal{T}(\lambda) = \tau(\lambda)\,\langle\,\Psi_\tau\,|$$

with
$$|\,\Psi_\tau\,\rangle = \sum_{\mathbf{h}\in\{0,1\}^L} \prod_{n=1}^{L} Q(\xi_n^{(h_n)})\; V_{\mathbf{h}}(\boldsymbol{\xi})\;|\,\mathbf{h}, \alpha, \beta+1\,\rangle_{\mathrm{Sk}}$$

$$\langle\,\Psi_\tau\,| = \sum_{\mathbf{h}\in\{0,1\}^L} \prod_{n=1}^{L} [f(\xi_n)^{h_n}\, Q(\xi_n^{(h_n)})]\; V_{1-\mathbf{h}}(\boldsymbol{\xi})\;{}_{\mathrm{Sk}}\langle\,\alpha, \beta-1, \mathbf{h}\,|$$

where Q and τ are solution of a discrete T-Q equation:

$$\tau(x)\,Q(x) = \mathbf{A}(x)\,Q(x+\eta) + \mathbf{A}(-x)\,Q(x-\eta), \quad x \in \cup_{n=1}^{L}\{\xi_n^{(0)}, \xi_n^{(1)}\}$$

- can always be rewritten in terms of solutions of the form

$$Q(\lambda) = \prod_{j=1}^{L} \sinh(\lambda-\lambda_j)\sinh(\lambda+\lambda_j)$$

of a continuous T-Q equation with additional term (cf. off-diagonal BA):

$$\tau(\lambda)\,Q(\lambda) = \mathbf{A}(\lambda)\,Q(\lambda-\eta) + \mathbf{A}(-\lambda)\,Q(\lambda+\eta) + \mathbf{F}(\lambda),$$

with $\mathbf{F}(\xi_n^{(0)}) = \mathbf{F}(\xi_n^{(1)}) = 0,\ n = 1, \ldots, L$ ($\to$ completeness)

Spectrum and eigenstates by SoV (general case)

- For α, β fixed in terms of the '$+$'-boundary parameters $\tau_+, \varphi_+, \psi_+$, eigenstates are special cases of separate states:

$$\mathcal{T}(\lambda)\,|\,\Psi_\tau\,\rangle = \tau(\lambda)\,|\,\Psi_\tau\,\rangle \quad \text{and} \quad \langle\,\Psi_\tau\,|\,\mathcal{T}(\lambda) = \tau(\lambda)\,\langle\,\Psi_\tau\,|$$

with
$$|\,\Psi_\tau\,\rangle = \sum_{\mathbf{h}\in\{0,1\}^L} \prod_{n=1}^{L} Q(\xi_n^{(h_n)})\; V_{\mathbf{h}}(\boldsymbol{\xi})\;|\,\mathbf{h}, \alpha, \beta+1\,\rangle_{\mathrm{Sk}}$$

$$\langle\,\Psi_\tau\,| = \sum_{\mathbf{h}\in\{0,1\}^L} \prod_{n=1}^{L} [f(\xi_n)^{h_n}\, Q(\xi_n^{(h_n)})]\; V_{1-\mathbf{h}}(\boldsymbol{\xi})\;{}_{\mathrm{Sk}}\langle\,\alpha, \beta-1, \mathbf{h}\,|$$

where Q and τ are solution of a discrete T-Q equation:

$$\tau(x)\,Q(x) = \mathbf{A}(x)\,Q(x+\eta) + \mathbf{A}(-x)\,Q(x-\eta), \quad x \in \cup_{n=1}^{L}\{\xi_n^{(0)}, \xi_n^{(1)}\}$$

