The homogeneous Robin boundary conditions for asymptotically Anti-de Sitter spaces

Ludovic Souêtre

Laboratoire Jacques-Louis Lions, Sorbonne Université

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- vacuum Einstein equations on a 4D Lorentzian manifold $(\widetilde{\mathcal{M}}, \widetilde{g})$ with cosmological constant $\Lambda = -3$

$$\mathrm{Ric}(\widetilde{g}) = -3\,\widetilde{g}\,, \tag{VEN}$$

- Anti-de Sitter manifold

$$\widetilde{\mathcal{M}}_{AdS} := \mathbb{R}^4\,, \quad \widetilde{g}_{AdS} := -(1+r^2)dt^2 + \frac{1}{1+r^2}dr^2 + r^2 d\sigma^2_{\mathbb{S}^2}\,,$$

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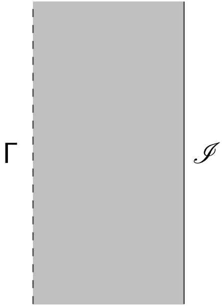
- With $\psi = \tan r \in [0, \pi/2)$,

$$(\cos\psi)^2\,\widetilde{g}_{AdS} = -dt^2 + d\psi^2 + \sin^2\psi\, d\sigma^2_{\mathbb{S}^2}$$

$\Longrightarrow$ conformorphism with the subset $\{\psi < \pi/2\}$ of the Einstein cylinder/universe

$$\mathcal{M}_{EC} := \mathbb{R}\times\mathbb{S}^3\,, \quad g_{EC} := -dt^2 + d\sigma^2_{\mathbb{S}^3}\,,$$

- Conformal boundary $\mathscr{I} = \{\psi = \pi/2\}$ **timelike**



Penrose diagram

Lemma

If $(\widetilde{\mathcal{M}}, \tilde{g})$ *is a solution to* (VEN) *admitting a conformal boundary* $\mathscr{I}$ *(with* $\mathcal{C}^2$ *unphysical metrics) then* $\mathscr{I}$ *is timelike.*

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Definition

An **asymptotically Anti-de Sitter** manifold is a triple $(\mathcal{M}, \mathscr{I}, \tilde{g})$ where

- $\mathcal{M}$ is a 4D smooth manifold with corners,
- $\mathscr{I}$ is a non-empty set of disjoint boundary hypersurfaces of $\mathcal{M}$,
- $\tilde{g}$ is a smooth Lorentzian metric on $\mathcal{M} \backslash \mathscr{I}$,

such that there exists a boundary defining function x of $\mathscr{I}$ verifying

i) $x^2\tilde{g}$ extends as a $\mathcal{C}^2$ metric up to $\mathscr{I}$,

ii) $(x^2\tilde{g})^{-1}(dx, dx) = 1$ on $\mathscr{I}$.

- $\mathrm{Ric}(\tilde{g}) + 3\tilde{g} = O(x^{-1})$
- perturbations of AdS (like Kerr-AdS) are aAdS
- topology and conformal class of $\mathscr{I}$ are not imposed

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Goal To find aAdS solutions to (VEN) by solving the Cauchy problem

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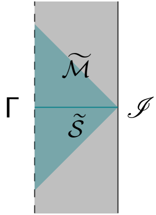
Theorem (Choquet-Bruhat '52, Choquet-Bruhat and Geroch '69)

For a given initial data set $(\widetilde{\mathcal{S}}, \widetilde{h}, \widetilde{K})$ verifying the constraint equations, there exists a unique (up to isometries) maximal globally hyperbolic development $(\widetilde{\mathcal{M}}, \widetilde{g})$ solution to the (VEN).

- aAdS manifolds are not globally hyperbolic
- in our case, this theorem is insufficient
  example $\{t = 0\}$ in AdS
- timelike $\mathscr{I}$ $\rightarrow$ initial boundary value problem



Subproblem What could be a geometric boundary condition on $\mathscr{I}$?

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Toy model 2D wave equation

$$-\partial_t^2 u + \partial_x^2 u = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}_+$$

If

$$u(t,x) = \sum_{k \in \mathbb{N}_0} u^{(k)}(t) x^k \,,$$

for some functions $u^{(k)}$, only two functions are free:

- $u^{(0)} = u|_{x=0}$ Dirichlet data
- $u^{(1)} = \partial_x u|_{x=0}$ Neumann data

Potential boundary conditions: Dirichlet, Neumann, Robin ...

Theorem (Fefferman and Graham '85)

Formal solutions of (VEN)*: in a coordinate system* (x, y^0, y^1, y^2) *where* x *is a boundary defining function of* $\mathscr{I}$ *and such that*

$$x^2\widetilde{g} = dx^2 + \underline{\mathfrak{h}}_{ij}(x, y)dy^i dy^j \,,$$

one has

$$\underline{\mathfrak{h}}(x, .) = \sum_{k\in\mathbb{N}_0} \mathfrak{h}^{(2k)} x^{2k} + \sum_{k\in\mathbb{N}_0} \mathfrak{h}^{(3+2k)} x^{3+2k} \,. \qquad \text{(FG)}$$

Two **free data***:*

- $\mathfrak{h} := \mathfrak{h}^{(0)}$ *Lorentzian metric on* $\mathscr{I}$
- $\mathfrak{t} := \mathfrak{h}^{(3)}$ *TT-tensor for* $\mathfrak{h}$ *(boundary/holographic stress-energy tensor)*

