COLLÈGE DE FRANCE 1530

# New symmetries at spatial infinity

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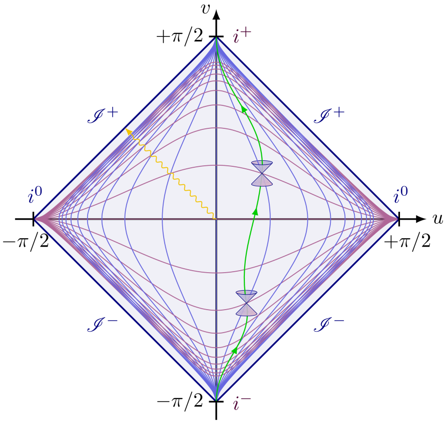
# 4d asymptotically flat spacetimes

- Spacetimes that are asymptotically Minkowski
- Why? Detectors are very far from the sources (BH mergers, etc)
- Spatial infinity ($i^0$)

  [see also Maël's talk]




[Figure Neutelings]

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# Observables in gravity need a boundary

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- How to define energy in gravity?
- Locally, it is impossible (equivalence principle)
- One needs a BOUNDARY to define a charge/generator associated to a symmetry via Noether procedure
- Same for electromagnetism: electric charge is measured using Gauss law

# Asymptotic symmetries - a brief sketch

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   If non-vanishing, it is PHYSICAL (not pure gauge)
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Method: covariant phase space [review: Fiorucci '21] or Hamiltonian [review: Henneaux's lectures at College de France '21-'22 & '22-'23]

Caution: answer depends on the choice of the asymptotic structure

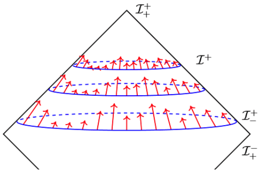
# Example: BMS algebra [Bondi et al. '61, Sachs '61]

- Exact symmetries of Minkowski
  Poincaré algebra
  $\mathfrak{iso}(1,3) = \mathfrak{so}(1,3) \inplus$ translations

$$x \to x + \text{constant}$$

- Asymptotic symmetries of asymptotically flat spacetimes [Compère-Gralla-Wei '23]
  BMS $= \mathfrak{iso}(1,3) \inplus$ supertranslations

$$x \to x + c(x)$$




[Action of supertranslation by Lim Zheng Liang]

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- Quantum Gravity: quantum states form a representation of the asymptotic symmetry algebra $\rightarrow$ Non-perturbative handle on Quantum Gravity
- Connection with low energy physics (infrared triangle)

# Symmetries at spatial infinity

# Main result

- New asymptotic symmetries at spatial infinity: *extension of BMS by logartihmic supertranslations (no parity conditions imposed)*

$$\text{log-BMS:} \qquad \mathfrak{iso}(1,3) \oplus \text{Heisenberg}(\text{supertranslation}, \text{log-supertranslation})$$

$$Q_{\text{supertranslation}} \sim \text{mass aspects}$$
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# Main result

- New asymptotic symmetries at spatial infinity: *extension of BMS by logartihmic supertranslations (no parity conditions imposed)*

$$\text{log-BMS:} \qquad \mathfrak{iso}(1,3) \subsetplus \text{Heisenberg}(\text{supertranslation}, \text{log-supertranslation})$$

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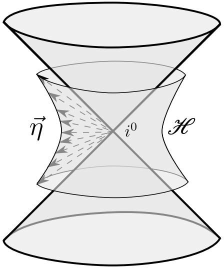
- BMS frame invariant notion of angular momentum (using the construction of [Fuentealba-Henneaux-Troessaert '22])

$$\mathfrak{iso}(1,3) \oplus \text{Heisenberg}(\text{supertranslation}, \text{log-supertranslation})$$

# Spatial infinity

- In Penrose's conformal compactification is a point but the fields are not single valued
- Blow-up of spatial infinity [Ashtekar-Hansen '77], such that it asymptotes to an unit-hyperboloid.

