Approximate description of Gravitational Memory

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Abstract: The large-distance behaviour of a sandwich gravitational wave can be approximated by a continuous but not necessarily smooth profile, providing us with a simplified description of particle motion. The approximate model is consistent with the Carroll symmetry. Our strategy is illustrated by the Pöschl-Teller profile.

Introduction

V. B. Braginsky and K. S. Thorne, [*] detect gravitational waves by observing motion of particles hit by the waves $\equiv$ *Memory Effect* (ME)

J. Ehlers and W. Kundt [†] : particles initially at rest fly off with *non-zero constant velocity* $\equiv$ *Velocity Effect* (VM).

Zel'dovich and Polnarev [‡]: final velocity is zero $\equiv$ *Displacement Effect* (DM).

Confirmed for Gaussian, Pöschl-Teller, their derivatives, Scarf, square wave profiles, provided wave amplitude takes "magic" value $\mathbf{k} = \mathbf{k_{crit}}$ $\sim$ quantization conditions in QM.

[*] "Gravitational-wave bursts with memory and experimental prospects," Nature **327** (1987), 123-125

[†] "Exact solutions of the gravitational field equations," (1962)

[‡] "Radiation of gravitational waves by a cluster of superdense stars," Astron. Zh. **51**, 30 (1974)

Sturm-Liouville eqn

Geodesics in $D = 1$ dim ($U \sim$ NR time)

$$\boxed{\frac{d^2X}{dU^2} + \mathcal{A}(U)X = 0\,.} \qquad (1)$$

Sturm-Liouville $\sim$ zero-energy Schrödinger eqn. Normalizability not required.

Particle at rest before hit by wave $\rightsquigarrow$ init cond :

$$X(-\infty) = X_0 = \text{const} \qquad \frac{dX}{dU}(-\infty) = 0\,. \qquad (2)$$

DM when, in addition,

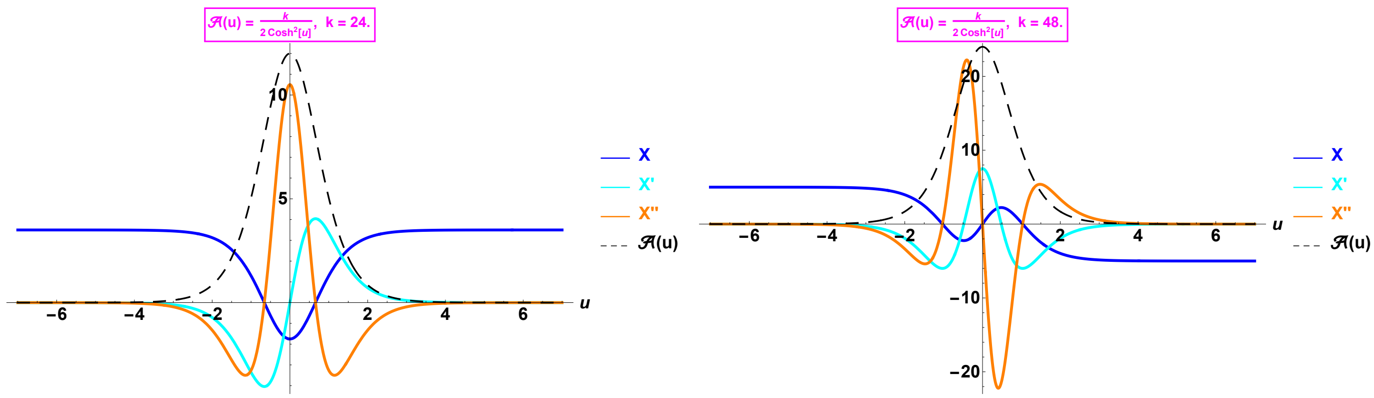
$$X(+\infty) = X_\infty = \text{const}, \qquad \frac{dX}{dU}(+\infty) = 0\,. \qquad (3)$$

System overdetermined $\rightsquigarrow$ sol only for wave amplitude $k = k_{crit}$, found numerically, or using special fcts ($\sim$ confl. Heun, Nikiforov-Uvarov . . .).

Exemple : Pöschl-Teller profile,

$$\mathcal{A}^{PT}(U) = \frac{m(m+1)}{\cosh^2 U}\,, \qquad (4)$$

where m positive integer. Ampli : $k^2 = m(m+1)$. Tragectories $\sim$ Legendre polynomials.

