Nonlinear stability of Minkowski spacetime for massive Dirac spinor fields

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Main system

We study the Einstein field equations of general relativity coupled with a massive Dirac spinor field

$$
\begin{aligned}
R_{\mu\nu} - \tfrac{1}{2} R_g g_{\mu\nu} &= 8\pi T[\Psi]_{\mu\nu}, \\
\mathfrak{D}\Psi + \mathrm{i} M \Psi &= 0,
\end{aligned}
$$

with the *Dirac operator*

$$
\mathfrak{D}\Psi := g^{\alpha\beta} \partial_\alpha \cdot \nabla_\beta \Psi,
$$

and the energy-momentum tensor T is defined as

$$
\begin{aligned}
T[\Psi]_{\mu\nu} :=& \frac{\mathrm{i}}{4} \Big(\langle \Psi, \partial_\mu \cdot \nabla_\nu \Psi \rangle_{\mathrm{D}} + \langle \Psi, \partial_\nu \cdot \nabla_\mu \Psi \rangle_{\mathrm{D}} \Big) \\
& - \frac{\mathrm{i}}{4} \Big(\langle \partial_\mu \cdot \nabla_\nu \Psi, \Psi \rangle_{\mathrm{D}} + \langle \partial_\nu \cdot \nabla_\mu \Psi, \Psi \rangle_{\mathrm{D}} \Big).
\end{aligned}
$$

Literature Review

Vacuum case:

Christodoulou–Klainerman [1993]: geometric proof of nonlinear stability.

Lindblad–Rodnianski [2005]: wave-coordinate approach; reduced Einstein equations.

Massive scalar fields:

LeFloch–Ma [2018(arXiv:1511.03324)]: Einstein–Klein–Gordon; Euclidean–hyperboloidal method.

Ionescu–Pausader [2019]: Einstein–Klein–Gordon; Z-norm framework; KG modified scattering.

Wang [2020]: Intrinsic hyperboloid approach; maximal foliation gauge.

Einstein-Dirac system

Chen [2025]: Massless case; coordinate-based formulation (concrete tetrad constructed); global well-posedness.

Zhao-Wu [2025]: characteristic initial value problem.

LeFloch-Ma-Zhang [2025]: Massive Einstein-Dirac; gauge-invariant formulation; global well-posedness.

Theorem

Let g_{r} be an $(N, \theta, \varepsilon_\star, \ell)$–admissible reference metric in the original asymptotically Euclidean chart $(\widetilde{t}, \widetilde{x})$, with N sufficiently large, $\theta, \varepsilon_\star$ sufficiently small, and $\ell \in (0, 1/2)$. After applying the coordinate transformation

$$t = \widetilde{t}, \quad x^a = (r/\widetilde{r})\widetilde{x}^a,$$

write the metric, the reference metric, and the spinor field in the new coordinates (t, x), and set

$$u_{\alpha\beta} = g_{\alpha\beta} - g_{\mathrm{r}\alpha\beta}, \quad r = |x|.$$

Assume that the initial data on $\{t = 1\}$ satisfy the Einstein constraint equations and the Dirac compatibility conditions. Suppose moreover that

$$\sum_{|I| \leq N} \left\| \langle r \rangle^{\kappa + |I|} \left(|\partial_x^I \partial_x u_0| + |\partial_x^I \partial_t u_0| \right) \right\|_{L^2(\mathbb{R}^3)} \leq \varepsilon, \quad \kappa \in (1/2, 3/4),$$

where $u_0 = u|_{\{t=1\}}$, and that the spinor initial data satisfy

$$\left\| \langle r \rangle^{\mu + N + 1} |\widehat{\partial}_x^J \Psi_0|_{\vec{n}} \right\|_{L^2(\mathbb{R}^3)} \leq \varepsilon, \quad |J| \leq N + 1, \quad \mu \in (3/4, 1),$$

where $\Psi_0 = \Psi|_{\{t=1\}}$ and $\vec{n}$ is the future-oriented unit normal of $\{t = 1\}$. Then the maximal globally hyperbolic Cauchy development of the data is future causally geodesically complete and remains asymptotically Euclidean in all causal directions.

Main Challenge: Transplanting scalar analysis to the spinor bundle

What is different from the Einstein–Klein–Gordon (scalar) case?