$$\lim_{\rho\to\infty} \frac{ds^2}{\rho^2} = (-\mathrm{d}\tau^2 + \cosh^2\tau\,\mathrm{d}\Omega^2) = \mathbf{h}_{ab}\mathrm{d}x^a\mathrm{d}x^b .$$




[Figure from 1902.08200]

Parity: $\quad F(\tau,\theta,\phi) \to F(-\tau,\pi-\theta,\pi+\phi) = \begin{cases} +F & \text{even} \\ -F & \text{odd} \end{cases}$

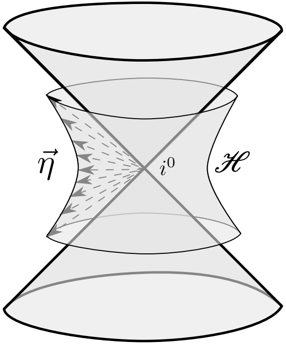
# Beig-Schmidt coordinate chart

- Minkowski:

$$\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}\rho^2 + \rho^2 \mathbf{h}_{ab}\mathrm{d}x^a \mathrm{d}x^b$$

- Fall-offs:

$$g_{\mu\nu} - \eta_{\mu\nu} = o(\rho)^0$$




[Figure from 1902.08200]

# Charge analysis

- Covariant phase space
  - [Ashtekar-Bombelli-Reula '91] Poincaré
  - [Compère-Dehouck '11] Lorentz + log translations + supertranslations
  - [Troessaert '17] BMS and connection to smooth null infinity upon imposing parity conditions (see also [Prabhu '19, Prabhu-Shehzad '21, Capone-Nguyen-Parisini '22])

$$\mathfrak{so}(1,3) \inplus \mathbb{R}_{\omega^{\text{super}}_{\text{odd}}}$$

- Hamiltonian
  - [Regge-Teitelboim '88] Poincaré
  - [Henneaux-Troessaert '18] BMS
  - [Fiorucci-Matulich-Ruzziconi '24] gBMS

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  - [Fiorucci-Matulich-Ruzziconi '24] gBMS
  - [Fuentealba-Henneaux-Troessaert '22] = BMS + logarithmic supertranslations $H$

$$\text{FHT log-BMS:} \quad \mathfrak{iso}(1,3) \inplus \mathfrak{h}(\omega^{\text{super}}_{\text{odd}}, H^{\text{super}}_{\text{even}})$$

# Project:

Goal: Log-supertranslations in covariant phase space in Beig-Schmidt coordinates

Why:

- easier to map spatial infinity to null infinities
  *useful for the gravitational S-matrix*
- no need to impose parity conditions
  *usually imposed for obtaning finitness and/or for connecting to smooth null infinity*
- independent derivation of FHT results

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- independent derivation of FHT results

**We will obtain an enlarged symmetry algebra**

$$\text{log BMS:} \quad \mathfrak{iso}(1,3) \oplus\!\!\!\!\!\!+ \left[\mathfrak{h}(\omega^{\text{super}}_{\text{odd}}, H^{\text{super}}_{\text{even}}) \oplus \mathfrak{h}(\omega^{\text{super}}_{\text{even}}, H^{\text{super}}_{\text{odd}}) \oplus \mathfrak{h}(\omega_{\text{sing}}, H_{\text{reg}})\right]$$

Set-up: Beig–Schmidt coordinates chart

- without imposing parity conditions
- with a generic polyhomogeneous radial expansion, starting at first order

Criteria and method:

- *Finitness* obtained using the regularization scheme of [McNees-Zwikel '23, '24]
- *Conserved charges* obtained by imposing new boundary conditions

