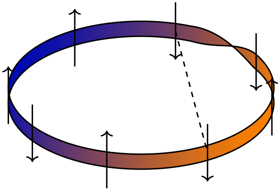
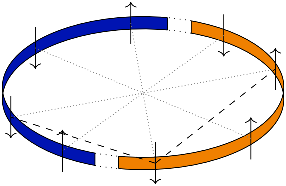
Fukui–Kawakami chains

Rob Klabbers





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Based on arXiv:2609.04378 with Antoine Lefebvre and on WiP with Thomas Scheutz

Two realms

Short-range integrability

- Ex: $H_{\mathrm{XXX}} = \sum_{i=1}^{N}(1 - P_{i,i+1})$
- Ubiquitous in physics
- Simple

Long-range integrability

- Ex: $H_{\mathrm{HS}} = \sum_{i<j}^{N} \frac{1 - P_{ij}}{\sin^2 \frac{\pi}{N}(i-j)}$
- FQHE, nanographene, ballistic transport

Quantum integrable due to $Y(\mathfrak{sl}_2)$-rep:

$V_a \otimes V^{\otimes N}$	Space	$\mathbb{C}[x_1, \ldots, x_N] \otimes V^{\otimes N}$
trivial	Centre	$\mathrm{qdet}\, M(u) \ni$ hams
$\mathrm{Tr}(M(u)) \ni$ hams	Commuting set	$M(u) \ni$ symmetries
Off-diagonal $M(u) \ni B(u), C(u)$	Raising Lowering	affine Yangian(?) (degenerate toroidal algebra)(?)

Elliptic bridge of spin chains

$$\begin{tikzcd}
& \textit{nearest-neighbour} & \textit{elliptic} & \textit{trig} \\
\textit{elliptic} & \text{Heisenberg XYZ} \arrow[d] & & \\
\textit{trig} & \begin{array}{c}\text{Heisenberg XXZ}\\ \text{with braid-translate}\end{array} \arrow[d] & \begin{array}{c}\text{[RK, Lamers, '24]}\\ q\text{-Inozemtsev}\end{array} \arrow[l] \arrow[r] \arrow[d] & \begin{array}{c}\text{[Uglov, 97]}\\ \text{[Lamers, '18]}\\ q\text{-Haldane–Shastry}\end{array} \arrow[d] \\
\textit{rational} & \text{Heisenberg XXX} & \begin{array}{c}\text{Inozemtsev}\\ \text{[Inozemtsev, '90–'97]}\end{array} \arrow[l] \arrow[r] & \begin{array}{c}\text{Haldane–Shastry}\\ \text{[Haldane, '88; Shastry '88]}\end{array}
\end{tikzcd}$$

Spin interaction $\longrightarrow$

potential $\longrightarrow$

interaction-length

Elliptic bridge of spin chains

$$
\begin{tikzcd}
 & \textit{nearest-neighbour} & \textit{elliptic} & \textit{trig} \\
\textit{elliptic} & \text{Heisenberg XYZ} \arrow[d] & & \\
\textit{trig} & \text{Heisenberg XXZ with braid-translate} \arrow[d] & q\text{-Inozemtsev} \arrow[l] \arrow[r] \arrow[d] & \text{dynamical } q\text{-Haldane–Shastry} \arrow[d] \\
\textit{twisted} & S^z\text{-twisted XXX} \arrow[d] & \text{twisted Inozemtsev} \arrow[l] \arrow[r] \arrow[d] & \text{Fukui-Kawakami} \arrow[d] \\
\textit{rational} & \text{Heisenberg XXX} & \text{Inozemtsev} \arrow[l] \arrow[r] & \text{Haldane–Shastry}
\end{tikzcd}
$$

[RK, Lamers, '24]

[Fukui, Kawakami, '96]

[RK, Scheutz, to appear]

[Inozemtsev, 1990-1997]

[Haldane (1988), Shastry (1988)]

$\uparrow$ Spin interaction

potential $\longrightarrow$

interaction-length

Questions

- How does integrability work for long-range twisted chains?
- Can it help to understand the elliptic (even untwisted) case?
- Towards general $\mathfrak{g}$, boundaries, perturbations
- But also further develop the representation theory of quantum algebras: Dunkl operators, Hecke algebras