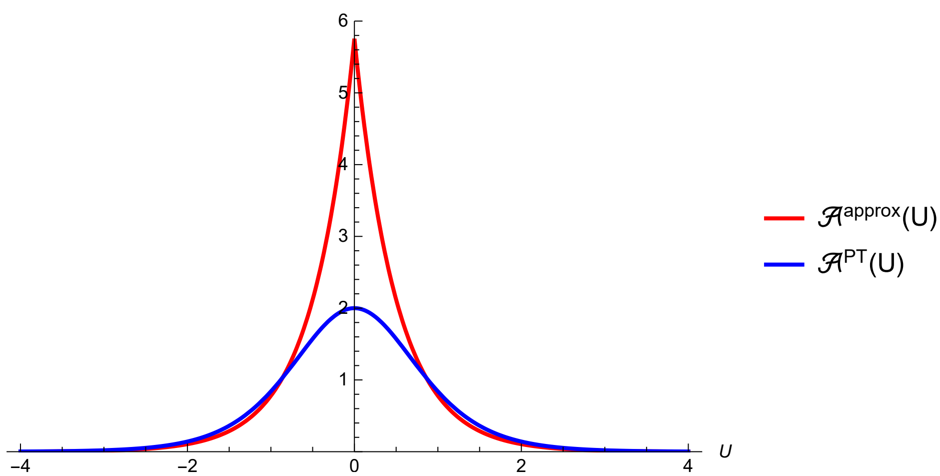


Particle trajectories for PT profile with $\mathbf{m=2}$ *and* $\mathbf{m=3}$.

Approximate model *

For large $|U|$, $\cosh^{-2}(U) \approx 4e^{-2|U|} \;\Rightarrow\;$ PT $\rightsquigarrow$

$$\boxed{\mathcal{A}^{appr}(U) = k^2 e^{-2|U|}\,,} \tag{5}$$



Approximate profile (5) *with amplitude* $k \approx 2.4$ *is close to Pöschl-Teller* $\mathcal{A}^{PT}$ (4) *with* $m = 1$.

Approx breaks down near origin.

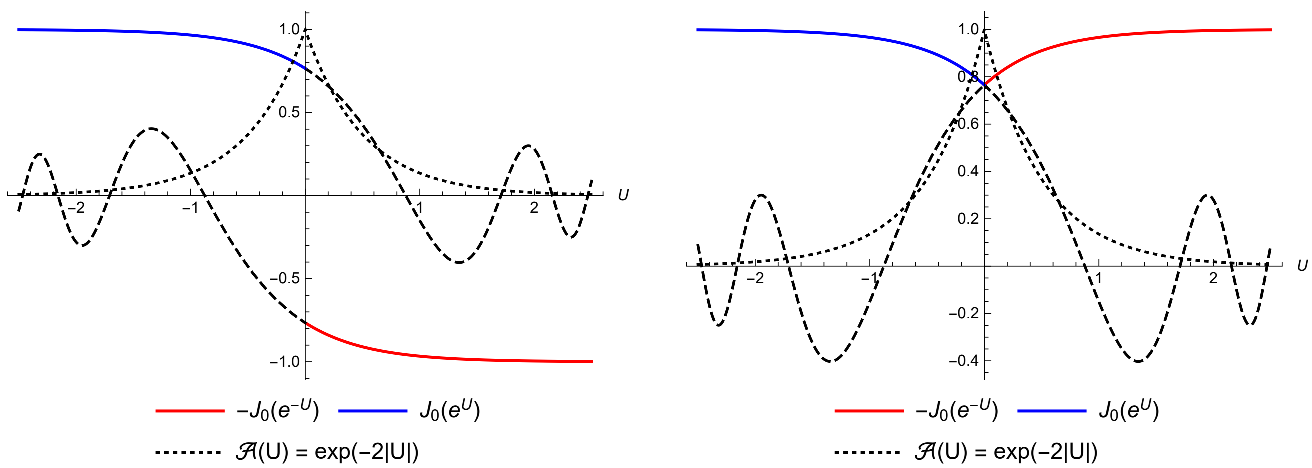
*Q. L. Zhao, P. M. Zhang, M. Elbistan and P. A. Horvathy, "Approximate Models for Gravitational Memory," Phys. Lett. B (2026). [arXiv:2603.15442 [gr-qc]].