- Levi-Civita connection $\rightarrow$ spin connection ;
- wave/Klein-Gordon operator $\rightarrow$ Dirac operator;

Main technical points of the vector field method (for E-KG case)

1. good commutation relations between the Killing fields and wave/Klein-Gordn operator;
2. appropriate foliation & energy estimates for wave operator;
3. Klainerman–Sobolev inequalities related to the foliation;
4. pointwise estimate for Klein–Gordon equation;
5. estimates for the matter-metric coupling terms in the Einstein equations

Gauge-invariant differentiation of spinor fields

∇ of Levi-Civita connection

1. Linearity

$$\nabla_{fX} Y = f \nabla_X Y$$

2. Leibniz rule

$$\nabla_X (fY) = df(X) Y + f \nabla_X Y$$

3. Metric compatibility

$$X\langle Y, Z\rangle_g = \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g$$

∇ of spin connection

1. Linearity

$$\nabla_{fX} \Psi = f \nabla_X \Psi$$

2.a Leibniz rule (scalar product)

$$\nabla_X (f\Psi) = \mathrm{d}f(X)\Psi + f \nabla_X \Psi$$

2.b Leibniz rule (Clifford product)

$$\nabla_X (Y \cdot \Psi) = Y \cdot \nabla_X \Psi + \nabla_X Y \cdot \Psi$$

4. Dirac form compatibility

$$X\langle \Phi, \Psi\rangle_{\mathrm{D}} = \langle \nabla_X \Phi, \Psi\rangle_{\mathrm{D}} + \langle \Phi, \nabla_X \Psi\rangle_{\mathrm{D}}$$

Commuting with the Dirac operator

Proposition (Commutator, invariant form)

For any spinor field Ψ *and any vector field* X*, with the adapted derivative* $\widehat{X}\Psi := \nabla_X \Psi - \frac{1}{4} g^{\alpha\gamma}\partial_\alpha \cdot \nabla_\gamma X \cdot \Psi$*, one has*

$$[\widehat{X}, \mathfrak{D}]\Psi = -\frac{1}{2}\pi[X]^{\alpha\beta}\partial_\alpha \cdot \nabla_\beta \Psi - \frac{1}{4} g^{\alpha\beta}\mathrm{Ric}(X, \partial_\alpha)\partial_\beta \cdot \Psi - \frac{1}{4}\Box_g X \cdot \Psi,$$

where $\Box_g := g^{\alpha\beta}\nabla^2_{\alpha\beta}$ *and* $\pi[X]$ *is the deformation tensor of* X*, defined as* $\pi[X]_{\alpha\beta} = (\mathcal{L}_X g)_{\alpha\beta}$.

Proposition (Commutator in generalized wave gauge)

With the wave-gauge vector field $W = W^\alpha \partial_\alpha = g^{\alpha\beta}\nabla_\alpha \partial_\beta$*, for Killing vector fields* $Z \in \{L_a, \Omega_{ab}, \partial_\alpha\}$*, one has*

$$[\widehat{Z}, \mathfrak{D}]\Psi = -\frac{1}{2}\pi[Z]^{\alpha\beta}\partial_\alpha \cdot \widehat{\partial_\beta}\Psi - \frac{1}{8} g^{\mu\nu}\pi[Z]^{\alpha\beta}\partial_\alpha \cdot \partial_\mu \cdot \nabla_\nu \partial_\beta \cdot \Psi$$
$$- \frac{1}{4}\pi[Z]^{\alpha\beta}\nabla_\alpha \partial_\beta \cdot \Psi - \frac{1}{4}[Z, W] \cdot \Psi.$$

The Euclidean-hyperboloidal foliation

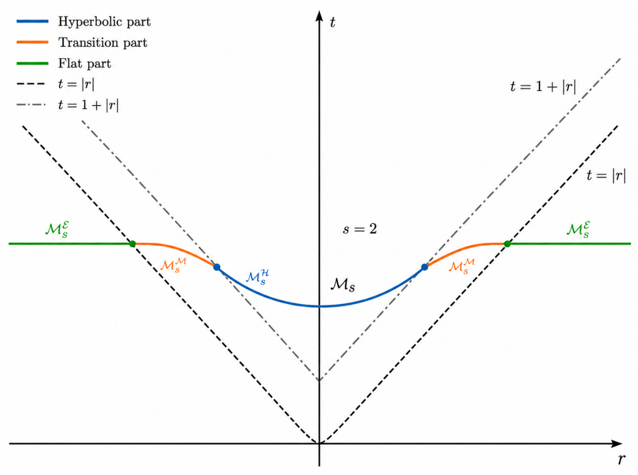


Figure: The Euclidean hyperboloidal slice

The uniform-spacelike conditions

In order to guarantee that the hypersurfaces are (sufficiently) spacelike, for a curved metric $g = \eta + H$, we impose the following **uniform spacelike conditions**:

$$H^{\mathcal{N}00} := g(\mathrm{d}t - \mathrm{d}r, \mathrm{d}t - \mathrm{d}r) < 0 \quad \text{in } \{3t/4 \leq r \leq r^{\mathcal{E}}(s)\} \cap \mathcal{M}_{[s_0,s_1]} \tag{1a}$$

with, for a sufficiently small $\varepsilon_s > 0$,

$$|H| := \max_{\alpha,\beta} |H_{\alpha\beta}| \leq \varepsilon_s \zeta \quad \text{in } \mathcal{M}_{[s_0,s_1]}, \tag{1b}$$

where ζ is the Minkowski-induced volume factor on $\mathcal{M}_s$, defined by

$$\zeta^2 = 1 - |\partial_r T|^2.$$

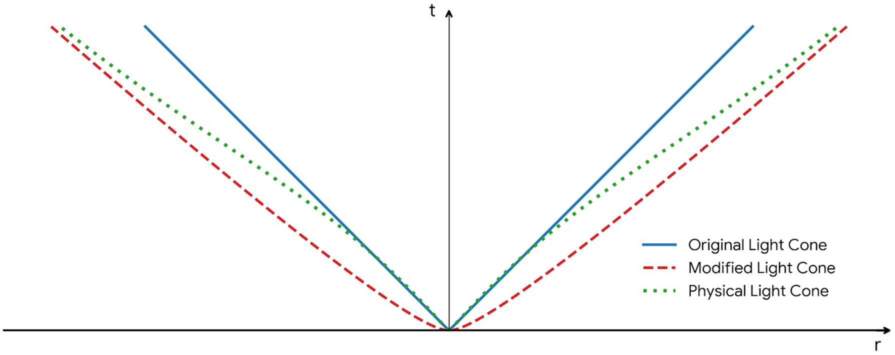


Figure: Comparison of the light-cones: $H^{\mathcal{N}00} < 0$

Dirac form and positive spinor norm

Let $\langle \cdot, \cdot \rangle$ be the standard Hermitian inner product on $\mathbb{C}^4$ and define the Dirac form by

$$\langle \Phi, \Psi \rangle_{\mathrm{D}} := \langle \gamma_0 \phi, \psi \rangle.$$

Here the Clifford multiplication is defined as follows:

$$v \cdot \psi = v^{\alpha} \gamma_{\alpha} \psi, \quad \psi \in \mathbb{C}^4.$$

We define the $\vec{n}$-dependent sesquilinear form

$$\langle \Phi, \Psi \rangle_{\vec{n}} := \langle \Phi, \vec{n} \cdot \Psi \rangle_{\mathrm{D}} = \langle \gamma_0 \phi, \vec{n} \cdot \psi \rangle,$$

and the induced pointwise norm

$$|\Psi|^2_{\vec{n}} := \langle \Psi, \Psi \rangle_{\vec{n}} = \langle \Psi, \vec{n} \cdot \Psi \rangle_{\mathrm{D}}.$$

For a future-directed unit timelike $\vec{n}$, this defines a positive definite inner product, and hence Cauchy–Schwarz yields

$$\left| \langle \Phi, \vec{n} \cdot \Psi \rangle_{\mathrm{D}} \right| = |\langle \Phi, \Psi \rangle_{\vec{n}}| \leq |\Phi|_{\vec{n}} |\Psi|_{\vec{n}}.$$

Energy flux

Consider the inhomogeneous Dirac equation

$$\mathfrak{D}\Psi + \mathrm{i}M\Psi = \Phi.$$

Then the Dirac current:

$$V[\Psi] := \langle \Psi, g^{\alpha\beta}\partial_\alpha \cdot \Psi \rangle_{\mathrm{D}} \partial_\beta,$$

which has a positive normal flux:

$$g(V[\Psi], \vec{n}) = \langle \Psi, \vec{n} \cdot \Psi \rangle_{\mathrm{D}} = |\Psi|^2_{\vec{n}}.$$

Energy identity (differential form):

$$\mathrm{div}(V[\Psi]) = 2\,\Re\big(\langle \Phi, \Psi \rangle_{\mathrm{D}}\big).$$

Proposition (Exterior energy estimates)