Today: study FK chains

- connection to Haldane–Shastry chain
- Description of the spectrum
- Hidden supersymmetry

Haldane–Shastry chain

Rational case: $\displaystyle \frac{1}{2}\sum_{i\in\mathbb{Z}} h_i\,, \quad h_i = \sum_{j\neq i} \frac{1-P_{ij}}{(i-j)^2}$

Periodic wrapping:

- Set $\sigma^{\alpha}_{i+N} = \sigma_i$ for all $i \in \mathbb{Z}$ and $\alpha \in \{x, y, z\}$
- Then

$$h_i^{(N)} = \frac{1}{4}\sum_{j=1}^{N-1}\sum_{k\in\mathbb{Z}} \frac{1-P_{i,i+j+kN}}{(n+kN)^2} \propto \sum_{\substack{j=1\\ j\neq i}}^{N-1} \frac{1-P_{i,i+j \bmod N}}{4\sin^2\frac{\pi}{N}(i-j)}$$

- yielding $\displaystyle H_{\mathrm{HS}} = \frac{1}{2}\sum_{i=1}^{N} h_i^{(N)} = \sum_{i=1}^{N}\sum_{j>i}^{N} \frac{1-P_{ij}}{4\sin^2\frac{\pi}{N}(i-j)}$
- Doing this for $h_i = 1 - P_{i,i+1}$ yields (periodic) Heisenberg XXX chain

Haldane–Shastry chain

- Commutes with an action of $Y(\mathfrak{sl}_2)$ on the physical Hilbert space: global $\mathfrak{sl}_2$ ($S^\pm$ and S^z) and

$$Q^\pm := \mp\frac{1}{2}\sum_{i<j}^{N} \cot(\tfrac{\pi}{N}(i-j))(\sigma_i^\pm\sigma_j^z - \sigma_i^z\sigma_j^\pm)$$

$$Q^z := \frac{1}{2}\sum_{i<j}^{N} \cot(\tfrac{\pi}{N}(i-j))(\sigma_i^+\sigma_j^- - \sigma_i^-\sigma_j^+)$$

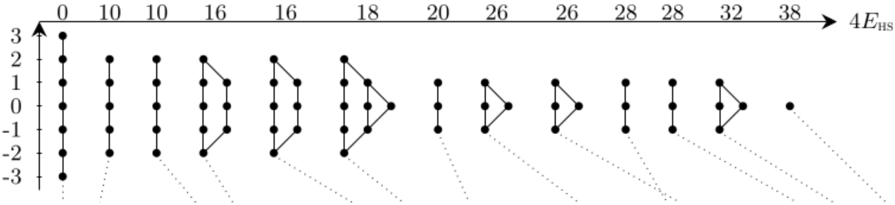
- $\mathcal{H} = \bigoplus_\mu V_\mu$, with V_μ $Y(\mathfrak{sl}_2)$-irrep labelled by motifs:

$$\mu_i \in \{1,\ldots,N-1\}\,, \quad \mu_i + 1 < \mu_{i+1}$$

- Each V_μ is generated from a Yangian highest-weight state $|\mu\rangle$ by acting with S^- and Q_-.
- All states have
  - energy $E_{\text{HS}} = \sum_i \varepsilon_0(\mu_i)$, $\varepsilon_0(\mu) = \mu(N-\mu)/2$
  - momentum $\sum \mu_i$

HS-spectrum

Example at $N = 6$:



Twisted wrapping [Fukui, Kawakami '96]

Rational case: $\frac{1}{2}\sum_{i\in\mathbb{Z}} h_i\,, \quad h_i = \sum_{j\neq i} \frac{1-P_{ij}}{(i-j)^2}$

