Testing General Relativity with the ringdown

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Introduction

- Detection of gravitational waves opened the possibility of **direct tests of black hole physics**
- New opportunity to test the theory of gravity in its **strong field regime**
- The *black hole spectroscopy* program focuses on tests done using the ringdown phase
- Going beyond a null-hypothesis check requires theoretical predictions **beyond General Relativity**

Main goal: understand how one can describe the ringdown in modified gravity theories and compute observables

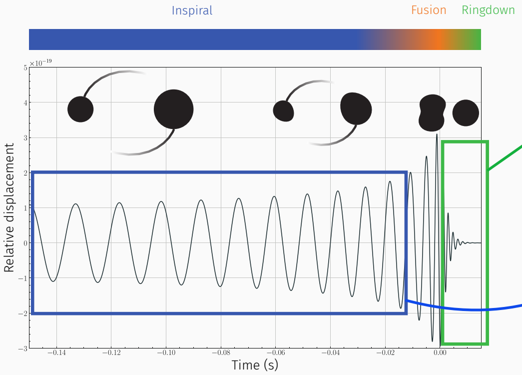
Outline

1. Quasinormal modes in GR
    - Black hole spectroscopy
    - Principle of computation
2. Modified gravity theories and hairy black holes
    - Necessity for modified gravity
    - Black hole solutions
3. Quasinormal modes in modified gravity
    - Numerical results
    - Mathematical description of the algorithm
    - Application to new black hole solutions in modified gravity

Quasinormal modes in GR

Parts of a gravitational wave signal

- First direct detection in 2015 by the LIGO/Virgo collaboration [Abbott et al. '16]
- New window for **direct tests of the strong gravity regime** of General Relativity



- **Ringdown** : perturbed black hole settles down by emission of gravitational waves at discrete frequencies
- **Inspiral** : binary system emits gravitational waves through the variation of the mass quadrupole

Why the ringdown?

It is a simple system...

- Perturbed system settling to equilibrium
- **Perturbation theory** describes the dynamics
- Linear order: stationary black hole + gravitational waves around it
- Can go beyond for higher precision

... with rich dynamics

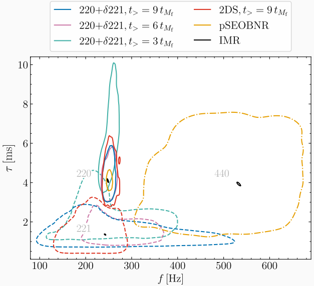
- Two boundary conditions: eigenvalue problem
- **Quasi-normal mode spectrum**

$$\omega_{\ell mn}(M, J) = f_{\ell mn} - i\tau_{\ell mn}$$

- Sensitive to both the black hole solution and the underlying theory

QNM computation and comparison with experiment

Black hole spectroscopy program: extraction of the modes from GW data and comparison with theoretical predictions



[A. G. Abac et al. ’26]

- Experimental and theoretical challenges [Abedi et al. ’25]
- GW250114: recent detection with highest signal-to-noise ratio [A. G. Abac et al. ’26]
- Very promising future: $1\,\%$ precision with Einstein Telescope and LISA [A. Abac et al. ’25; Colpi et al. ’24]
- Null-hypothesis check of General Relativity

Perturbations around a static black hole

Linearized GR

- Schwarzschild background $\bar{g}_{\mu\nu}$
- Perturbations:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon h_{\mu\nu}$$

$\longrightarrow$ obtain **10 linearized equations** for the 10 components of $h_{\mu\nu}$

Symmetries

- Decompose $h_{\mu\nu}$ over Y_ℓ^m
- Separate by parity: polar (even) and axial (odd) modes
- Fourier transform: functions of r only

$\longrightarrow$ obtain **two decoupled systems** of ODEs in r

Final result

- Fix the gauge
- **Technical step:** remove redundant equations [Regge, Wheeler '57; Zerilli '70]

$\longrightarrow$ two 2×2 systems:

$$\frac{\mathrm{d}X_{\mathrm{odd}}}{\mathrm{d}r} = M_{\mathrm{odd}}(r) X_{\mathrm{odd}}$$

$$\frac{\mathrm{d}X_{\mathrm{even}}}{\mathrm{d}r} = M_{\mathrm{even}}(r) X_{\mathrm{even}}$$