Under the **uniform-spacelike conditions** *with a sufficiently small* $\varepsilon_s > 0$

$$H^{\mathcal{N}00} < 0 \quad in\ \{r \geq 3t/4\} \cap \mathcal{M}_{[s_0,s_1]}, \qquad |H| := \max_{\alpha,\beta} |H| \leq \epsilon_s \zeta \quad in\ \mathcal{M}_{[s_0,s_1]},$$

the following energy estimates hold for any solution to the equation

$$\mathfrak{D}\Psi + \mathrm{i}M\Psi = \Phi$$

with sufficient decay rate at spatial infinity:

$$\mathbf{E}_\kappa^{\mathcal{E}\mathcal{M}}(s_1,\Psi) - \mathbf{E}_{g,\kappa}^{\mathcal{E}\mathcal{M}}(s_0,\Psi) + \kappa \int_{s_0}^{s_1} s \int_{\mathcal{M}_s^{\mathcal{E}\mathcal{M}}} \frac{\big(\omega^\kappa \zeta |\Psi|_{\vec{\gamma}}\big)^2}{\langle r-t\rangle} \mathrm{d}x\mathrm{d}s + \mathbf{E}^{\mathcal{C}}(s_0,s_1;\Psi)$$

$$\lesssim \int_{s_0}^{s_1} s \int_{\mathcal{M}_s^{\mathcal{E}\mathcal{M}}} \omega^{2\kappa}\zeta^2 \big|\langle \Phi,\Psi\rangle_{\mathrm{D}}\big| \,\mathrm{d}x\mathrm{d}s \lesssim \int_{s_0}^{s_1} s\mathbf{E}_\kappa^{\mathcal{E}\mathcal{M}}(s,\Psi)^{1/2} \|\omega^\kappa\zeta^2\bar{\zeta}^{-1/2}|\Phi|_{\vec{n}}\|_{L^2(\mathcal{M}_s^{\mathcal{E}\mathcal{M}})}\,\mathrm{d}s,$$

where $\vec{\gamma} := \mathrm{grad}(r-t)$, $|\Psi|_{\vec{\gamma}}^2 = \langle \Psi, \vec{\gamma}\cdot\Psi\rangle_{\mathrm{D}}$, $\bar{\zeta} = (\zeta^2 + |H^{\mathcal{N}00}|)^{1/2}$, *and*

$$\mathbf{E}_\kappa^{\mathcal{E}\mathcal{M}}(s,\Psi) := \int_{\mathcal{M}_s^{\mathcal{E}\mathcal{M}}} \mathbf{e}_\kappa[\Psi]\mathrm{d}x, \quad \mathbf{E}^{\mathcal{C}}(s_0,s_1;\Psi) := \int_{\mathcal{C}_{[s_0,s_1]}} \langle \Psi, \vec{n}_{\mathcal{C}}\cdot\Psi\rangle_{\mathrm{D}}\, dV_{\mathcal{C}} \geq 0,$$

and $\vec{n}_{\mathcal{C}}$ *is the future-oriented unit normal vector of* $\mathcal{C}_{[s_0,s_1]}$, *while* $dV_{\mathcal{C}}$ *is the associated volume form.*

Klainerman-Sobolev inequalities

Theorem (K-S inequalities for spinor fields)

Assume the **uniform spacelike conditions** *with sufficiently small* ε_s*. Consider any sufficiently regular spinor field* Ψ *defined in the spacetime slab* $\mathcal{M}_{[s_0,s_1]}$*.*
Provided the metric satisfies the decay condition for some $0 \leq \delta \leq 1$*,*

$$(s/t)^{-1}|H|_1 + \frac{(s/t)^{-2}|H^{\mathcal{N}00}|_1}{1+(s/t)^{-2}|H^{\mathcal{N}00}|} \lesssim (s/t)^{-\delta}, \quad \zeta^{-1}|H|_1 + \frac{\zeta^{-2}|H^{\mathcal{N}00}|_1}{1+\zeta^{-2}|H^{\mathcal{N}00}|} \lesssim \zeta^{-\delta}.$$

- **Hyperboloidal region.**

$$(s/t)^{1/2+2\delta} t^{3/2} |\Psi|_{\vec{n}} \lesssim \sum_{|J|\leq 2} \left\| (s/t)^{1/2} |\widehat{L}^J \Psi|_{\vec{n}} \right\|_{L^2(\mathcal{M}_s^{\mathcal{H}})}.$$

- **Merging–Euclidean domain.** *One has, for* $0 \leq \kappa \leq 1$ *and in* $\mathcal{M}^{\mathcal{EM}}_{[s_0,s_1]}$*,*

$$\zeta^{1/2+2\delta}(2+r-t)^{\kappa}(1+r)|\Psi|_{\vec{n}} \lesssim_\delta \sum_{j+|K|\leq 2} \|(2+r-t)^{\kappa}\zeta^{1/2}|\widehat{\partial}_r^j \widehat{\Omega}^K \Psi|_{\vec{n}}\|_{L^2(\mathcal{M}_s^{\mathcal{EM}})}.$$