Twisted wrapping: $\varphi = p/q \in \mathbb{Q}$

- Set
  $\sigma^{\pm}_{i+kN} = \mathrm{e}^{\pm\mathrm{i}\pi k\varphi}\sigma^{\pm}_i\,, \quad \sigma^z_{i+kN} = \sigma^z_i \qquad \sigma^{\alpha}_{i+kN} = \mathrm{e}^{\mathrm{i}\pi k\varphi S^z}\sigma^{\alpha}_i\mathrm{e}^{-\mathrm{i}\pi k\varphi S^z}$
- Then

$$H_{\mathrm{FK}}(p/q) := \sum_{i<j}\left[\frac{1}{4\sin^2\frac{\pi}{N}(i-j)}\frac{1-\sigma^z_i\sigma^z_j}{2} - \sum_{m=0}^{q-1}\frac{e^{2\pi\mathrm{i}m\frac{p}{q}}\sigma^+_i\sigma^-_j + e^{-2\pi\mathrm{i}m\frac{p}{q}}\sigma^-_i\sigma^+_j}{4q^2\sin^2\frac{\pi}{qN}(i-j+mN)}\right],$$

- For the Heisenberg chains we get

$$H^{\mathrm{tw}}_{\mathrm{XXX}} = \frac{1}{2}\sum_{i=1}^{N}(1-P_{i,i+1}) \text{ with } \sigma^{\pm}_{N+1} = \mathrm{e}^{\pm\mathrm{i}\pi\varphi}\sigma^{\pm}_1\,, \quad \sigma^z_{N+1} = \sigma^z_1$$

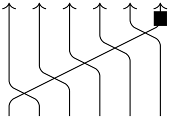
Fukui-Kawakami chains [Fukui, Kawakami '96] [RK, Lefebvre, '26] [RK, Scheutz, to appear]

$$H_{\text{FK}}(p/q) = \frac{1}{2}\sum_{i<j}\Big(A_{-}(i-j)\sigma_i^{+}\sigma_j^{-} + A_{+}(i-j)\sigma_i^{-}\sigma_j^{+} + \frac{1-\sigma_i^z\sigma_j^z}{4\sin^2\frac{\pi}{N}(i-j)}\Big)$$

with

$$A_{\pm}(x) = -\frac{1}{2}\Big(\sin^{-2}\tfrac{\pi}{N}x \pm 2\varphi(\cot\tfrac{\pi}{N}x \mp \mathrm{i})\Big)\mathrm{e}^{\pm 2\pi\mathrm{i}\varphi x/N}$$

- φ-symmetries:
  - $H_{\text{FK}}(\varphi+1) = H_{\text{FK}}(\varphi)$
  - $S_{\mathsf{sf}}H_{\text{FK}}(\varphi)S_{\mathsf{sf}} = H_{\text{FK}}(1-\varphi)$

  So $\varphi \in [0, 1/2] \cap \mathbb{Q}$.
- Commutes with
  - S^z: magnons
  - Translation operator: $T(\varphi) :=$  , with $\mathrm{e}^{\mathrm{i}\pi\varphi\,\sigma^z} =$  ,
  - Extra charge: a twisted $Q^z(p/q)$
- Integrability formally unknown

End points: periodic and antiperiodic cases [RK, Lefebvre,'26]

- $H_{\rm FK}(0) = H_{\rm HS}$ and [Sechin, Zotov, '18]
  $$H_{\rm FK}(1/2) = H_{\rm SZ} = -\frac{1}{2}\sum_{i<j} \frac{\cos\frac{\pi}{N}(i-j)(\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y) + \sigma_i^z\sigma_j^z - 1}{4\sin^2\frac{\pi}{N}(i-j)}$$
- $H_{\rm SZ}$ has higher hamiltonians by a matrix-valued Lax construction that should follow from [Matushko, Chalykh, '25].
- but no known connection to quantum groups
- but it does have something else [Liashyk, Sechin, private communication at Raqis '24]

$$U\big(H_{\rm SZ} + \tfrac{1}{2}Q^z(1/2)\big)\,U^{-1} = H_{\rm HS}\,, \quad \text{with } U = \prod_{j=1}^{N} e^{\mathrm{i}\pi\sigma_j^z j/(2N)}$$

- General (as also suggested in [Fukui, Kawakami, '97])

$$U\big(H_{\rm FK}(\varphi) + \varphi Q^z(\varphi)\big)\,U^{-1} = H_{\rm HS}\,, \quad \text{with } U = \prod_{j=1}^{N} e^{\mathrm{i}\pi\varphi\sigma_j^z j/N}$$