Geodesics

Approximate model has exact geodesics $\sim$ Bessel fcts. **DM** boundary conds (2)-(3) used separately in $\mathcal{I}_+ = \{U > 0\}$ & $\mathcal{I}_- = \{U < 0\}$

$$X(U) = \begin{cases} X_+(U) &=& \alpha_1 J_0\big(ke^{-U}\big) & \text{in } \mathcal{I}_+ \\ X_-(U) &=& \alpha_2 J_0\big(ke^{U}\big) & \text{in } \mathcal{I}_- \end{cases} \tag{6}$$

positive / negative U-branches do not match :



For antisymmetric fitting $X_-(0) \neq X_+(0)$. For symmetric fitting $X'_-(0) \neq X'_+(0)$. DM but not "trajectory" !

Strategy: increasing amplitude pulls branches apart. Smooth matching obtained for $k = k_{krit}$ when left & right sols match smoothly at $U = 0$.

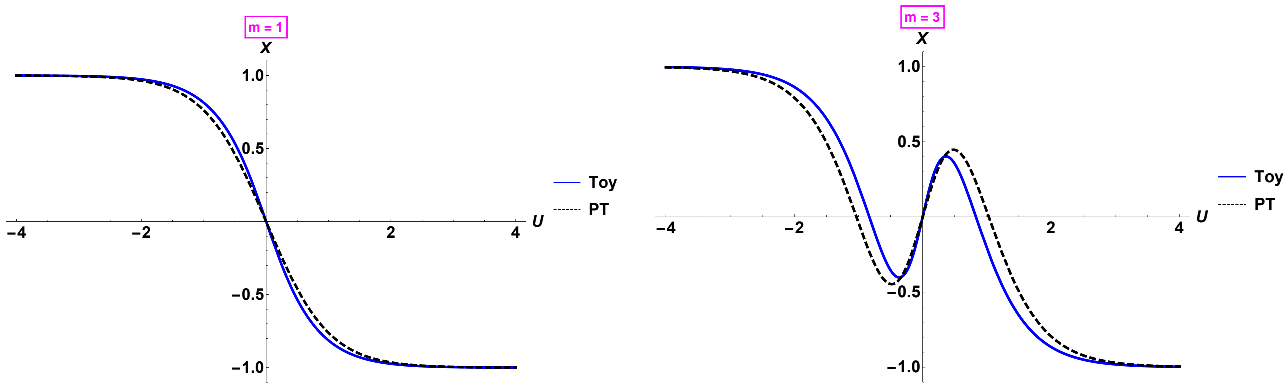
• When $J_0(k) = 0$ get *odd* wave number, $m = 2\ell+1$. Branches joined at origin and slopes equal,

$$X_-(0) = X_-(0) = 0\,, \quad X'_-(0) = X'_+(0)\,. \qquad (7)$$

Require to glue branches antisymmetrically

$$X^{odd}(U) = \begin{cases} X_+(U) &= -\alpha\, J_0\Big(ke^{-U}\Big) \\ X_-(U) &= \alpha\, J_0\Big(ke^{U}\Big) \end{cases} . \qquad (8)$$

Approx trajectories (8) ≈ Pöschl-Teller with odd half-wave number



DM trajectories (dashed) for toy model (5) *approximate those of* **Pöschl-Teller** (4) *with odd half-wave number* $m = 2\ell + 1$.

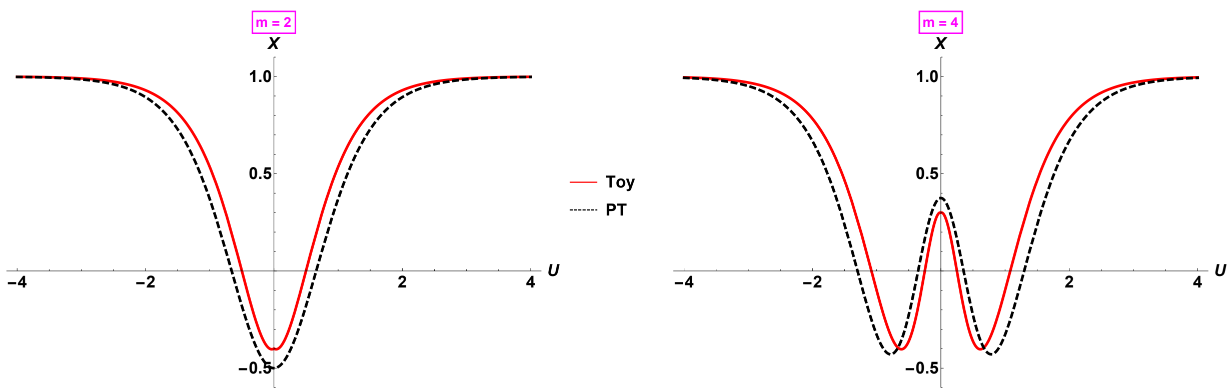
• For $J_0(k) \neq 0$ trajectory be smooth at junction,