- **Euclidean domain.** *In* $\mathcal{M}^{\mathcal{E}}_{[s_0,s_1]}$*,*

$$(2+r-t)^{\kappa}(1+r)|\Psi|_{\vec{n}} \lesssim \sum \|(2+r-t)^{\kappa}|\widehat{\partial}_r^J \widehat{\Omega}^K \Psi|_{\vec{n}}\|_{L^2(\mathcal{M}_s^{\mathcal{E}})}.$$

Existence of a light-bending coordinate chart

Consider a reference metric g_{r} in the original coordinates $(\widetilde{t}, \widetilde{x})$.

- <u>Asymptotic flatness</u>:

$$|\widetilde{H}_{r\alpha\beta}|_{\widetilde{\mathscr{K}},N+2} \lesssim \varepsilon_\star (\tilde{t} + \tilde{r} + 1)^{-1+\theta},$$

where $\widetilde{\mathscr{K}} = \{\widetilde{S}, \widetilde{\partial}_\alpha, \widetilde{L}_a, \widetilde{\Omega}_{ab}\}$.

$$|\tilde{\partial} \widetilde{\not{H}}_{\mathrm{r}}| \lesssim \varepsilon_\star (\tilde{t} + \tilde{r} + 1)^{-1},$$

near the light cone, i.e., for $(1-\ell)\tilde{t} \leq \tilde{r} \leq (1-\ell)^{-1}\tilde{t}$.

- <u>Generalized wave coordinate condition</u>:

$$\left|\widetilde{\mathscr{K}}^I \widetilde{W}_{\mathrm{r}}^\alpha\right| \lesssim \varepsilon_\star (\tilde{t} + \tilde{r} + 1)^{-4-(p-k)+\theta},$$

with $\Box_{g_{\mathrm{r}}} \widetilde{x}^\alpha = -\widetilde{W}_{\mathrm{r}}^\alpha$.

- <u>Almost Ricci-flat condition</u>:

$$\left|\widetilde{\mathscr{K}}^I \widetilde{U}_{\mathrm{r}\,\alpha\beta}\right| \lesssim \varepsilon_\star^2 (\tilde{t} + \tilde{r} + 1)^{-4-(p-k)+\theta},$$

with $\widetilde{R}_{\mathrm{r}\,\alpha\beta} = \widetilde{U}_{\mathrm{r}\,\alpha\beta}$.

Existence of a light-bending coordinate chart

Example: the reference metric applied in Lindblad-Rodnianski [2010]

$$g^{\mathrm{ref}}_{\mu\nu} = \eta_{\mu\nu} + \chi\left(\frac{r}{t}\right) \chi(r) \frac{M}{r} \delta_{\mu\nu}.$$

The actual metric is decomposed as

$$g_{\mu\nu} = \eta_{\mu\nu} + h^0_{\mu\nu} + h^1_{\mu\nu} = g^{\mathrm{ref}}_{\mu\nu} + h^1_{\mu\nu}.$$

This reference captures the Schwarzschild-type ADM mass tail $\frac{M}{r}\delta_{\mu\nu}$, while the remainder $h^1_{\mu\nu}$ is the finite-energy perturbation estimated by energy method.

Proposition (Existence of a light-bending coordinate chart)

Consider a spacetime $(\mathcal{M}, g_{\mathrm{r}})$ *where* $\mathcal{M}$ *is diffeomorphic to* $\mathbb{R}^{1+3}_{+} = \{(t,x)\,|\,x \in \mathbb{R}^3,\ t \geq 0\}$. *Assume that there is a global coordinate chart* $\{\tilde{x}^\alpha\}$ *such that* g_{r} *is an* $(N, \theta, \varepsilon_\star, \ell)$*–admissible reference metric with* $(N+3)\theta \leq 1$. *Then there exist two universal constants* K_0, K_1 *and some* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon_\star \leq \varepsilon_0$ *and any* C_L *satisfying*

$$0 \leq C_L \leq \frac{1}{K_1 \varepsilon_\star} - K_0,$$

so that in the new coordinate chart $\{x^\alpha\}$ *defined by*

$$t = \tilde{t}, \quad x^a = (r/\tilde{r})\tilde{x}^a$$

with

$$r = \tilde{r} - K_{\mathrm{r}} \varepsilon_\star \chi(\tilde{r}/\tilde{t})\tilde{r}^\theta, \quad K_{\mathrm{r}} = (K_0 + C_L)\theta^{-1},$$

where χ *is a cut-off function, the reference metric* g_{r} *satisfies the following properties, provided that* $K_{\mathrm{r}}\varepsilon_\star$ *is sufficiently small.*

