Compactifying Gauge Theories

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Maxim Grigoriev, M.M. 2512.06576, 2310.09637.

Background

- Theories defined on manifolds with boundaries, particularly asymptotic boundaries, play an important role in modern field theory. Gauge field theories in such settings are of special interest.

- In the GR context, asymptotic boundaries can be effectively treated as ordinary boundaries using the conformal compactification technique *[Penrose, Geroch, Friedrich, Kroon,...]*. Extra structure: a boundary defining function.

Background

- Fefferman-Graham construction: diffeomorphism-invariant equations in asymptotically AdS spacetimes $\rightarrow$ Weyl-invariant equations on the conformal boundary.

- In the AdS/CFT context, it is known *[Skenderis, Solodukhin, Henningson]* that the anomaly term in the renormalized on-shell action can be interpreted as an action functional for the Fefferman–Graham equations induced on the boundary.

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- The Batalin-Vilkovisky (BV) formalism provides a powerful framework for the systematic analysis of gauge theories.
- Gauge PDE: a geometric object encoding the BV formalism [*Barnich, Grigoriev 2010; Grigoriev, Kotov 2019*]. It behaves well under restriction to submanifolds and boundaries in a manifestly coordinate-independent manner.
- A closely related approach to the boundary structure of gauge field theories in the AdS space based on ambient space techniques: [*Bekaert, Grigoriev, Chekmenev*].
- Related formalism employing BV and related structures:
  Unfolded approach of HS theories *Vasiliev...*
  AKSZ construction *Alexandrov, Kontsevich, Schwarz, Zaboronsky*
  BV/BFV formalism with boundaries *Cattaneo et al*

Goal of this talk

To show how Penrose's idea of compactification can be interpreted from the broader gauge-theoretical perspective of the gPDE framework, and to demonstrate a natural generalization of the idea of an "anomalous" boundary action for the boundary values of a theory to a general setting.

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Example. Given a smooth manifold X with local coordinates x^μ, one can construct the supermanifold $T[1]X$ with coordinates (x^μ, θ^μ), where the θ^μ are anticommuting: $\theta^\mu\theta^\nu = -\theta^\nu\theta^\mu$. It is equipped with the vector field

$$d_X \equiv \theta^\mu \frac{\partial}{\partial x^\mu}, \quad d_X^2 = 0.$$

One easily sees that the complex $(C^\infty(T[1]X), d_X)$ is isomorphic to the de Rham complex $(\Lambda(X), d_{\mathrm{dR}})$.

Q-manifolds with Q-boundaries

Definition

A submanifold $\partial^Q M \subset M$ is called a Q-boundary if there exists a local coordinate system (Ω, ξ, y^A) with $gh(\Omega) = 0$ and $\Omega \geq 0$, such that $\partial^Q M$ is locally specified by $\Omega = \xi = 0$ and

$$Q\Omega = \xi.$$

Example. Given a smooth manifold with boundary X, its shifted tangent bundle $(T[1]X, d_X)$ is naturally a Q-manifold with a Q-boundary. If (Ω, y^a) are near-boundary coordinates on X, then the Q-boundary is locally specified by

$$\Omega = d_X \Omega = 0.$$

Moreover,

$$\partial^Q T[1]X = T[1]\partial X.$$

Cocycle map

Theorem

Let (M, Q) *be a* Q*-manifold with a* Q*-boundary, and let* $\tilde{v} \in \bigwedge^\bullet(Int(M))$ *be a* Q*-cocycle that diverges as one approaches the boundary. Assume that there exists* $p \in \mathbb{N}$ *such that*

$$v = \Omega^p \tilde{v}$$

smoothly extends $\partial^Q M$*. Then there exist two* Q*-cocycles on* $\partial^Q M$*, denoted by* $v^{\mathcal{A}}$ *and* $v^{\mathcal{R}}$*, such that*

$$gh(v^{\mathcal{A}}) = gh(\tilde{v}) - 1, \qquad gh(v^{\mathcal{R}}) = gh(\tilde{v}).$$

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Moreover, one can show that there exists a form $v_{ren} \in \bigwedge^{\bullet}(M)$ such that

$$v_{ren} = \tilde{v} + L_Q(\dots) \quad \text{on } Int(M)\,, \qquad b^* v_{ren} = v^{\mathcal{R}}\,,$$

(cf. holographic renormalization *[Henningson, Skenderis, Solodukhin,…]* and homotopic renormalization *[Costello,…]*).