- under the constraint (for a given $M \le L$ & given signs), part of the spectrum/eigenstates can be rewritten in terms of solutions

$$Q(\lambda) = \prod_{j=1}^{M} \sinh(\lambda - \lambda_j)\sinh(\lambda + \lambda_j)$$

of the usual (i.e. continuous, without additional term) T-Q equation

$\rightsquigarrow$ in terms of usual Bethe equations

Computation of scalar products

- Eigenstates are special cases of separate states :

$$\langle P \,| \equiv \langle \alpha, \beta-1, P \,| = \sum_{\mathbf{h}\in\{0,1\}^L} \prod_{n=1}^{L} [f(\xi_n)^{h_n} P(\xi_n^{(h_n)})] \, V_{1-\mathbf{h}}(\boldsymbol{\xi}) \;{}_{\mathrm{Sk}}\langle \alpha, \beta-1, \mathbf{h} \,|$$

$$|\, Q \rangle \equiv |\, Q, \alpha, \beta+1 \rangle = \sum_{\mathbf{h}\in\{0,1\}^L} \prod_{n=1}^{L} Q(\xi_n^{(h_n)}) \, V_{\mathbf{h}}(\boldsymbol{\xi}) \;|\, \mathbf{h}, \alpha, \beta+1 \rangle_{\mathrm{Sk}}$$

  where P and Q are arbitrary functions

- The scalar products of separate states can be expressed (by construction) as determinants:

$${}_{\mathrm{Sk}}\langle \alpha, \beta-1, \mathbf{h} \,|\, \mathbf{k}, \alpha, \beta+1 \rangle_{\mathrm{Sk}} \propto \delta_{\mathbf{h},\mathbf{k}} \, \frac{e^{2\sum_{j=1}^{L} h_j \xi_j}}{V_{\mathbf{h}}(\boldsymbol{\xi})}$$

  with $V_{\mathbf{h}}(\boldsymbol{\xi}) = V(\xi_1^{(h_1)}, \ldots, \xi_L^{(h_L)}) = \det_L \left[\sinh^{2(j-1)}(\xi_i^{(h_i)}) \right]$

$$\rightsquigarrow \;\; \langle P \,|\, Q \rangle \propto \det_{1\le i,j\le L} \left[\sum_{h\in\{0,1\}} f(\xi_i^{(h_i)}) P(\xi_i^{(h_i)}) Q(\xi_i^{(h_i)}) \, \sinh^{2(j-1)}(\xi_i^{(1-h_i)}) \right]$$

  However non directly usable for the consideration of the homogeneous/thermodynamic limit...

- For P and Q of the form $\prod_{j=1}^{M} \sinh(\lambda-\lambda_j)\sinh(\lambda+\lambda_j)$, these determinants can be transformed into some generalisation of Slavnov determinants [Kitanine, Maillet, Niccoli, VT 18]

Computation of overlaps ?

Consider the boundary quench $h_+ \longrightarrow \widetilde{h}_+$, i.e. $K_+ \longrightarrow \widetilde{K}_+$

$\longrightarrow$ gauge parameters (fixed to diagonalise K_+) are changed: $(\alpha, \beta) \longrightarrow (\widetilde{\alpha}, \widetilde{\beta})$

$\longrightarrow$ SoV bases are changed : $|\, \mathbf{h}, \alpha, \beta + 1 \,\rangle_{\rm Sk} \longrightarrow |\, \mathbf{h}, \widetilde{\alpha}, \widetilde{\beta} + 1 \,\rangle_{\rm Sk}$

$\longrightarrow$ The computation of the overlap would imply the computation of scalar products of separate states expressed in different SoV bases:

$$\langle\, \alpha, \beta - 1, P \,|\, Q, \widetilde{\alpha}, \widetilde{\beta} + 1 \,\rangle$$

This is still an open problem...

Question: Can we nevertheless consider some kind of boundary quench towards a configuration involving general boundary fields ?

Yes if we start from a very special configuration [Niccoli, VT, arXiv:2608.22365]

Special boundary condition at site 1

Taking $\xi_+ \to +\infty$:

$$K_+^{(Inv)}(\lambda) = \begin{pmatrix} e^{\lambda+\eta/2} & 0 \\ 0 & e^{-(\lambda+\eta/2)} \end{pmatrix} = e^{(\eta/2-\lambda)\sigma^z} \quad \text{i.e.} \quad h_+^{(Inv)} = \sinh\eta\sigma_1^z$$

the other boundary matrix being generic, then

$$\mathcal{T}^{(\mathrm{Inv})}(\lambda) = \mathrm{tr}\left[K_+^{(\mathrm{Inv})}(\lambda)\,\mathcal{U}_-(\lambda)\right] = e^{\lambda+\eta/2}\,\mathcal{A}_-(\lambda) + e^{-(\lambda+\frac{\eta}{2})}\,\mathcal{D}_-(-\lambda)$$