Proposition

- *Almost Minkowski condition:*

$$\left|h_{\mathrm{r}\alpha\beta}\right|_{\mathscr{K},p} + \left|h_{\mathrm{r}}{}^{\alpha\beta}\right|_{\mathscr{K},p} \lesssim_N K_{\mathrm{r}}\varepsilon_\star(t+r+1)^{-1+(p+1)\theta},$$

Other almost Minkowski bounds for g_r and its derivatives.

- *Almost Ricci-flat condition:*

$$\left|R[g_{\mathrm{r}}]_{\alpha\beta}\right|_N + (1+r+t)\left|\partial_\gamma R[g_{\mathrm{r}}]_{\alpha\beta}\right|_{N-1} \lesssim_N \varepsilon_\star^2(1+t+r)^{-4+\theta},$$

- *Generalized wave coordinate condition:*

$$\left|\Gamma_{\mathrm{r}}{}^{\lambda}\right|_{N+1} + (1+|r-t|)\left|\partial_\gamma \Gamma_{\mathrm{r}}{}^{\lambda}\right|_N \lesssim_N K_{\mathrm{r}}\varepsilon_\star(t+r+1)^{-2+\theta},$$

- *Light-bending condition:*

$$h_{\mathrm{r}}{}^{\mathcal{N}00} := (\mathrm{d}t-\mathrm{d}r, \mathrm{d}t-\mathrm{d}r)_{g_{\mathrm{r}}} < -C_L\varepsilon_\star r^{-1+\theta} \qquad \text{in } \mathcal{M}_{[s_0,s_1]} \cap \{r \geq (3/4)t\}.$$

Spinorial source estimates

Lemma (Derivation of Dirac form)

For any sufficiently regular spinor fields Φ, Ψ and any vector field X defined in (a subset of) $\mathcal{M}_{[s_0,s_1]}$, one has

$$X\big(\langle \Phi, \Psi\rangle_{\mathrm{D}}\big) = \langle \widehat{X}\Phi, \Psi\rangle_{\mathrm{D}} + \langle \Phi, \widehat{X}\Psi\rangle_{\mathrm{D}} - \frac{1}{2}\mathrm{div}(X)\langle \Phi, \Psi\rangle_{\mathrm{D}}.$$

Furthermore,

$$\mathscr{Z}^I(\langle \Phi, \Psi\rangle_{\mathrm{D}}) \simeq \langle \widehat{\mathscr{Z}}^J\Phi, \widehat{\mathscr{Z}}^K\Psi\rangle_{\mathrm{D}} + \textit{cubic and higher-order terms}$$

where the sum is taken over multi-indices J, K satisfying

$$\mathrm{ord}(J) + \mathrm{ord}(K) \leq p, \quad \mathrm{rank}(J) + \mathrm{rank}(K) \leq k.$$

Coupled term

$u_{\alpha\beta} := g_{\alpha\beta} - g_{r\alpha\beta}$: metric perturbation
$\mathbb{F}$: metric nonlinearities
$\mathbb{S}$: spinorial source term
$\mathbb{G}$: gauge terms

$$\begin{aligned} g^{\mu\nu}\partial_\mu\partial_\nu u_{\alpha\beta} &= \mathbb{F}(g, g; \partial u, \partial u)_{\alpha\beta} + \mathbb{S}(g, \partial g, \Psi, \widehat{\partial}\Psi)_{\alpha\beta} \\ &\quad + 2\mathbb{F}(g, g; \partial g_{\mathrm{r}}, \partial u)_{\alpha\beta} + 2\mathbb{F}(u, g_{\mathrm{r}}; \partial g_{\mathrm{r}}, \partial g_{\mathrm{r}})_{\alpha\beta} \\ &\quad + \mathbb{F}(u, u; \partial g_{\mathrm{r}}, \partial g_{\mathrm{r}})_{\alpha\beta} + u^{\mu\nu}\partial_\mu\partial_\nu g_{\mathrm{r}\alpha\beta} \\ &\quad - 2\big(\mathbb{G}(g, \partial g, \Gamma, \partial\Gamma)_{\alpha\beta} - \mathbb{G}(g_{\mathrm{r}}, \partial g_{\mathrm{r}}, \Gamma_{\mathrm{r}}, \partial\Gamma_{\mathrm{r}})_{\alpha\beta}\big), \\ \mathfrak{D}\Psi + iM\Psi &= 0, \end{aligned}$$

with generalized wave coordinate conditions:

$$\Gamma^\lambda = \Gamma_{\mathrm{r}}{}^\lambda + \partial_\alpha\big(\tilde{\Phi}^\delta_\beta\big)\tilde{\Phi}^\lambda_\delta u^{\alpha\beta}.$$

Thank you!