$$X'(0) = 0 \;\Rightarrow\; J_1(k) = 0\,. \tag{9}$$

DM trajectory with initial position X_0 is,

$$X^{even}(U) = J_0\Big(k_{(2\ell)}e^{-|U|}\Big)\,X_0\,. \tag{10}$$

Good approximation of Pöschl-Teller for *even* half-wave number $m = 2\ell$. Obtained by gluing branches *symmetrically*, $X(-U) = X(U)$, *



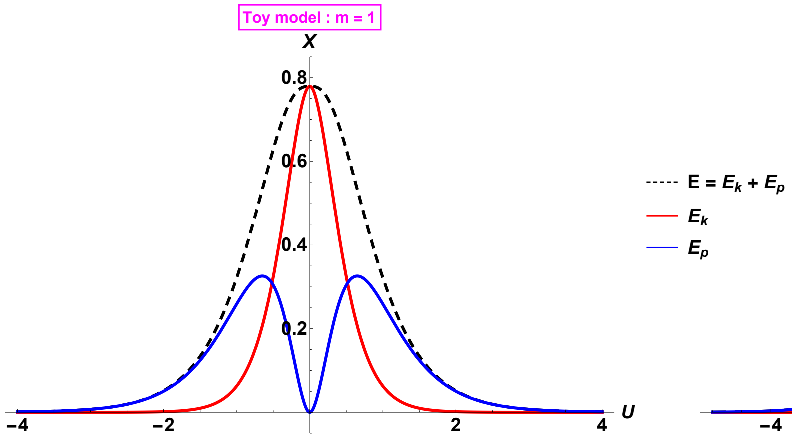
DM toy trajectories (dashed) obtained when Wavezone contains an even number, $m = 2\ell$, of symmetrically glued half-waves. They approximate those for **Pöschl-Teller** *.*

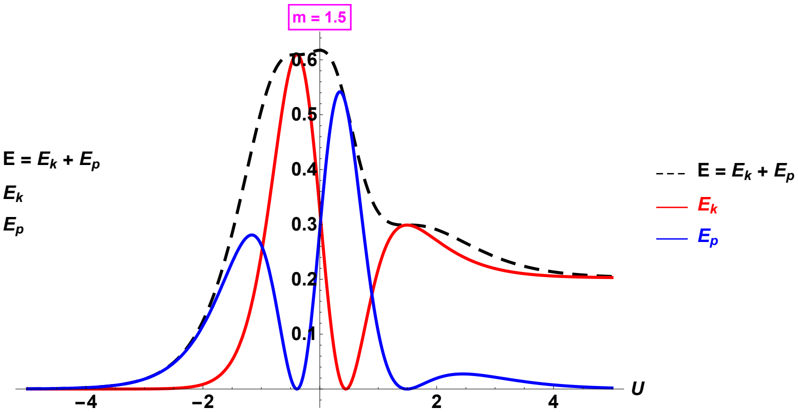
Conclusion: **DM** when:

$$X_-(0) = X_-(0) = 0 \;\text{ or }\; (X_-)'(0) = (X_+)'(0) = 0.$$

* "DM trajectories with no final displacement" .