# Beig-Schmidt polyhomogenous expansion

$$\mathrm{d}s^2 = \sigma^2\,\mathrm{d}\rho^2 + \rho^2 h_{ab}\mathrm{d}x^a\mathrm{d}x^b + o\Big(\rho^{-1}\Big)\mathrm{d}\rho\mathrm{d}x^a$$

Expansion

$$\sigma = 1 + \frac{1}{\rho}(\bar\sigma + \log\rho\,\tilde\sigma) + o\big(\rho^{-1}\big),$$
$$h_{ab} = h^0_{ab} + \frac{1}{\rho}\big(\bar h_{ab} + \log\rho\,\tilde h_{ab}\big) + o\big(\rho^{-1}\big),$$

$$\bar\tau_{ab} = \bar h_{ab} - \tilde h_{ab} - \mathbf{h}_{ab}(\bar h - \tilde h + 4\bar\sigma)\,,$$
$$\tilde\tau_{ab} = \tilde h_{ab} - \mathbf{h}_{ab}\tilde h\,,$$
$$\bar\tau = -2(\bar h - \tilde h + 6\bar\sigma)\,,$$
$$\tilde\tau = -2\tilde h\,.$$

Kerr-Taub-NUT: $\bar\sigma = GM\cosh 2\tau\,\mathrm{sech}\,\tau\,,\qquad \bar\tau_{ab} = -2GNk_{(0)ab}$

# Boundary conditions for conserved charges (for $\tilde{\sigma} = 0$)

New boundary conditions:

$$h^0_{ab} = \mathbf{h}_{ab}\,, \quad \tilde{\tau} = 0 = \bar{\tau}\,, \quad \tilde{\tau}_{\langle ab \rangle} = 2D_{\langle a}D_{b\rangle}\tilde{\beta}\,, \quad (D^2+3)\tilde{\beta} = 0$$

Consequences: electric $E_{ab}$ and magnetic part of the Weyl tensor $B_{ab}$

$$E_{ab} = \frac{1}{\rho}\bar{E}_{ab} + \frac{\log\rho}{\rho}{\color{red}0} + o\left(\rho^{-1}\right) \qquad B_{ab} = \frac{1}{\rho}\bar{B}_{ab} + \frac{\log\rho}{\rho}{\color{red}0} + o\left(\rho^{-1}\right)$$

$$\bar{E}_{ab} \approx -D_{\langle a}D_{b\rangle}\bar{\sigma} + \frac{1}{2}\tilde{\tau}_{\langle ab\rangle}, \qquad \bar{B}_{ab} \approx -\frac{1}{2}\boldsymbol{\nabla}\times\bar{\tau}_{ab}$$

# Residual symmetries

$$\xi^\rho = \omega(x^c) + \log\rho\, H(x^c) + o\left(\rho^0\right) \qquad \xi^a = \mathcal{Y}^a(x^c) + o\left(\rho^0\right)$$

- $\mathcal{Y}^a$ are diffeomorphisms of $h^0_{ab}$
- $\omega$ are supertranslations of both parities and translations
- $H$ are log-supertranslations of both parities and translations

Algebra

$$\mathfrak{X}(\text{hyperboloid})_{\mathcal{Y}^a} \niplus [(\mathbb{R})_\omega \oplus (\mathbb{R})_H]$$

$[\xi_1, \xi_2] = \xi_{12}$ with $\xi_i = \xi_{(\omega_i, H_i, \mathcal{Y}_i)}$,

$$\omega_{12} = \mathcal{Y}^a_1 \partial_a \omega_2 - \mathcal{Y}^a_2 \partial_a \omega_1 \,, \quad H_{12} = \mathcal{Y}^a_1 \partial_a H_2 - \mathcal{Y}^a_2 \partial_a H_1 \,, \quad \mathcal{Y}^a_{12} = \mathcal{Y}^b_1 \partial_b \mathcal{Y}^a_2 - \mathcal{Y}^b_2 \partial_b \mathcal{Y}^a_1$$