Appendix: Local representation of the spinor objects

In the expression of $T_{\mu\nu}$, one has:

- The spinor field Ψ: $\mathbb{C}^4-$valued functions ψ under a fixed *tetrad* $\{e_0, e_1, e_2, e_3\}$;
- The spin covariant derivatives ∇. Under a fixed tetrad:

$$\nabla\psi := \mathrm{d}\psi + \frac{1}{4}\omega_{\alpha\beta}\gamma^\alpha\gamma^\beta\psi \tag{2}$$

  where $\omega_{\alpha\beta}$ are the connection forms for of $\{e_\alpha\}$:

$$\omega_{\alpha\beta}\eta^{\beta\gamma} \otimes e_\gamma = \nabla e_\alpha,$$

  and $\gamma^\alpha = \eta^{\alpha\beta}\gamma_\beta$ the Dirac matrices.
- The Clifford multiplication. Under $\{e_\alpha\}$:

$$X \cdot \psi := X^\alpha\gamma_\alpha\psi, \quad X = X^\alpha e_\alpha. \tag{3}$$

Appendix: Dirac matrix

The Dirac matrices: basis for the linear representation of a Clifford algebra on $\mathrm{End}(\mathbb{C}^4)$. The Dirac matrices γ_i are defined as (not unique):

$$\gamma_0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3,$$

where $\sigma_1, \sigma_2, \sigma_3$ denote the standard 2×2 Pauli matrices, defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The famous anti-commutation relation of these matrices is

$$\begin{aligned} &\{\gamma_i, \gamma_j\} = \gamma_i \gamma_j + \gamma_j \gamma_i = -2\,\eta_{ij}\, I_4 && (i, j = 0, 1, 2, 3), \\ &\gamma_0^\dagger = \gamma_0, \qquad \gamma_{\hat\imath}^\dagger = -\gamma_{\hat\imath} && (\hat\imath = 1, 2, 3), \end{aligned}$$

where I_2 and I_4 denote the 2×2 and 4×4 identity matrices, respectively, and the † symbol denotes the Hermitian transpose of a matrix.

Appendix: The Euclidean-hyperboloidal foliation

Precisely,

$$\partial_r T(s,r) = \frac{r\xi(s,r)}{(s^2+r^2)^{1/2}}, \quad T(s,0) = s.$$

where $\xi = \xi(s,r) \in [0,1]$ is a *cut-off function* satisfying

$$\xi(s,r) = \begin{cases} 1, & r < r^{\mathcal{H}}(s), \\ 0, & r > r^{\mathcal{E}}(s), \end{cases} \tag{4}$$

here the hyperboloidal radius $r^{\mathcal{H}}(s) := \frac{1}{2}(s^2-1)$ and the Euclidean radius $r^{\mathcal{E}}(s) = \frac{1}{2}(s^2+1)$.

Appendix: The Euclidean-hyperboloidal foliation

Direct calculation shows that ($r := |x|$)

$$T(s,r) = \begin{cases} (r^2+s^2)^{1/2}, & 0 \le r \le r^{\mathcal{H}}(s), \\ s + \displaystyle\int_0^r \frac{\xi(s,\rho)\rho}{\sqrt{s^2+\rho^2}} d\rho, & r^{\mathcal{H}}(s) < r < r^{\mathcal{E}}(s), \\ T(s) = s + \displaystyle\int_0^{r^{\mathcal{E}}(s)} \frac{\xi(s,\rho)\rho}{\sqrt{s^2+\rho^2}} d\rho, & r^{\mathcal{E}}(s) \le r < \infty. \end{cases}$$

$$\mathcal{M}_s := \{(t,r) | t = T(s,r)\}, \quad \mathcal{M}_{[s_0,s_1]} := \{(t,r) | T(s_0,r) \le t \le T(s_1,r)\}.$$