Special connection

$$U\big(H_{\mathrm{FK}}(\varphi) + \varphi Q^z_{\mathrm{FK}}(\varphi)\big)\,U^{-1} = H_{\mathrm{HS}}\,, \quad \text{with } U = \prod_{j=1}^{N} e^{\mathrm{i}\pi\varphi\sigma^z_j j/N}$$

$\Longrightarrow$ Shared eigenbasis of H_{FK} and Q^z_{FK} would yield basis for HS

$\Longleftarrow$ Yangian highest-weight states of HS can yield eigenvectors of H_{FK}

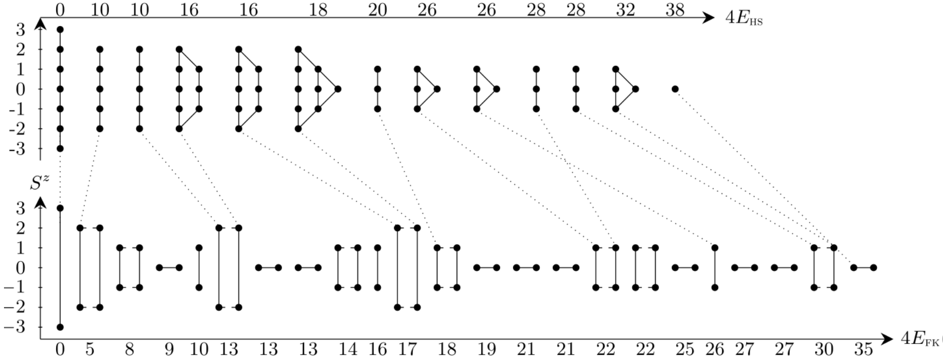
Recipe

- Take such a vector $|\mu\rangle$

$$H_{\mathrm{HS}}|\mu\rangle = E_{\mathrm{HS}}|\mu\rangle\,, \quad S^z|\mu\rangle = \left(\frac{N}{2} - M\right)|\mu\rangle\,,$$
$$Q^z|\mu\rangle = \lambda^z|\mu\rangle\,, \qquad S^+|\mu\rangle = 0\,,$$

- Use that $UQ^z_{\mathrm{FK}}U^{-1} = Q^z + \frac{\mathrm{i}}{2}(S^z + S^-S^+ - \frac{N}{2})$
- To find that $H_{\mathrm{FK}}U^{-1}|\mu\rangle = \left(E_{\mathrm{HS}} - \varphi(\mathrm{i}\lambda^z + M/2)\right)U^{-1}|\mu\rangle\,.$
  Energy:
  $$E_{\mathrm{FK}} = \sum_i \varepsilon_\varphi(\mu_i)\,, \qquad \varepsilon_\varphi(\mu) = \frac{1}{2}(\mu_i + \varphi)(N - \mu_i - \varphi) + \frac{\varphi(\varphi-1)}{2}$$

Comparing spectra



Quadruplets for $H_{\mathrm{FK}}(1/2)$: Spin-flip and parity (not for all φ)

Haldane's Bethe-ansatz-like equations

The 'ABA equations' in [Fukui, Kawakami, '96] can be rewritten as Haldane's Bethe-ansatz-like equations: $k_i = \frac{2\pi\mu_i}{N}$ and non-coinciding $I_i \in \mathbb{Z}_N$

$$k_i N = 2\pi I_i - (M-1)\pi + \pi \sum_{j \neq i} \operatorname{sign}(k_i - k_j)\,,$$