Schrödinger equation of propagation

- Schwarzschild black hole: non-rotating of mass M, horizon at $r = 2M$ in natural units

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)$$

- Change of variables $X = P(r)\tilde{X}$ for each sector [Regge, Wheeler ’57; Zerilli ’70]
- “tortoise coordinate” such that $r = 2M$ is equivalent to $r_* = -\infty$ $(\mathrm{d}r/\mathrm{d}r_* = 1 - 2M/r)$
- Obtain two Schrödinger-like equations with potentials V

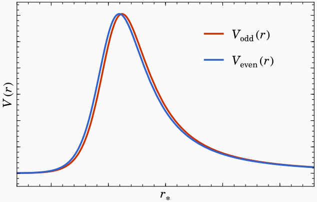
$$\boxed{\frac{\mathrm{d}^2\tilde{X}_{\text{odd}}}{\mathrm{d}r_*^2} + \left[\omega^2 - V_{\text{odd}}(r)\right]\tilde{X}_{\text{odd}} = 0} \qquad \boxed{\frac{\mathrm{d}^2\tilde{X}_{\text{even}}}{\mathrm{d}r_*^2} + \left[\omega^2 - V_{\text{even}}(r)\right]\tilde{X}_{\text{even}} = 0}$$

Interpretation of the equation

Horizon

$$(r = 2M, r_* = -\infty)$$

$$\tilde{X} \sim e^{-i\omega(t \pm r_*)}$$

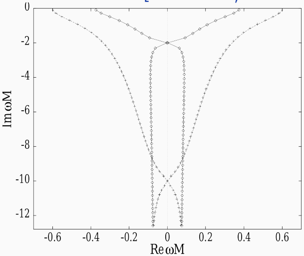


Infinity

$$\tilde{X} \sim e^{-i\omega(t \pm r_*)}$$

- Propagation of GWs at the speed of light $c = 1$
- Recover the boundary conditions useful for QNM computation
- Numerical resolution of problem available using methods from quantum mechanics [Leaver '85]

[Kokkotas, Schmidt '99]



Summary

- **Ringdown:** sum of quasi-normal modes (damped sinusoids)
- **Black hole spectroscopy:** extract mode frequencies from GW data and compare with theory
- **Drawback:** this is a null-hypothesis check, theoretically motivated parametrized deviations are lacking
- **Computation setup:** decouple the two degrees of freedom, cast them into Schrödinger-like equations, solve the eigenvalue problem

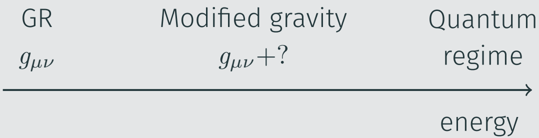
To go beyond the null hypothesis check, we need **theoretical predictions for the spectrum** in beyond-GR theories.

Modified gravity theories and hairy black holes

Motivations for modified gravity

Heuristic approach

- Design new tests of GR beyond a null hypothesis check
- EFT of some high energy theory



GR $g_{\mu\nu}$

Modified gravity $g_{\mu\nu} + ?$

Quantum regime

energy

Issues of GR

- Big Bang singularity
- Black hole center singularity
- Cosmic expansion

$\Rightarrow$ Important to look for extensions of GR

$\Rightarrow$ Need to develop tests of these modified theories

Various theories of modified gravity

Lovelock's theorem for gravity

- Four dimensional spacetime
- Only field is the metric $g_{\mu\nu}$
- Theory is local and diffeomorphism invariant
- Second order derivatives in equations

} General Relativity is the **only possible theory**

$\longrightarrow$ this yields starting points for modifying General Relativity

Building a modified gravity theory

General procedure to construct a modified gravity theory:

Break one or several hypotheses of **Lovelock's theorem** → Make sure the new theory is **relevant** (ghosts, instabilities, no-hair theorems...) → Take **experimental constraints** into account