Field transformations:

$$\delta_\xi \bar\sigma = \pounds_{\mathcal{Y}} \bar\sigma + H\,, \qquad \delta_\xi \tilde\sigma = \pounds_{\mathcal{Y}} \tilde\sigma$$
$$\delta_\xi \bar\tau_{\langle ab\rangle} = \pounds_{\mathcal{Y}} \bar\tau_{\langle ab\rangle} + 2 D_{\langle a} D_{b\rangle}\omega\,, \qquad \delta_\xi \bar\tau = \pounds_{\mathcal{Y}} \bar\tau - 4(D^2+3)\omega$$
$$\delta_\xi \tilde\tau_{\langle ab\rangle} = \pounds_{\mathcal{Y}} \tilde\tau_{\langle ab\rangle} + 2 D_{\langle a} D_{b\rangle} H\,, \qquad \delta_\xi \tilde\tau = \pounds_{\mathcal{Y}} \tilde\tau - 4(D^2+3)H,$$

Boundary conditions:

$$h^0_{ab} = \mathbf{h}_{ab}\,, \quad \tilde\tau = 0 = \bar\tau\,, \quad \tilde\tau_{\langle ab\rangle} = 2 D_{\langle a} D_{b\rangle}\tilde\beta$$

Consequences:

- $\mathcal{Y}^a$ are diffeomorphisms of the 6 Killing vectors of $\mathbf{h}_{ab}$, ie $D_a \mathcal{Y}_b + D_b \mathcal{Y}_a = 0$
- $(D^2+3)H = 0 = (D^2+3)\omega$

# Solutions to $(D^2+3)f=0$ on the hyperboloid

- Odd *Regular functions* are given in terms of $\ell = 0,1$ harmonics satisfying $(D_a D_b + \mathbf{h}_{ab})f = 0$.

$$\zeta_0^{\text{odd}} = -\sinh(\tau)\, Y_{0,0}(x^A)\,, \quad \zeta_{1m}^{\text{odd}} = \cosh(\tau) Y_{1,m}(x^A)\,, \quad m = -1,0,1,$$

- Even *Singular functions* are given in terms of $\ell = 0,1$ harmonics, but do not satisfy $(D_a D_b + \mathbf{h}_{ab})f = 0$

$$\zeta_0^{\text{even}} = \frac{\cosh(2\tau)}{\cosh(\tau)} Y_{0,0}(x^A)\,, \quad \zeta_{1m}^{\text{even}} = \left(2\sinh(\tau) + \frac{\tanh(\tau)}{\cosh(\tau)}\right) Y_{1,m}(x^A)\,, \quad m = -1,0,1$$

  and they are parity-time even. We denote them as $f_{\text{sing}}$

- Odd/even *Super functions* correspond to higher harmonics ($\ell \geq 2$)

Example: ordinary translations are $\omega_{\text{reg}}$.

Note: $H_{\text{sing}}$ is disallowed, by setting $\tilde{\beta}_{\text{sing}} = 0$, for preventing reaching negative mass solution.

# Intermediate summary

- Subleading solution space:

$$\bar{\sigma}\,, \quad \bar{\tau}_{\langle ab\rangle} = \bar{\wp}_{\langle ab\rangle} + D_{\langle a}D_{b\rangle}\bar{\beta}\,, \quad \tilde{\tau}_{\langle ab\rangle} = D_{\langle a}D_{b\rangle}\tilde{\beta}, \quad \tilde{\beta}_{\mathsf{sing}} = 0$$

  where $f = (\bar{\sigma}\,, \bar{\beta}\,, \tilde{\beta})$ are solutions of $(D^2+3)f = 0$.
- Subsubleading solution space $\ni T_{ab}$, carrying information about angular momentum
- Boundary conditions

$$h^0_{ab} = \mathbf{h}_{ab}\,, \quad \tilde{\tau} = 0 = \bar{\tau}\,, \quad \tilde{\wp}_{\langle ab\rangle} = 0$$