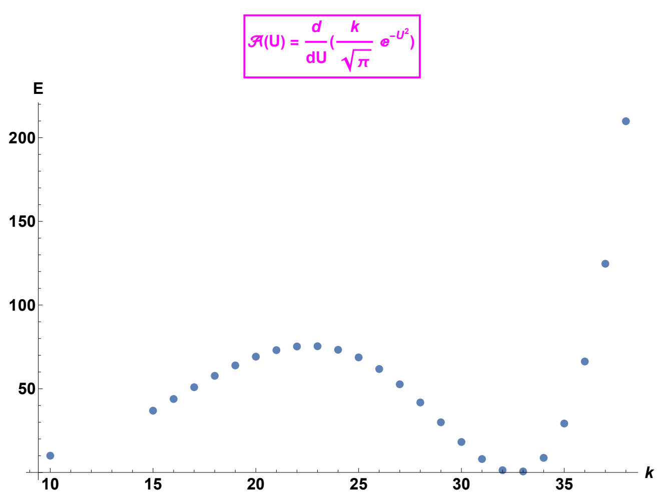
Energy balance

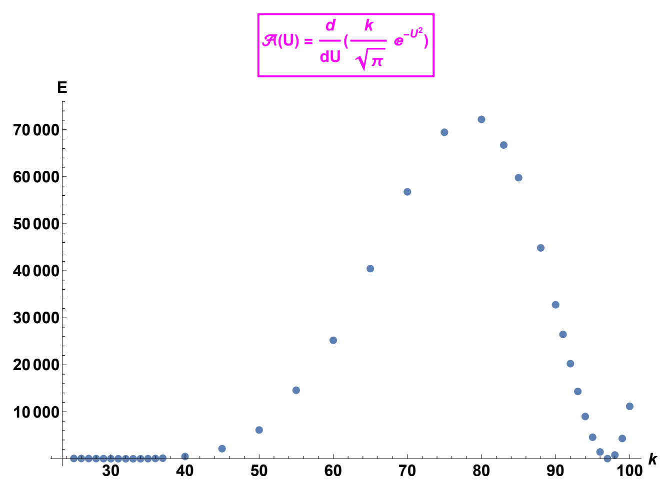




For toy model (5) total (transverse) particle energy balance is zero for **DM** *, positive for* **VM** *.*

Off critical amplitude $k \neq k_{crit}$ we have (**VM**) : outgoing velocity does not vanish, implying increased transverse particle energy.





Energy dissipation ? Propagation ??

Carroll symmetry

$D = 1$ dim Sturm-Liouville eqn (1),

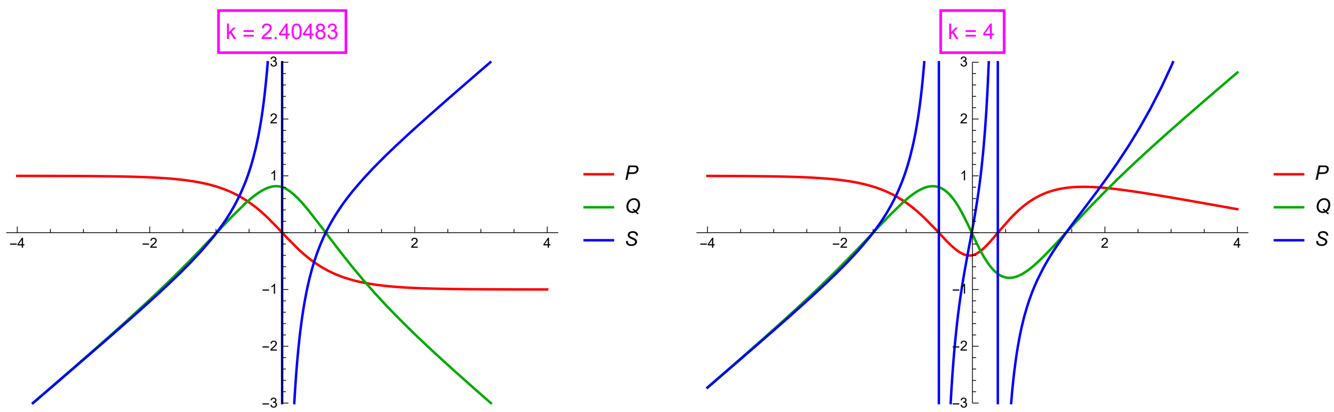
$$\frac{d^2X}{dU^2} + \mathcal{A}(U)X = 0\,, \tag{11}$$

has two indept solutions. P has init conds

$$P(-\infty) = 1 \qquad \frac{dP}{dU}(-\infty) = 0\,, \tag{12}$$

2nd, indept sol $\sim$ Souriau-Arnold matrix*

$$\mathbf{Q(U) = P(U)S(U)}\,, \quad S(U) = \int_{U_0}^{U} dv P^{-2}(v)\,. \tag{13}$$



$\mathbf{P}$, *non-DM 2nd solution* $\mathbf{Q = PS}$, *and* **Souriau matrix** $\mathbf{S}$, *shown for toy model* (5) *with (a)* **DM** *amplitude* $k = k_{crit}$ *and (b)* **VM** *amplitude* $k = 4$.