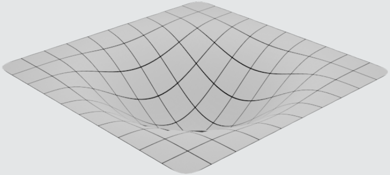
Existing constraints

- Solar System tests, binary pulsars: no deviation from GR measured
- GW170817: $c_{\mathrm{GW}} = c$ in the LIGO frequency band [Abbott et al. '17; Rham, Melville '18]

An example: scalar-tensor theories

General relativity

- Gravity from spacetime curvature
- 2 degrees of freedom



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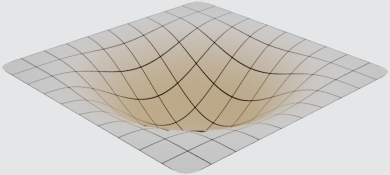
Scalar field

- New degree of freedom: scalar defined everywhere
- Can be seen as a fifth force



Horndeski: principle of construction

- Proposed in 1974 [Horndeski '74]
- Add scalar field ϕ to break Lovelock
- Action contains first and second derivatives of ϕ
- Specific combinations to have second order equations of motion
- Obtain all possible terms and classify by powers of derivatives

$$\text{Horndeski} = \boxed{\text{GR}} \times \boxed{\text{Coupling}} + \boxed{\begin{array}{c}\text{Orders 0 and}\\ \text{1 in } \nabla\nabla\phi\end{array}} + \boxed{(\nabla\nabla\phi)^2} + \boxed{(\nabla\nabla\phi)^3}$$

Horndeski theory of gravity

Shift-symmetric Horndeski theory

$$S[g_{\mu\nu},\phi] = \int \mathrm{d}^4x \left[F(X)R + P(X) + Q(X)\Box\phi + 2F'(X)\left(\phi_{\mu\nu}\phi^{\mu\nu} - (\Box\phi)^2\right)\right],$$

$$\phi_\mu = \nabla_\mu\phi\,, \qquad X = \phi_\mu\phi^\mu$$

- New scalar field: additional propagating degree of freedom
- Easier to evade no-hair theorems [Hui, Nicolis '13]: new black hole solutions [Babichev et al. '17; Van Aelst et al. '20; Ben Achour et al. '20]
- More involved dynamics in vacuum: no Schrödinger reformulation available

Hairy black holes in Horndeski

Choice of Horndeski parameters [Babichev et al. '17]:

$$F(X) = f_0 + f_1\sqrt{X} \quad P(X) = -p_1 X\,, \quad Q(X) = 0$$

Metric sector

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{1}{A(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

$$A(r) = 1 - \frac{r_m}{r} - \xi\frac{r_m^2}{r^2}\,, \quad \xi = \frac{f_1^2}{2f_0 p_1 r_m^2}$$

Scalar sector ("hair")

$$\phi = \psi(r)\,, \quad \psi'(r) = \pm\frac{f_1}{p_1 r^2\sqrt{A(r)}}$$

$$X(r) = \frac{f_1^2}{p_1^2 r^4}$$

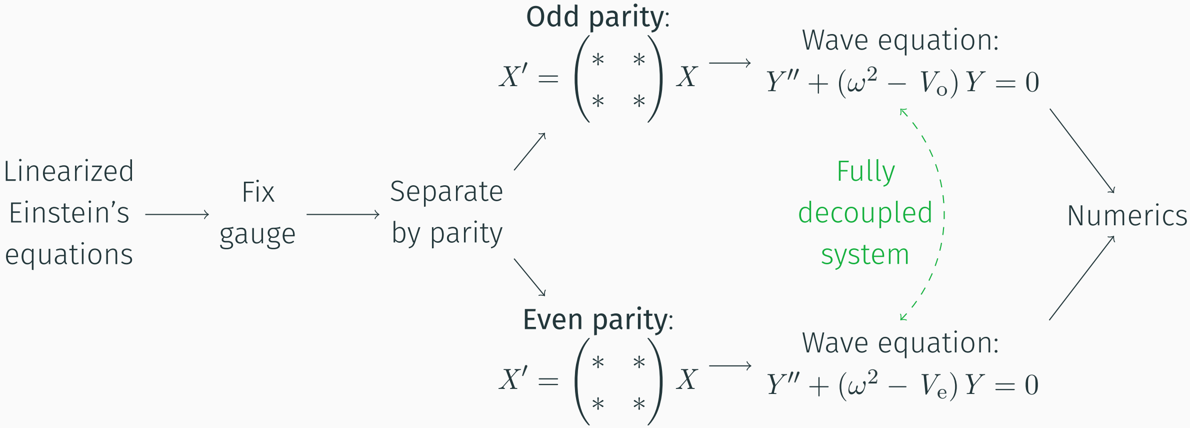
Minimal theory of massive gravity (MTMG)