- Residual symmetries $\xi(\mathcal{Y}^a, \omega_{\mathsf{reg}}, \omega_{\mathsf{sing}}, \omega_{\mathsf{super}}, H_{\mathsf{reg}}, H_{\mathsf{super}})$

$$\xi = \omega\partial_\rho + H\log\rho\,\partial_\rho + \mathcal{Y}^a\partial_a M \qquad (D^2+3)\omega = 0 = (D^2+3)H$$

# Symplectic current

Renormalization scheme [McNees-Zwikel '23, '24]: *Prescription of corner ambiguities that guarantee finite symplectic current and charges.*

$$\Theta^\rho = \frac{\sqrt{-h_0}}{2\kappa^2}\left(T^{ab}\delta h^0_{ab} + \bar{\sigma}\delta\bar{\tau} + \bar{\beta}\delta\tilde{\tau} + \frac{1}{2}\bar{\wp}^{\langle ab\rangle}\delta\tilde{\wp}_{\langle ab\rangle}\right)$$

As announced, enforcing the boundary conditions yields $\Theta^\rho = 0$ hence conserved charges.

# Charges

[Iyer-Wald '94]: $\delta_\xi \Theta^\rho = \partial_a k_\xi^{\rho a} + \text{eom}$, $\delta Q_\xi = \oint_{S^2} k_\xi$

$$\boxed{Q_\xi = \frac{2}{\kappa^2} \oint_{S^2} \cosh^2 \tau \, s_a \left[ \omega D^a \bar{\sigma} - \bar{\sigma} D^a \omega + H D^a \bar{\beta} - \bar{\beta} D^a H + \frac{1}{2} T^{ab} \mathcal{Y}_b \right]}$$

$s_a$ timelike normal

- Conserved
- Kerr black holes, $Q_{\text{ time translation}} \sim$ mass , $Q_{\text{rotation}} \sim$ angular momentum
- $\bar{\beta}$ corresponds to the supertranslation Goldstone at null infinity [Compere-Gralla-Wei '23]

# Asymptotic symmetry algebra

Representation theorem:

$$\{Q_{\xi_1}, Q_{\xi_2}\} = \delta_{\xi_2} Q_{\xi_1} = Q_{[\xi_1,\xi_2]_\star} + K_{(\xi_1,\xi_2)}\,.$$

$$K_{(\xi_1,\xi_2)} = \frac{2}{\kappa^2}\oint_{S^2} \cosh^2\tau\, s_a[\omega_1 D^a H_2 - H_2 D^a \omega_1 + H_1 D^a \omega_2 - \omega_2 D^a H_1]$$

$$\boxed{\text{log BMS:} \quad \mathfrak{iso}(1,3)_{(\mathcal{Y}^a,\omega_{\text{reg}})} \oplus\!\!\!\!+ \left[\mathfrak{h}(\omega^{\text{super}}_{\text{odd}}, H^{\text{super}}_{\text{even}}) \oplus \mathfrak{h}(\omega^{\text{super}}_{\text{even}}, H^{\text{super}}_{\text{odd}}) \oplus \mathfrak{h}(\omega_{\text{sing}}, H_{\text{reg}})\right]}$$

Time translation commutes with $(H_{\text{reg}}, H_{\text{super}}, \omega_{\text{sing}}, \omega_{\text{super}})$, so the vacuum is degenerate along these directions in the phase space.