*J-M. Souriau, "Ondes et radiations gravitationnelles," Colloques Internationaux du CNRS No 220, pp. 243-256. Paris (1973); V. I. Arnold, *Dopolnitelnye glavy teorii obyknovennykh differentsialnykh uravnenii*. Nauka (1978).

P & Q span *Carroll symmetry** generated by $\partial/\partial V$ and

$$\underbrace{c\left(P\frac{\partial}{\partial X}-P'X\frac{\partial}{\partial V}\right)}_{translations}+\underbrace{b\left(Q\frac{\partial}{\partial X}-Q'X\frac{\partial}{\partial V}\right)}_{Carroll\,boosts}. \quad (14)$$

Assoc conserved quantities

$$\mathbf{p}_0=P\boldsymbol{X}'-\big(P\big)'\boldsymbol{X}, \quad \mathbf{k}_0=-Q\boldsymbol{X}'+\big(Q\big)'\boldsymbol{X} \quad (15)$$

determine geodesics†

$$\boldsymbol{X}(U)=P(U)\,\mathbf{k}_0+Q(U)\,\mathbf{p}_0\,. \quad (16)$$

DM for $\mathbf{p}_0=0$.

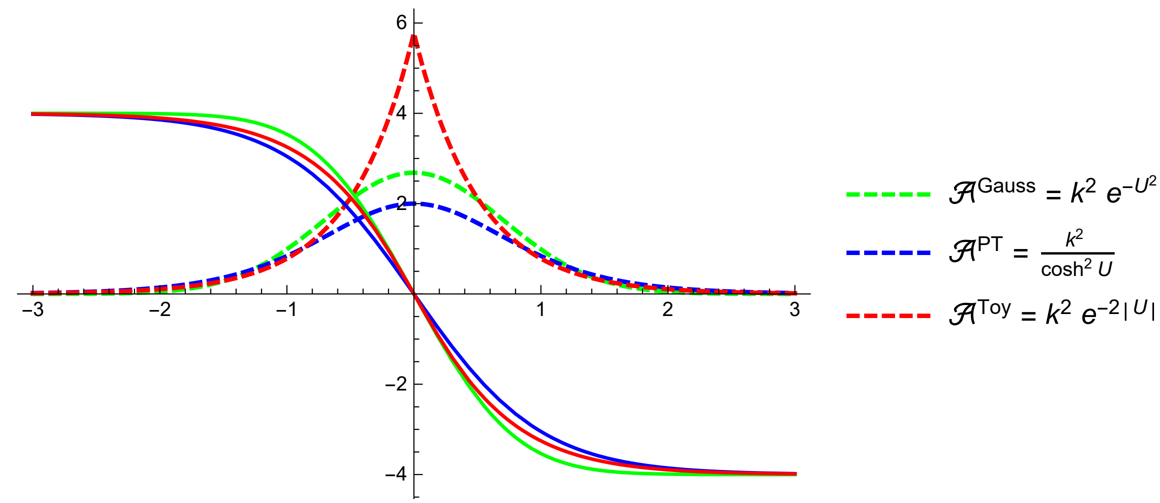
Approx not limited to PT. Gaussian prof

$$\mathcal{A}^{Gauss}(U)\;\propto\;e^{-U^2}. \quad (17)$$

For large $|U|$, approximated by toy profile. Gaussian & Pöschl-Teller geodesics are similar despite different critical amplitudes :

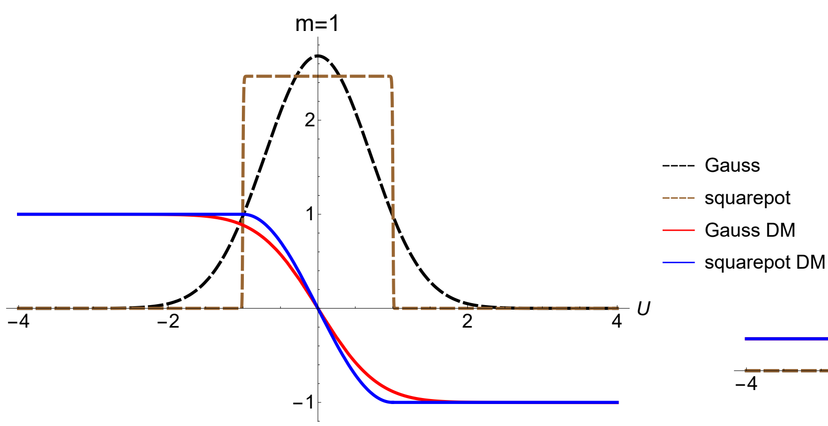
*J. M. Lévy-Leblond, "Une nouvelle limite non-relativiste du groupe de Poincaré," Ann. Inst. H Poincaré **3** (1965) 1