AKSZ-like Gauge PDEs

It is known [*Grigoriev, Barnich, Kotov*] that, disregarding global aspects, the field content of any gauge field theory at the level of equations of motion can be realized as the space of degree-preserving maps

$$\sigma : (T[1]\tilde{X}, d_{\tilde{X}}) \to (\tilde{M}, \tilde{Q}) \quad \text{satisfying} \quad d_{\tilde{X}} \circ \sigma^* = \sigma^* \circ \tilde{Q}.$$

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Suppose, in addition, that we have $\tilde{\omega} \in \bigwedge^2(\tilde{M})$ such that

$$L_{\tilde{Q}}\tilde{\omega} = 0, \qquad gh(\tilde{\omega}) = dim(\tilde{X}) - 1\,.$$

Then there is a natural functional on the space of maps [*Grigoriev, Alkalaev*]

$$S[\sigma] = \int_{T[1]\tilde{X}} \sigma^* \tilde{\chi}(d_{\tilde{X}}) + \sigma^* \tilde{H},$$

where $\tilde{\omega} = d\tilde{\chi}$ and $i_{\tilde{Q}}\tilde{\omega} + d\tilde{H} = 0$. In the case where the target $\tilde{M}$ is finite-dimensional, this is precisely the content of the celebrated AKSZ construction for topological field theories.

Compactification of Gauge PDEs

Suppose that

$$\tilde{X} = Int(X), \qquad (\tilde{M}, \tilde{Q}) = (Int(M), Q)$$

for some manifold with boundary X, and a Q mainfold with a Q-boundary (M, Q). We then obtain a compactification of the original theory.
The space of boundary values is naturally identified with the space of maps

$$\sigma_\partial : (T[1]\partial X, d_{\partial X}) \to (\partial^Q M, Q_\partial) \quad \text{satisfying} \qquad d_{\partial X} \circ \sigma_\partial^* = \sigma_\partial^* \circ Q_\partial .$$

We note that a given theory may admit different compactifications corresponding to different asymptotic behaviors of the fields.

Compactification of Gauge PDE's

For a given compactification, if the presymplectic structure $\tilde{\omega}$ on $Int(M)$ is polynomially divergent near the boundary, then one has two cocycles

$$\omega^{\mathcal{A}}, \omega^{\mathcal{R}} \in \bigwedge^2(\partial^Q M), \quad gh(\omega^{\mathcal{A}}) = dim(\partial X) - 1, \quad gh(\omega^{\mathcal{R}}) = dim(\partial X).$$

$\omega^{\mathcal{R}}$ is the analogue of the renormalized covariant phase-space structure on the boundary, but we will not discuss it in this talk.

Instead, we note that the form $\omega^{\mathcal{A}}$ has precisely the right degree to define an action functional on the space of boundary values.

AAdS examples

In the case of asymptotically AdS gravity and fields propagating on its background, the construction reproduces the known results.

- $D = 5, 7$ AAdS gravity $\longrightarrow$ an action for $d = 4, 6$ conformal gravity.
- Klein–Gordon fields on an AAdS gravitational background $\longrightarrow$ action functionals for GJMS equations (with the conformal scalar as a special case).

By construction, all resulting action functionals are of first order in derivatives.