Motivations

- Natural way to **change the long-range behavior** of gravity (interesting for expansion of the Universe)
- Difficult to obtain a healthy formulation (no ghosts) [Rham, Gabadadze, et al. '11]
- Reformulation with only two propagating degrees of freedom: MTMG [De Felice, Mukohyama '16]

- Self-accelerating cosmological branch
- Mimicking theorem: solutions of GR are solutions of MTMG
- New non-dynamical fields make the Schrödinger reformulation unavailable
- Lovelock's theorem broken because theory is not diffeomorphism invariant

Summary: computation of modes in GR



Linearized Einstein's equations → Fix gauge → Separate by parity

Odd parity:

$$X' = \begin{pmatrix} * & * \\ * & * \end{pmatrix} X \longrightarrow$$

Wave equation:

$$Y'' + (\omega^2 - V_\mathrm{o})\, Y = 0$$

Even parity:

$$X' = \begin{pmatrix} * & * \\ * & * \end{pmatrix} X \longrightarrow$$

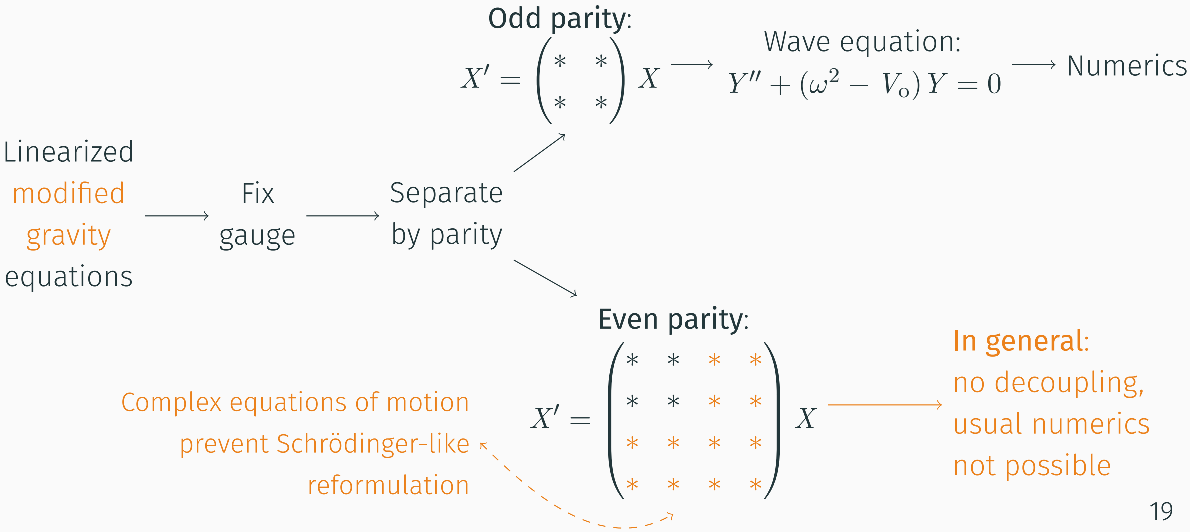
Wave equation:

$$Y'' + (\omega^2 - V_\mathrm{e})\, Y = 0$$

Fully decoupled system

Numerics

What changes in modified gravity



Linearized modified gravity equations → Fix gauge → Separate by parity

Odd parity:

$$X' = \begin{pmatrix} * & * \\ * & * \end{pmatrix} X \longrightarrow$$

Wave equation:

$$Y'' + (\omega^2 - V_{\mathrm{o}})\,Y = 0 \longrightarrow$$

Numerics

Even parity:

$$X' = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} X \longrightarrow$$

In general: no decoupling, usual numerics not possible

Complex equations of motion prevent Schrödinger-like reformulation

Summary

- Modified gravity theories are motivated either by the **need for parametrized deviations** from GR or the possibility to **solve issues of GR**
- Constructing new theories requires evading both Lovelock's theorem and no-hair theorems
- More complex equations of motion lead to systems that cannot be cast into Schrödinger-like equations: usual computation setup no longer available

How can one find a way to compute QNM spectra in modified gravity?