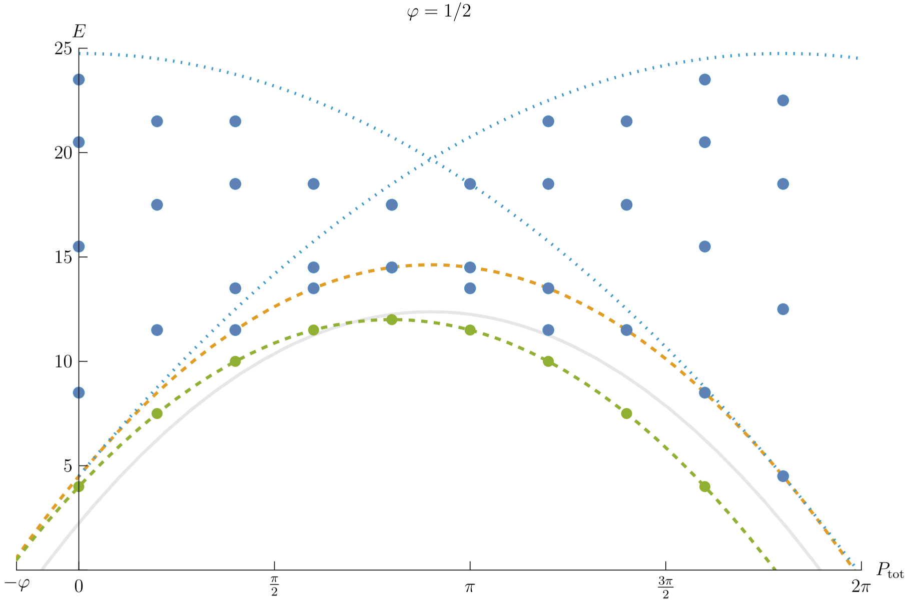
which can be solved by $\mu_j = I_j - M + j$ and includes all motifs.

- We can allow $\mu_1 = 0$ to include $\mathfrak{sl}_2$-like descendants
- but these equations seemingly do not describe the full spectrum: there are energies E_{FK} already for $M = 2$ such that

  $$\varepsilon_\varphi(\mu_1) + \varepsilon_\varphi(\mu_2) = E_{\mathrm{FK}}$$

  cannot be solved for with $\mu_i \in \mathbb{C}$.

Two-magnon spectrum at $N = 10$



Deformed descendants

The two-magnon spectrum consists of

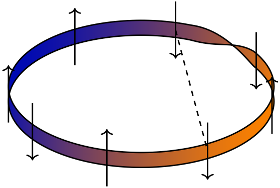
- $\binom{N-2}{2}$ Motif-states: $\mu = \{\mu_1, \mu_2\}$ with eigenvector $U^{-1}|\mu\rangle$ and energy $\varepsilon_\varphi(\mu_1) + \varepsilon_\varphi(\mu_2)$
- $N-2$ 'zero-momentum descendants': $\mu_1 = 0$ and $2 \leq \mu_2 \leq N-1$ and energy $\varepsilon_\varphi(0) + \varepsilon_\varphi(\mu_2)$
- $N-1$ 'shifted-momentum descendants': $\mu_1 = -\varphi$ and $\mu_2 = \mu$ with $\mu \in \{\varphi, \ldots, N-2+\varphi\}$ with energy $3\varepsilon_\varphi(-\varphi) + \varepsilon_\varphi(\mu+\varphi)$

Observations:

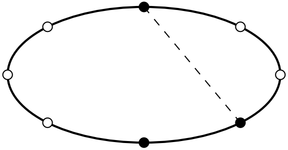
- The two types of descendants cannot easily be associated to the S^- $(N-1+1)$ and Q^- $(N-3)$ of Haldane–Shastry
- Begs for an algebraic explanation
- We see some extra structure in the elliptic case

Tale of four spin chains

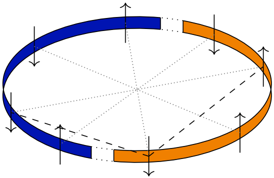
Consider the Haldane–Shastry pair interaction $\frac{\mathrm{Antisym}(i,j)}{d(i-j)^2}$ on these configurations:



$H_{\mathrm{FK}}(1/2) = H_{\mathrm{SZ}}$



H_{sHS}



H_{doubled}

[Basu–Mallick, Finkel, González–López, '20]