but we also have

$$\mathcal{T}^{(\mathrm{Inv})}(\lambda) = \mathrm{tr}\left[\widehat{K}_+^{(\mathrm{Inv})}(\lambda|\alpha,\beta)\,\widehat{\mathcal{U}}_-(\lambda|\alpha,\beta)\right],$$

with

$$\widehat{K}_+^{(\mathrm{Inv})}(\lambda|\alpha,\beta) \equiv S^{-1}(\lambda+\tfrac{\eta}{2}|\alpha,\beta)\;K_+^{(\mathrm{Inv})}(\lambda)\;S(-\lambda-\tfrac{\eta}{2}|\alpha,\beta) = e^{-(\lambda+\eta/2)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

being diagonal and independent of α and β, so that

$$\begin{aligned}\mathcal{T}^{(\mathrm{Inv})}(\lambda) &= e^{-\lambda+\eta/2}\,\frac{\sinh(2\lambda+\eta)}{\sinh 2\lambda}\,\mathcal{A}_-(\lambda|\alpha,\beta-1) + e^{\lambda+\eta/2}\,\frac{\sinh(2\lambda-\eta)}{\sinh 2\lambda}\,\mathcal{A}_-(-\lambda|\alpha,\beta-1) \\ &= e^{-\lambda+\eta/2}\,\frac{\sinh(2\lambda+\eta)}{\sinh 2\lambda}\,\mathcal{D}_-(\lambda|\alpha,\beta+1) + e^{\lambda+\eta/2}\,\frac{\sinh(2\lambda-\eta)}{\sinh 2\lambda}\,\mathcal{D}_-(-\lambda|\alpha,\beta+1)\end{aligned}$$

for any values of α and β

Family of SoV bases for $\mathcal{T}^{(\mathrm{Inv})}(\lambda)$

All bases $|\,\mathbf{h},\alpha,\beta+1\,\rangle_{\mathrm{Sk}}$ and ${}_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,|$ for $\mathbf{h}\equiv(h_1,\ldots,h_N)\in\{0,1\}^N$ such that

$$\mathcal{B}_-(\lambda|\alpha,\beta-1)\,|\,\mathbf{h},\alpha,\beta-1\,\rangle_{\mathrm{Sk}}=\mathsf{b}_R(\lambda|\alpha,\beta)\,a_{\mathbf{h}}(\lambda)\,a_{\mathbf{h}}(-\lambda)\,|\,\mathbf{h},\alpha,\beta+1\,\rangle_{\mathrm{Sk}},$$
$${}_{\mathrm{Sk}}\langle\,\alpha,\beta+1,\mathbf{h}\,|\,\mathcal{B}_-(\lambda|\alpha,\beta+1)=\mathsf{b}_L(\lambda|\alpha,\beta)\,a_{\mathbf{h}}(\lambda)\,a_{\mathbf{h}}(-\lambda)\,{}_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,|,$$

for any values of α and β can be used to construct the eigenstates of $\mathcal{T}^{(\mathrm{Inv})}(\lambda)$

Remark: it can be shown (for well-chosen normalisation) that

$$|\,\mathbf{h},\alpha,\beta+1\,\rangle_{\mathrm{Sk}}=|\,\mathbf{h},\alpha-\beta-1\,\rangle,\qquad {}_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,|=\langle\,\alpha-\beta+1,\mathbf{h}\,|,$$

depend on $\alpha-\beta$ only (since $\det S(\frac{\eta}{2}-\lambda|\alpha,\beta)\,\mathcal{B}_-(\lambda|\alpha,\beta)$ depends on $\alpha-\beta$ only)

$\rightsquigarrow$ eigenstates of $\mathcal{T}^{(\mathrm{Inv})}(\lambda)$ can be constructed as separate states in any of these bases

Spectrum and eigenstates of $\mathcal{T}^{(\mathrm{Inv})}(\lambda)$

$\longrightarrow$ $\mathcal{T}^{(\mathrm{Inv})}(\lambda)$ is isospectral to an open spin chain with both longitudinal boundary fields with boundary parameters $\xi_+^{(D)} = \varphi_-$, $\quad \xi_-^{(D)} = \psi_-$