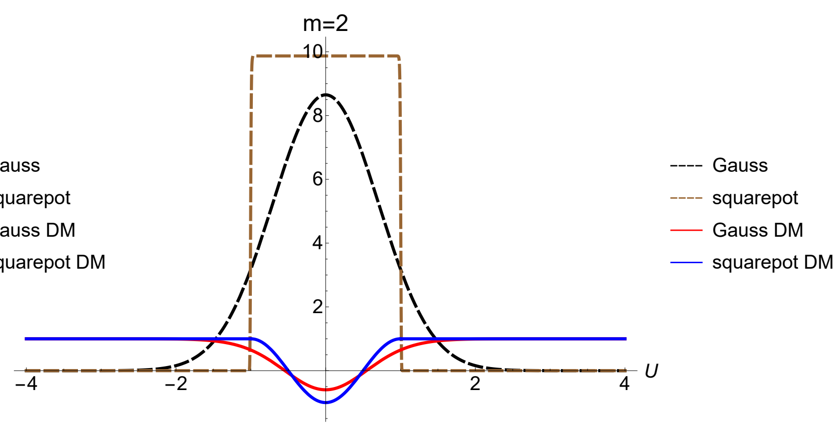
†M. Elbistan, P. M. Zhang and P. Horvathy, *"Globally defined Carroll symmetry of gravitational waves,"* Nucl. Phys. B **1024** (2026), 117354.



Gaussian profile and trajectory approx PT.

Approx scheme works also for square profiles,

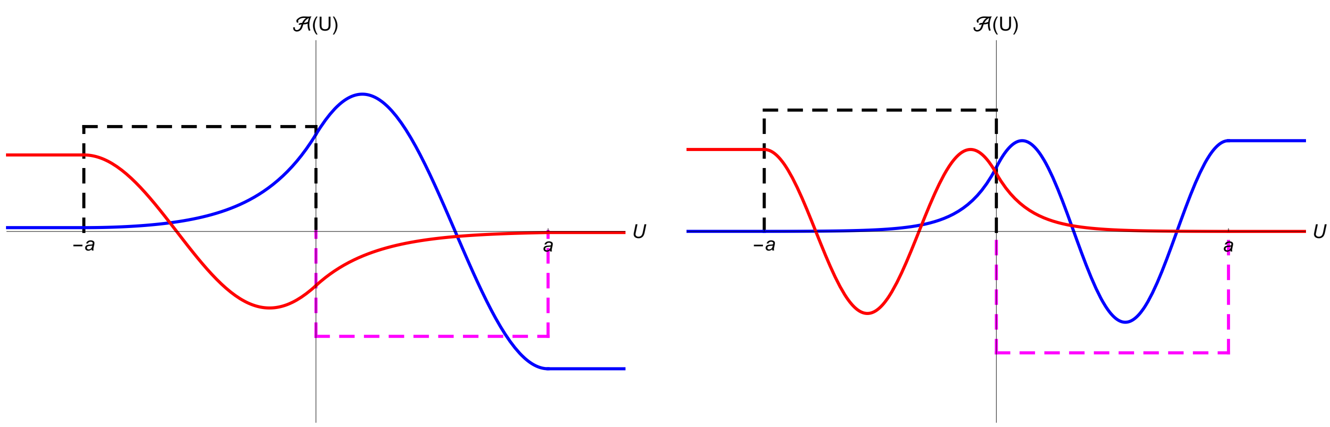




Gaussian (17) *and square profiles have similar trajectories.*

Square profiles could also be combined antisymmetrically yielding double-square approximation for flyby profile

$$\mathcal{A} \equiv \mathcal{A}^G = \frac{d}{dU}\left(\frac{k}{\sqrt{\pi}}e^{-U^2}\right) \qquad (18)$$



DM *trajectories for double-square approximation of flyby profile* (18) *with wave numbers* $\mathbf{m = 1}$ *and* $\mathbf{m = 2}$.

Parity-dependent U-inversion antisymmetry/symmetry manifest for both components $X^{\pm}(U)$.