Quasinormal modes in modified gravity

State of the art

Theoretical results

- Results rely on Schrödinger reformulation
- Possible in some subcases but not in general [Blázquez-Salcedo et al. '17; Takahashi et al. '19]
- Perturbative regime: build such an equation order by order
- Several theoretical approaches for the non-rotating case [Cano et al. '23; Chung, Wagle, et al. '23]

Data analysis

- Theory-agnostic measurement
- GR waveforms required for increased precision
- Very few beyond GR theories investigated yet [Chung, Lam, et al. '25; A. G. Abac et al. '26]

First-order system and boundary conditions

Main idea

Get boundary conditions and perform numerical computations **from the first-order system**

Steps to perform

- Find asymptotic behaviour at the horizon and infinity
- Impose the correct boundary conditions
- Use a numerical method that does not require Schrödinger equations

Main result of this work: a systematic way to obtain the physical boundary conditions **without a Schrödinger-like reformulation**

“Recipe” for the computation of quasinormal modes

① Linearized Einstein’s eqs → ② Gauge choice → ③ Background → ④ First-order system → ⑤ Asymptotical behaviour → ⑥ Numerical computation

- Generic algorithm that should work for any modified gravity theory
- Go around the technical difficulties of finding a wave equation
- Caveat: we do not get the full decoupled equations for the modes
- Asymptotical behaviour is enough to obtain boundary conditions for numerical resolution

Numerical method

- Outgoing waves at infinity: $X \sim_{+\infty} e^{-i\omega t + i\omega r_*}$
- Ingoing waves at the horizon r_+: $X \sim_{-\infty} e^{-i\omega t - i\omega r_*} = e^{-i\omega t}(r - r_+)^{-i\omega}$

Ansatz

$$X = \underbrace{e^{+i\omega r} r^{i\mu\omega}}_{\text{Infinity}} \underbrace{\left(\frac{r - r_+}{r}\right)^{-i\omega}}_{\text{Horizon}} \sum_{n=0}^{\infty} \boldsymbol{X}_n(\omega) \left(\frac{r - r_+}{r}\right)^{n}$$

- Singular behaviour can differ depending on the background and the theory but will always be available using the algorithm

ω is a QNM $\leftrightarrow$ the power series converges

Condition for convergence

Convergence criterion: continued fraction method [Rosa, Dolan ’12; Leaver ’85; Gautschi ’67]

Recursion relation from the equations of perturbations

$$\boldsymbol{\alpha}_0 \boldsymbol{X}_1 + \boldsymbol{\beta}_0 \boldsymbol{X}_0 = 0$$
$$\boldsymbol{\alpha}_n \boldsymbol{X}_{n+1} + \boldsymbol{\beta}_n \boldsymbol{X}_n + \boldsymbol{\gamma}_n \boldsymbol{X}_{n-1} = 0\,.$$

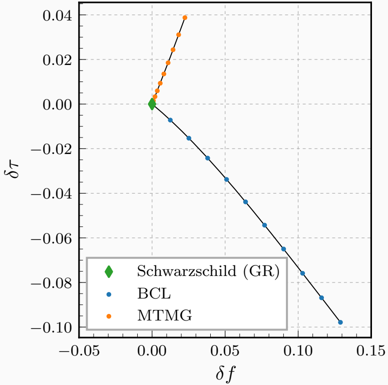
Matrix-valued continued fraction method

$$\boldsymbol{M}\boldsymbol{X}_0 = 0 \qquad \Longrightarrow \qquad \det(\boldsymbol{M}) = 0\,,$$
$$\boldsymbol{M} = \boldsymbol{\beta}_0 - \boldsymbol{\alpha}_0 \left[\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1 [\ldots]^{-1} \boldsymbol{\gamma}_2\right]^{-1} \boldsymbol{\gamma}_1$$