Flat examples

An important feature of the formalism presented here is that the (asymptotically) AdS case is in no way distinguished. Analogous results hold for any gauge field theory with an asymptotic boundary. For example,

Fields in the Minkowski bulk

$\Downarrow$

Conformally Carroll-invariant actions on null infinity

Flat examples: scalar

Starting with a scalar field exhibiting the asymptotic behaviour

$$\varphi \sim \Omega^{\frac{3-D}{2}}(\mathrm{reg}),$$

one obtains

$$S[\varphi, \varphi^{(1)}] = \int du d^{D-2}y\sqrt{\gamma}(\varphi^{(1)}\partial_u\varphi + \frac{1}{2}\partial^i\varphi\partial_i\varphi + \frac{D-1}{8D}\varphi^2 R[\gamma])$$

where γ is the metric on the $(D-2)$-dimensional celestial sphere, with coordinates y. This is the well-known action for the magnetic Carroll scalar.

Flat examples: scalar

For more general asymptotic behaviour

$$\varphi \sim \Omega^{\frac{r-D}{2}}(\mathrm{reg}), \qquad r \in \mathbb{N}_{\geq 3}$$

and working in a flat patch of the celestial sphere, one obtains the general action

$$S[\varphi^{(i)}] = \sum_{i=0}^{r-3} C_{r-3}^{i} \int du d^{D-2}y \left(\varphi^{(r-2-i)}\partial_u \varphi^{(i)} + \frac{1}{2}\partial_A \varphi^{(r-3-i)}\partial^A \varphi^{(i)}\right)$$

which is a first-order action for the equations

$$(2N - r + 2)\partial_u \varphi^{(N)} + N\partial_A \partial^A \varphi^{(N-1)} = 0\,, \quad 0 \leq N \leq r-2\,.$$

The untruncated version of these equations can be obtained from an analysis of the wave equation on Minkowski space, see *[Satishchandran, Wald; Bekaert, Raj]*.

Flat examples: Maxwell

Starting with the Maxwell field on a $D = 5$ dimensional Minkowski, we obtain

$$S[A, F, J] = \int du d^3y \sqrt{\gamma} (2J^i(\partial_u A_i - \partial_i A_u) + 2F^{ij}\partial_i A_j - \frac{1}{2} F_{ij}F^{ij}) \,,$$

which is a first-order action for the magnetic limit of Maxwell theory.

Flat examples: Maxwell

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$$S[A, F, J] = \int dud^3y\sqrt{\gamma}(2J^i(\partial_u A_i - \partial_i A_u) + 2F^{ij}\partial_i A_j - \frac{1}{2}F_{ij}F^{ij})\,,$$

which is a first-order action for the magnetic limit of Maxwell theory. Starting with $D = 6$, in flat coordinates we obtain an action

$$S[A, J^{(0)}, J^{(1)}] = \int dud^4y(J^{(1)|B}\partial_{[u}A_{B]} + \partial^{[A}J^{(0)B]}\partial_{[A}A_{B]})$$

with equations of motion

$$\begin{aligned} &\partial_u A_i = 0, \qquad \partial_i F^{ij}[A] = 0, \\ &\partial^i J^{(1)}_i = 0, \qquad \partial_u J^{(1)}_i + \partial^j(\partial_j J^{(1)}_i - \partial_i J^{(0)}_j) = 0. \end{aligned}$$

Conclusions

What has been shown here:

- The notion of a Q-boundary of a Q-manifold and a $1 \to 2$ map for divergent cocycles.
- A geometric, coordinate-independent, BV-AKSZ approach to asymptotic boundaries of gauge field theories.
- A general construction of action principles for boundary-value equations, together with applications to AAdS and flat-space theories.

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- A geometric, coordinate-independent, BV-AKSZ approach to asymptotic boundaries of gauge field theories.
- A general construction of action principles for boundary-value equations, together with applications to AAdS and flat-space theories.

What has not been shown here:

- Applications to general asymptotically flat background.
- Asymptotic symmetries in this formalism, including the BMS group *[M.G., M.M. 2310.09637]*.
- $\omega^{\mathcal{R}}$-structure $\Rightarrow$ BV version of the covariant phase space formalism, asymptotic charges, etc.