Tale of four spin chains [RK, Lefebvre, '26] [Basu–Mallick, Finkel, González–López, '20]

$$H_{\mathrm{FK}}(1/2) = H_{\mathrm{SZ}} = -\frac{1}{2}\sum_{i<j}\frac{\cos\frac{\pi}{N}(i-j)(\sigma_i^x\sigma_j^x+\sigma_i^y\sigma_j^y)+\sigma_i^z\sigma_j^z-1}{4\sin^2\frac{\pi}{N}(i-j)}$$

$$H_{\mathsf{sHS}} = -\frac{1}{2}\sum_{i<j}\frac{1-\mathbb{P}_{ij}}{4\sin^2\frac{\pi}{N}(i-j)} \quad \text{on } (\mathbb{C}^2)^{\otimes N} \text{ after JW-transformation}$$

$$H_{\mathsf{doubled}} = \frac{1}{16}\sum_{i<j}\left(\frac{1}{\sin^2\frac{\pi}{2N}(i-j)}(1-P_{ij}) + \frac{1}{\cos^2\frac{\pi}{2N}(i-j)}(1-P_{ij}\sigma_i^x\sigma_j^x)\right)$$

- Algebra and a basis transformation show: $H_{\mathrm{FK}}(1/2) \equiv H_{\mathsf{doubled}}$
- Since $Z_{\mathsf{doubled}} = Z_{\mathsf{sHS}}$, we even have

$$H_{\mathrm{SZ}} = H_{\mathrm{FK}}(1/2) \equiv H_{\mathsf{doubled}} \equiv H_{\mathsf{sHS}}$$

- So all these hamiltonians commute with some $\mathfrak{gl}(1|1)$-action: hidden supersymmetry

$\mathfrak{gl}(1|1)$-HS-chain

- Commutes with an action of $Y(\mathfrak{gl}(1|1))$ on the physical Hilbert space: global $\mathfrak{gl}(1|1)$ + one extra charge

$$N_G = \sum_{i=1}^{N} f_i^\dagger f_i\,, \quad E_G = 1\,, \quad F_G = \sum_{i=1}^{N} f_i\,,$$

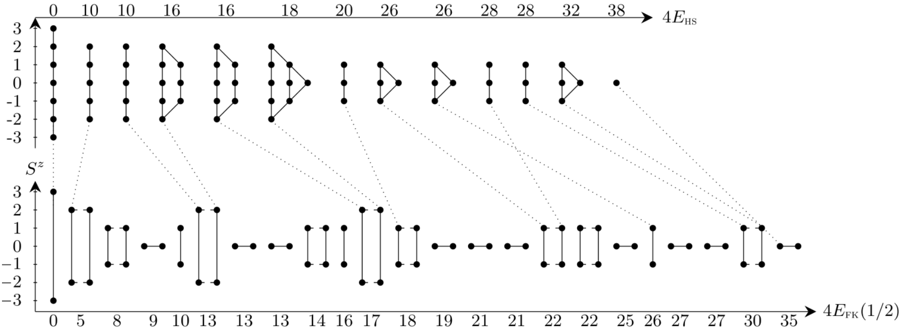
$$F_G^\dagger = \sum_{i=1}^{N} f_i^\dagger\,, \quad Q^z_{\mathrm{sHS}} = \frac{1}{2}\sum_{i<j} \cot\frac{\pi}{N}(i-j)\left(f_i^\dagger f_j - f_j^\dagger f_i\right)\,,$$

- $\mathcal{H} = \bigoplus_\nu V_\nu$, with V_ν $Y(\mathfrak{gl}(1|1))$-irrep labelled by $\mathfrak{gl}(1|1)$-motifs:

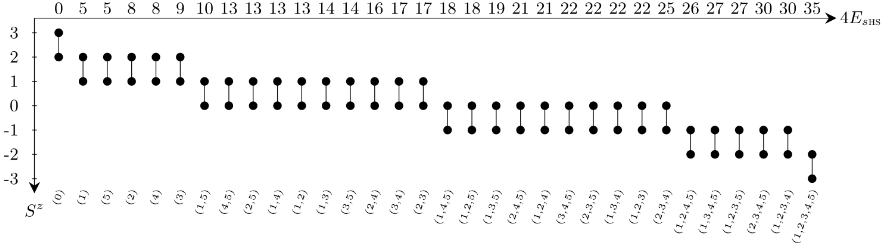
$$\nu_i \in \{1, \ldots, N-1\}\,, \quad \nu_i + 1 \leq \nu_{i+1}$$