Results

Tracking of the fundamental QNM

- Can obtain very precisely the shift for each overtone
- The fundamental mode is the **least sensitive one**
- Clear qualitative differences between modifications of gravity



Summary

- Main difficulty in beyond GR theories: identifying the correct boundary conditions from a complex system of equations
- We present a generic algorithm that allows one to do this **for any system**
- Compute spectra for **several modified gravity theories** adapting existing QNM computation techniques
- Not applicable for rotating black holes yet

Example for axial Schwarzschild

Investigate $\frac{\mathrm{d}X_{\text{odd}}}{\mathrm{d}r} = M_{\text{odd}}(r)X_{\text{odd}}$ when $r \longrightarrow +\infty$

$$M(r) = -i\begin{pmatrix} 0 & \omega^2 \\ 1 & 0 \end{pmatrix} + \frac{2}{r}\begin{pmatrix} 1 & 0 \\ -i\mu & 0 \end{pmatrix} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

Diagonalizing

$$\tilde{M}_{\text{odd}}(r) = \begin{pmatrix} -i\omega & 0 \\ 0 & +i\omega \end{pmatrix} + \frac{1}{r}\begin{pmatrix} 1 - i\mu\omega & 0 \\ 0 & 1 + i\mu\omega \end{pmatrix} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

$$\frac{\mathrm{d}\tilde{X}_{\text{odd}}}{\mathrm{d}r} = \tilde{M}_{\text{odd}}\tilde{X}_{\text{odd}} \implies \tilde{X}_{\text{odd}} \sim \begin{pmatrix} e^{-i\omega r} & 0 \\ 0 & e^{+i\omega r} \end{pmatrix} \tilde{X}_c$$

One recovers ingoing and outgoing modes!

Example for polar Schwarzschild

$$M_{\text{polar}}(r) = \begin{pmatrix} 0 & 0 \\ i\omega^2/\lambda & 0 \end{pmatrix} r^2 + \mathcal{O}(r)$$

$$\frac{\mathrm{d}X_{\text{polar}}}{\mathrm{d}r} = M_{\text{polar}} X_{\text{polar}} \implies X_{\text{polar}} \sim \begin{pmatrix} 1 & 0 \\ \frac{i\omega^2}{\lambda}\frac{r^3}{3} & 1 \end{pmatrix} X_c$$

- No diagonalizing possible
- Asymptotic behaviour does not give ingoing and outgoing waves
- Reason of this problem: leading order is nilpotent

Mathematical results

Solution from theory of meromorphic systems of linear ODEs [Wasow '65; Balser '99]

Mathematical algorithm

- Main idea: apply a change of variables $X = P\tilde{X}$
- New system has

$$\tilde{M} = P^{-1}MP - P^{-1}\frac{\mathrm{d}P}{\mathrm{d}r}$$

- $\tilde{M}$ can be diagonal even if M is not diagonalisable

$\Rightarrow$ important result: diagonalization is always possible order by order

Principle of the algorithm

Diagonalisable

$$P = I + \frac{1}{r}\Sigma_1 + \frac{1}{r^2}\Sigma_2 + \dots$$

Solve for Σ_i order by order → final system diagonal

Non-diagonalisable

$$M \sim \begin{pmatrix} \lambda & 0 & \dots \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix} r^p$$

1. $P = \exp\big(\lambda r^{p+1}/(p+1)\big) I$
2. $P = \mathrm{diag}(1, r, r^2, \dots)$
3. Repeat

Several eigenvalues

$$P = I + \frac{1}{r}\Sigma_1 + \frac{1}{r^2}\Sigma_2 + \dots$$

Solve for Σ_i to decouple subsystems order by order

General result

$$M = M_p r^p + M_{p-1} r^{p-1} + \dots \longrightarrow \tilde{M} = D_q r^q + D_{q-1} r^{q-1} + \dots$$

$$X \sim e^{D(r)} r^{D_{-1}} F(r) X_c$$

Algorithm for polar Schwarzschild

$$M_{\text{polar}}(r) = \begin{pmatrix} 0 & 0 \\ i\omega^2/\lambda & 0 \end{pmatrix} r^2 + \mathcal{O}(r)$$