$\longrightarrow$ separate states of the form, for any choice of $\alpha - \beta$,

$$| \, Q, \alpha-\beta-1 \, \rangle = \sum_{\mathbf{h} \in \{0,1\}^L} \prod_{n=1}^{L} Q(\xi_n^{(h_n)}) \; e^{-\sum_j h_j \xi_j} \; V_{\mathbf{h}}(\boldsymbol{\xi}) \, | \, \mathbf{h}, \alpha-\beta-1 \, \rangle,$$

$$^{(\mathrm{Inv})}\langle \, \alpha-\beta+1, Q \, | = \sum_{\mathbf{h} \in \{0,1\}^L} \prod_{n=1}^{L} \Big[f^{(\mathrm{Inv})}(\xi_n)^{h_n} \, Q(\xi_n^{(h_n)}) \Big] \times e^{-\sum_j h_j \xi_j} \; V_{\mathbf{1}-\mathbf{h}}(\boldsymbol{\xi}) \; \langle \, \alpha-\beta+1, \mathbf{h} \, |,$$

for $\qquad Q(\lambda) = \prod_{j=1}^{M} \sinh(\lambda - \lambda_j) \sinh(\lambda + \lambda_j) \qquad (M \le L-1),$

are eigenstates of $\mathcal{T}^{(\mathrm{Inv})}(\lambda)$ with eigenvalue $\tau(\lambda)$ if

$$\tau(\lambda) \, Q(\lambda) = \mathbf{A}^{(\mathrm{Inv})}(\lambda) \, Q(\lambda - \eta) + \mathbf{A}^{(\mathrm{Inv})}(-\lambda) \, Q(\lambda + \eta)$$

(complete description of the spectrum in terms of an homogeneous T-Q-equation)

Computation of overlaps ?

For fixed (general, non-longitudinal) boundary field h_-, consider the boundary quench $h_+^{(\mathsf{Inv})} \longrightarrow h_+^{(\mathsf{gen})}$, i.e. $K_+^{(\mathsf{Inv})} \longrightarrow K_+^{(\mathsf{gen})}$, in which $h_+^{(\mathsf{gen})}$, $K_+^{(\mathsf{gen})}$ are generic (or conversely)

$\longrightarrow$ the transfer matrix $\mathcal{T}^{(\mathsf{gen})}$ for $(h_+^{(\mathsf{gen})}, h_-)$ can be diagonalised in the SoV bases $|\, \mathbf{h}, \alpha, \beta+1 \,\rangle_{\mathrm{Sk}} = |\, \mathbf{h}, \alpha-\beta-1 \,\rangle$, $\quad {}_{\mathrm{Sk}}\langle\, \alpha, \beta-1, \mathbf{h} \,| = \langle\, \alpha-\beta+1, \mathbf{h} \,|$, with gauge parameters α, β fixed by $h_+^{(\mathsf{gen})}, K_+^{(\mathsf{gen})}$

$\longrightarrow$ among all possible SoV bases for $\mathcal{T}^{(\mathsf{Inv})}$, we can choose the one with same parameters α, β as above

$\longrightarrow$ For P solution of the (homogeneous or inhomogeneous) T-Q equation for $\mathcal{T}^{(\mathsf{gen})}$, and Q solution of the (homogeneous) T-Q equation for $\mathcal{T}^{(\mathsf{Inv})}$, with

$$P(\lambda) = \prod_{j=1}^{M_P} \sinh(\lambda-\mu_j)\sinh(\lambda+\mu_j), \quad Q(\lambda) = \prod_{j=1}^{M_Q} \sinh(\lambda-\lambda_j)\sinh(\lambda+\lambda_j)$$

it is possible to compute the overlaps

$${}^{(\mathsf{gen})}\langle\, \alpha-\beta+1, P \,|\, Q, \alpha-\beta-1 \,\rangle, \quad {}^{(\mathsf{Inv})}\langle\, \alpha-\beta+1, Q \,|\, P, \alpha-\beta-1 \,\rangle$$

as determinants $\rightsquigarrow$ generalised Slavnov determinants (with possible extra lines. . .)