- Each V_ν is a doublet: $\{|\nu\rangle, F_G^\dagger|\nu\rangle\}$.
- Both states have energy $E_{s\mathrm{HS}} = \sum_i \frac{\varepsilon_0(\nu_i)}{2}$ and momentum $\sum_i \nu_i$

Spectra



0 10 10 16 16 18 20 26 26 28 28 32 38
$4E_{\text{HS}}$
3 2 1 0 -1 -2 -3
S^z
3 2 1 0 −1 −2 −3
0 5 8 9 10 13 13 13 14 16 17 18 19 21 21 22 22 25 26 27 27 30 35
$4E_{\text{FK}}(1/2)$



0 5 5 8 8 9 10 13 13 13 13 14 14 16 17 17 18 18 19 21 21 22 22 22 22 25 26 27 27 30 30 35
$4E_{s\text{HS}}$
3 2 1 0 -1 -2 -3
S^z
(0) (1) (5) (2) (4) (3) (1,5) (4,5) (2,5) (1,4) (1,2) (1,3) (3,5) (2,4) (3,4) (2,3) (1,4,5) (1,2,5) (1,3,5) (2,4,5) (1,2,4) (3,4,5) (2,3,5) (1,3,4) (1,2,3) (2,3,4) (1,2,4,5) (1,3,4,5) (1,2,3,5) (2,3,4,5) (1,2,3,4) (1,2,3,4,5)

Thickening supermotifs

Thickening (inspired by, but seemingly different to) [Jiang, Lamers, Miao, '26]:
$(\mu_1, \mu_2, \dots, \mu_M) \mapsto \nu_\mu = (\mu_1, \mu_1 + 1, \mu_2, \mu_2 + 1, \dots \mu_M, \mu_M + 1) \bmod N$.

- Heuristic: 1/2-twisted case covers the chain twice: μ_i and $\mu_i + \varphi = \mu_i + 1/2 \rightsquigarrow \mu_i + 1$
- Fits the dispersion relation:

$$\varepsilon_{1/2}(\mu) = \frac{\varepsilon_0(\mu) + \varepsilon_0(\mu + 1)}{2}$$

and thus:

$$E_{\mathrm{FK}}(\mu) = E_{\mathrm{sHS}}(\nu_\mu)\,.$$

An interpolating family of anyons

Periodic case
Haldane–Shastry ($\varphi = 0$)

- $Y(\mathfrak{gl}(2|0))$
- Motifs: $\mu_i + 1 < \mu_{i+1}$
- **free** $1/2$-anyons (semions)

$H_{\mathrm{FK}}(\varphi)$

- ?
- ?
- free(?) $1/2 + \varphi$-anyons

antiperiodic case
susy HS ($\varphi = 1/2$)

- $Y(\mathfrak{gl}(1|1))$
- Motifs: $\nu_i + 1 \leq \nu_{i+1}$
  Contains: $\nu_\mu \approx \mu + 1$
- **free** 1-anyons (fermions)

Free: only interacting via Haldane's generalised exclusion principle [Haldane, '91]

Lots of work left: Basis change $H_{\mathrm{FK}}(1/2) \equiv H_{s\mathrm{HS}}$

- is non-local
- does not preserve magnon number
- does not connect translation operators

Conclusions

Summary:

- FK-chains admit an unexpected connection to motifs, but this seems to not exhaust the spectrum
- Deformed descendants hint at extra structure
- $\varphi = 1/2$ has hidden supersymmetry
- Family of (free?) anyons

Future directions:

- Construct from a monodromy-matrix like object, cf. [Bernard, Gaudin, Haldane, Pasquier, '93]
- Determine a nice eigenbasis, compare with [Jiang, Lamers, Miao, '26]
- Understand the fermions for $\varphi = 1/2$
- Elliptic extension (WiP with Thomas Scheutz)
- Ballistic transport (WiP with Miłos Panfil and TU Wrocław)
- higher rank, q-deformation,...