Algorithm

$$P(r) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} r^2 + \begin{pmatrix} i & i \\ \frac{(2\lambda-3)\mu}{4\lambda} - \frac{i}{2\omega} & \frac{(2\lambda-3)\mu}{4\lambda} + \frac{i}{2\omega} \end{pmatrix} r + \mathcal{O}(1)$$

$$\tilde{M}_{\text{polar}}(r) = \begin{pmatrix} -i\omega & 0 \\ 0 & +i\omega \end{pmatrix} + \frac{1}{r} \begin{pmatrix} 1 - i\mu\omega & 0 \\ 0 & 1 + i\mu\omega \end{pmatrix} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

$$\frac{\mathrm{d}\tilde{X}_{\text{polar}}}{\mathrm{d}r} = \tilde{M}_{\text{polar}} \tilde{X}_{\text{polar}} \implies \tilde{X}_{\text{polar}} \sim \begin{pmatrix} e^{-i\omega r} & 0 \\ 0 & e^{+i\omega r} \end{pmatrix} \tilde{X}_c$$

One recovers ingoing and outgoing modes, and behaviour from axial modes

Algorithm for BCL black hole: infinity

BCL black hole

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + A(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\;, \qquad \phi = \psi(r)$$

$$A(r) = 1 - \mu/r - \xi\mu^2/r^2\;, \qquad \psi'(r) = \pm c\Big/ r^2\sqrt{A(r)}$$

$$\begin{tikzcd}
M = \begin{pmatrix} \omega^2 & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} r^2 + \begin{pmatrix} 2\mu\omega^2 & -i\omega & & \\ & & & \\ & & & \\ & & & \end{pmatrix} r + \dots \arrow[d, bend right, "\text{algorithm}"'] \\
D = \begin{pmatrix} -i\omega & & & \\ & +i\omega & & \\ & & -\sqrt{2}\omega & \\ & & & +\sqrt{2}\omega \end{pmatrix} + \dots
\end{tikzcd}$$

$$\mathfrak{g}^{\infty}_{\pm}(t,r) = a_{\pm}\, e^{-i\omega t \pm i\omega r}$$

$$\mathfrak{s}^{\infty}_{\pm}(t,r) = b_{\pm}\, e^{-i\omega t \pm \sqrt{2}\omega r}$$

Algorithm for BCL black hole: horizon

BCL black hole

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + A(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\ , \qquad \phi = \psi(r)$$

$$A(r) = 1 - \mu/r - \xi\mu^2/r^2\ , \qquad \psi'(r) = \pm c\Big/ r^2\sqrt{A(r)}$$

$$\begin{tikzcd}
M = \begin{pmatrix} a_0 & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \dfrac{1}{(r-r_+)^2} + M_{-1}\dfrac{1}{r-r_+} + \ldots \arrow[d, bend right, "\text{algorithm}"'] \\
D = \begin{pmatrix} -i\omega/c_0 & & & \\ & +i\omega/c_0 & & \\ & & 1/2 & 1 \\ & & & 1/2 \end{pmatrix} \dfrac{1}{r-r_+} + \ldots
\end{tikzcd}$$

$$\mathfrak{g}_\pm^{r_+}(t,r) = a_\pm (r-r_+)^{\pm i\omega/c_0}\, e^{-i\omega t}$$

$$\mathfrak{s}_1^{r_+}(t,r) = b_1\sqrt{r-r_+}\, e^{-i\omega t}$$

$$\mathfrak{s}_2^{r_+}(t,r) = b_2\sqrt{r-r_+}\,\log(r-r_+)\, e^{-i\omega t}$$

Conclusion

- Computing quasi-normal modes in beyond GR theories is crucial for the conception of tests
- Additional complexity in the equations (degrees of freedom, higher order...) make the computation of QNMs much more involved in these theories
- Algorithm to extract boundary behaviour without a Schrödinger-like reformulation
- Use it to understand the physics of propagation and to get boundary conditions
- Adapt existing numerical methods to compute deviations in QNMs

Thank you for your attention!