# Recovering former results

- [Compère-Dehouck '11] $\tilde{\tau}_{ab} = 0$, no log-supertranslations.
  Beig-Schmidt gauge, different renormalization techniques, same charges

- Restriction on parity to glue to smooth null infinity [Compère-Gralla-Wei '23]:
  - $\bar{\beta}$ odd and $\bar{\sigma}$ even for harmonics $\ell \geq 2$.
    Asymptotic symmetries: $\xi = \xi(\omega_{\text{reg}}, \omega^{\text{odd}}_{\text{super}}, H^{\text{even}}_{\text{super}})$

    $$\text{FHT - log BMS:} \qquad \mathfrak{iso}(1,3) \inplus \mathfrak{h}(\omega^{\text{super}}_{\text{odd}}, H^{\text{super}}_{\text{even}})$$

    Caution: The match with FHT at the level of boundary conditions/metric is hard as they use a different coordinates system. We match the algebra structure.
  - $\tilde{\tau}_{ab} = 0$, we recover BMS

    $$\text{BMS}_{(\mathcal{Y}_a, \omega_{\text{reg}}, \omega^{\text{super}}_{\text{odd}})}$$

# Results

- New finite and conserved charges associated to log-supertranslations in the covariant space formalism without imposing parity conditions
- New asymptotic symmetry algebra at spatial infinity

$$\mathfrak{iso}(1,3) \oplus\!\!\!\!+ \left[\mathfrak{h}(\omega^{\mathsf{super}}_{\mathsf{odd}}, H^{\mathsf{super}}_{\mathsf{even}}) \oplus \mathfrak{h}(\omega^{\mathsf{super}}_{\mathsf{even}}, H^{\mathsf{super}}_{\mathsf{odd}}) \oplus \mathfrak{h}(\omega_{\mathsf{sing}}, H_{\mathsf{reg}})\right]$$

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- Redefinition of the Lorentz generators such that Poincaré is an ideal (similar construction as in [FHT '22, Fuentealba-Henneaux '23]). "BMS frame independent" definition of angular momentum

$$\mathfrak{iso}(1,3) {\color{red}\oplus} \left[\mathfrak{h}(\omega^{\text{super}}_{\text{odd}}, H^{\text{super}}_{\text{even}}) \oplus \mathfrak{h}(\omega^{\text{super}}_{\text{even}}, H^{\text{super}}_{\text{odd}}) \oplus \mathfrak{h}(\omega_{\text{sing}}, H_{\text{reg}})\right]$$

# Future prospects:

- Log BMS as symmetries of null infinity? [WIP]
- We kept both parities. What is the other sector?
- Holographic interpretation of these new charges?
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Merci!

In modes, $F = \frac{1}{2\cosh(\tau)} \sum F^{O}_{\ell,m}\, f(\tau)^{O}_{\ell} Y_{\ell,m} + F^{E}_{\ell,m} f(\tau)^{E}_{\ell} Y_{\ell,m}$

$$Q_\omega = \frac{1}{4\pi G}\sum_{\ell m}(\hat\sigma^E_{\ell m}\hat\omega^O_{\ell,-m} - \hat\sigma^O_{\ell m}\hat\omega^E_{\ell,-m})\mathcal{C}_\ell$$
$$Q_H = \frac{1}{4\pi G}\sum_{\ell\geq 2,m}(\hat\beta^E_{\ell m}\hat H^O_{\ell,-m} - \hat\beta^O_{\ell m}\hat H^E_{\ell,-m})\mathcal{C}_\ell + \frac{1}{4\pi G}\sum_{\ell=0,1,m}(\hat\beta^E_{\ell m}\hat H^O_{\ell,-m})\mathcal{C}_\ell$$

$\mathcal{C}_\ell$ are numbers.

$$K_{(\xi_1,\xi_2)} = \frac{2}{\kappa^2}\sum_{\ell\geq 2,m}((\hat\omega_1)^E_{\ell,m}(\hat H_2)^O_{\ell,-m} - (\hat\omega_1)^O_{\ell,m}(\hat H_2)^E_{\ell,-m})\mathcal{C}_\ell + \frac{2}{\kappa^2}\sum_{\ell=0,1,m}(\hat\omega_1)^E_{\ell,m}(\hat H_2)^O_{\ell,-m}\mathcal{C}_\ell -$$