Conclusion, perspectives and open problems

- In the diagonal case, a boundary quench does not modify the algebraic form of the Bethe states

  $\rightsquigarrow$ normalised overlap: ratio of Slavnov / Gaudin determinants

  $\rightsquigarrow$ thermodynamic limit (GS-GS, excited states with holes)

  To do: excited states with complex roots (strings), sum over excited states. . .

- In the generic non-diagonal case, a boundary quench modifies the algebraic construction of the eigenstates through the parameters α, β of the Vertex-IRF transformation

  Question: scalar product involving two different SoV bases (with different gauge parameters) ?

- there exists a special configuration which is invariant with respect to a change of the Vertex-IRF parameters α, β

  $\rightsquigarrow$ family of SoV bases

  $\rightsquigarrow$ possibility to compute overlaps from this configuration to a more general one in the form of generalised Slavnov determinants

  To do: thermodynamic limit

Eigenstates as generalised Bethe states

- In the range of Sklyanin's approach, separate states can be reformulated as generalised Bethe states:

$$| \, Q \, \rangle_{\mathrm{Sk}} \propto \prod_{j=1\to M} \mathcal{B}_-(\lambda_j|\alpha,\beta-2j+1) \, | \, \Omega_{\alpha,\beta+1-2M} \, \rangle_{\mathrm{Sk}}$$

$$_{\mathrm{Sk}}\langle \, Q \, | \propto {}_{\mathrm{Sk}}\langle \, \Omega_{\alpha,\beta-1+2M} \, | \prod_{j=1\to M} \mathcal{B}_-(\lambda_j|\alpha,\beta+2M-2j+1)$$

for any $Q(\lambda) = \prod_{j=1}^{M} \sinh(\lambda-\lambda_j)\sinh(\lambda+\lambda_j)$

with $| \, \Omega_{\alpha,\beta+1-2M} \, \rangle_{\mathrm{Sk}}$ and ${}_{\mathrm{Sk}}\langle \, \Omega_{\alpha,\beta-1+2M} \, |$ special separate states

- For a special choice of $\alpha+\beta$ in terms of the K_- boundary parameters the reference state $| \, \Omega_{\alpha,\beta+1-2M} \, \rangle$ (with well-chosen normalisation) can be identified with the reference state of the generalised ABA construction of [Fan et al 96; Cao et al 03] :

$$| \, \eta, \alpha+\beta+L-1-2M \, \rangle \equiv \prod_{n=1}^{L} S_n(-\xi_n|\alpha,\beta+n-1-2M) \, | \, 0 \, \rangle$$

The new SOV approach [Maillet, Niccoli 19]

Under the hypothesis that

- $\xi_i \neq \xi_k \pm \eta \mod i\pi$ if $i \neq k$
- the two boundary matrices $K^{\pm}$ are not both proportional to the identity

one can construct, for almost any choice of the co-vector $\langle\, S \,|$, the following SOV basis:

$$
{}_S\langle\, \mathbf{h} \,| \propto \langle\, S \,| \prod_{n=1}^{L} \mathcal{T}(\xi_n - \eta/2)^{1-h_n}, \qquad \mathbf{h} \equiv (h_1, \ldots, h_L) \in \{0,1\}^L
$$

$$
|\, \mathbf{h} \,\rangle_S \propto \prod_{n=1}^{L} \mathcal{T}(\xi_n + \eta/2)^{h_n} |\, R \,\rangle, \qquad \mathbf{h} \in \{0,1\}^L
$$

where $|\, R \,\rangle$ is uniquely fixed by adequate orthogonality conditions:

$$
{}_S\langle\, \mathbf{h} \,|\, R \,\rangle = N(\{\xi\})\, \delta_{\mathbf{h},\mathbf{0}}
$$

They satisfy the following orthogonality conditions (same as previous basis):

$$
{}_S\langle\, \mathbf{h} \,|\, \mathbf{h}' \,\rangle_S \propto \delta_{\mathbf{h},\mathbf{h}'} \, \frac{e^{2\sum_{j=1}^{N} h_j \xi_j}}{V_{\mathbf{h}}(\boldsymbol{\xi})}
$$