Appendix: The Euclidean-hyperboloidal foliation

$$
\begin{aligned}
&\text{Hyperbolic part} \quad && \mathcal{M}_s^{\mathcal{H}} := \mathcal{M}_s \cap \{r \leq r^{\mathcal{H}}(s)\},\\
&\text{Transition part} \quad && \mathcal{M}_s^{\mathcal{M}} := \mathcal{M}_s \cap \{r^{\mathcal{H}}(s) \leq r \leq r^{\mathcal{E}}(s)\},\\
&\text{Flat part} \quad && \mathcal{M}_s^{\mathcal{E}} := \mathcal{M}_s \cap \{r \geq r^{\mathcal{E}}(s)\}.
\end{aligned}
$$

We also denote by

$$
\mathcal{M}^{\mathcal{H}}_{[s_0,s_1]} := \bigcup_{s_0 \leq s \leq s_1} \mathcal{M}_s^{\mathcal{H}}, \quad \mathcal{M}^{\mathcal{M}}_{[s_0,s_1]} := \bigcup_{s_0 \leq s \leq s_1} \mathcal{M}_s^{\mathcal{M}}, \quad \mathcal{M}^{\mathcal{E}}_{[s_0,s_1]} := \bigcup_{s_0 \leq s \leq s_1} \mathcal{M}_s^{\mathcal{E}}
$$

We also use the notation $\mathcal{M}_s^{\mathcal{EM}} = \mathcal{M}_s^{\mathcal{M}} \cup \mathcal{M}_s^{\mathcal{E}}$.

Appendix: Energy estimate

Proposition (Energy estimate, global version)

Suppose that $g_{\alpha\beta} = \eta_{\alpha\beta} + H_{\alpha\beta}$ is a sufficiently regular metric defined in $\mathcal{M}_{[s_0,s_1]}$. Then any C^1 solution Ψ to the Dirac equation

$$\mathfrak{D}\Psi + \mathrm{i}M\Psi = \Phi,$$

defined in the domain $\mathcal{M}_{[s_0,s_1]}$ and vanishing sufficiently fast at the spatial infinity, satisfies the energy identity

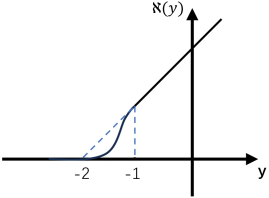
$$\begin{aligned}
&\int_{\mathcal{M}_{s_1}} w\langle\Psi, \vec{n}\cdot\Psi\rangle_{\mathrm{D}} dV_{\bar\sigma} + \int_{s_0}^{s_1}\int_{\mathcal{M}_s} \langle\Psi, \mathrm{grad}(w)\cdot\Psi\rangle_{\mathrm{D}}\, \bar{l}\, dV_{\bar\sigma}\mathrm{d}s \\
&= \int_{\mathcal{M}_{s_0}} w\langle\Psi, \vec{n}\cdot\Psi\rangle_{\mathrm{D}} dV_{\bar\sigma} - 2\int_{s_0}^{s_1}\int_{\mathcal{M}_s} w\Re\big(\langle\Phi,\Psi\rangle_{\mathrm{D}}\big)\, \bar{l}\, dV_{\bar\sigma}\mathrm{d}s.
\end{aligned}$$

Here, $\vec{n}$ is the future oriented, unit normal vector to $\mathcal{M}_s$, and w is a positive regular scalar weight.

Appendix: Energy estimate

We choose a ghost weight function

$$\omega(t, r) := \aleph(r - t), \tag{5}$$



where $\aleph$ is a smooth and non-decreasing function, satisfying

$$\aleph(y) = 0 \quad \text{for} \quad y \leq -2 \quad \text{and} \quad \aleph(y) = y + 2 \quad \text{for} \quad y \geq -1,$$
$$\aleph'(y) > 0 \quad \text{for} \quad y > -2.$$

Observe that this weight satisfies

$$\omega \begin{cases} \leq 1, & \mathcal{M}^{\mathcal{H}}_{[s_0, s_1]}, \\ = (2 + r - t), & \mathcal{M}^{\mathcal{EM}}_{[s_0, s_1]}. \end{cases}$$

Appendix: Klainerman-Sobolev inequalities

Turn L^2 (energy type) estimates to pointwise estimates. Original Klainerman-Sobolev type inequality for scalar field within a flat background spacetime (version of Hörmander[1997]):

$$|t^{3/2}\phi(t,x)| \lesssim \sum_{|J|\leq 2} \|L^J\phi\|_{L^2(\mathcal{H}_s)}, \quad s = \sqrt{t^2 - r^2}. \tag{6}$$