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Gauge-independent free data = class $[(\mathfrak{h},\mathfrak{t})]$ for $(\mathfrak{h}',\mathfrak{t}') \sim (\mathfrak{h},\mathfrak{t})$ if

$$\exists\, \phi \text{ diffeomorphism of } \mathscr{I}, \omega \in \mathcal{C}^\infty(\mathscr{I},\mathbb{R}^\star_+),\ \phi^\star\mathfrak{h}' = \omega^2\mathfrak{h} \text{ and } \phi^\star\mathfrak{t}' = \omega^{-1}\mathfrak{t}$$

Example Schwarzschild-AdS $[(-dt^2 + d\sigma^2_{\mathbb{S}^2}, 2m/3(2dt^2 + d\sigma^2_{\mathbb{S}^2}))]$

Definition

A geometric boundary condition on $\mathscr{I}$ is a condition on the class $[(\mathfrak{h}, \mathfrak{k})]$.

Potential geometric boundary conditions:

- Dirichlet: imposing the conformal class $[\mathfrak{h}]$
- Neumann: imposing the class $[\mathfrak{k}]$

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Lemma (S. '25)

The **homogeneous Robin boundary conditions** *write as follows*

$$\mathfrak{t} = \mu\, \mathfrak{y}(\mathfrak{h}) \qquad \textit{with } \mu \in \mathbb{R}\,, \tag{HR}$$

where $\mathfrak{y}(\mathfrak{h})$ *is the Cotton-York tensor of* $\mathfrak{h}$*.*

- Cotton-York = dual of Cotton (3D analogue of Weyl)
- homogeneous Neumann boundary condition for $\mu = 0$
- see also Fernández-Álvarez and Senovilla '25

Local existence and **local uniqueness** results for the Cauchy problem with **Dirichlet** boundary conditions

Local existence and **local uniqueness** results for the Cauchy problem with **Dirichlet** boundary conditions

Theorem (Friedrich '95)

For a given conformal structure $(\mathscr{I}, [\mathfrak{h}])$ *and a solution to the associated initial data problem, there exists a locally unique (up to isometries) aAdS solution to the Cauchy problem for* (VEN) *satisfying the* **Dirichlet** *boundary condition of boundary data* $(\mathscr{I}, [\mathfrak{h}])$.

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Some other results with Dirichlet boundary conditions:

- Holzegel and Smulevici '12: Einstein-Klein-Gordon in spherical symmetry
- Lübbe and Valiente Kroon '12, Carranza and Valiente Kroon '19: coupling with conformally invariant matter fields
- Enciso and Kamran '19: vacuum case in higher dimensions

Local existence and **local uniqueness** results for the Cauchy problem with **homogeneous Robin** boundary conditions

Local existence and **local uniqueness** results for the Cauchy problem with **homogeneous Robin** boundary conditions

Theorem (S. '25)

For $\mu \in \mathbb{R}$ and a given solution to the associated initial data problem, there exists a locally unique (up to isometries) aAdS solution to the Cauchy problem for (VEN) *satisfying the* **homogeneous Robin** *boundary condition of constant μ.*

In fact, it is possible to impose a different boundary condition (Dirichlet or homogeneous Robin) on each connected component of $\mathscr{I}$.

Definition

An **initial data set** is a triple composed of

- $\mathcal{S}$ a 3D compact smooth manifold with boundary,
- $\widetilde{h}$ a smooth Riemannian metric on $\mathring{\mathcal{S}}$,
- $\widetilde{K}$ a smooth symmetric 2-tensor on $\mathring{\mathcal{S}}$.

Definition

The **initial data problem** consists in finding an initial data set such that

H1. $(\mathcal{S}, \widetilde{h}, \partial\mathcal{S})$ is **asymptotically hyperbolic**,

H2. $(\mathring{\mathcal{S}}, \widetilde{h}, \widetilde{K})$ is solution to the **constraint equations** with $\Lambda = -3$,

H3. the **unphysical fields** are **smooth** up to $\partial\mathcal{S}$,

H4. the **corner conditions** associated to a given boundary condition hold.

- H3 are implicit conditions on the initial data set

Results on the initial data problem:

- Andersson and Chruściel and Friedrich '92: resolution of H1+H2+H3 for pure trace $\widetilde{K}$
- Andersson and Chruściel '96: resolution of H1+H2 (adapted version of the conformal method)
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Lemma (S. '25)

For an initial data set verifying H1+H2, the hypothesis H3 is equivalent to impose

i) *the second fundamental form of $\partial\mathcal{S}$ (in particular, the boundary is umbilical),*

ii) *conditions on the asymptotics of $\widetilde{K}$.*

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The **extended conformal Einstein equations** (Friedrich '95)

- tensorial equations on certain unphysical fields (including the rescaled Weyl tensor)
- conformal representation of the 4D Einstein equations
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Proof contains two main parts:

1. differential systems in the **bulk** for the **evolution** and **constraints**
2. differential system on the **boundary** to pass from the **gauge-dependent boundary conditions** to the **geometric** ones

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- nice structure $\rightarrow$ rely on the linear theory for boundary conditions
- gauge-dependent boundary conditions such that the energy-flux at the boundary is non-positive:

$$\langle \Pi E, \star \Pi H \rangle \leqslant 0 \tag{1}$$

where Π projection operator, (E, H) electric and magnetic parts of rescaled Weyl tensor. In particular, reflective boundary conditions

$$(\cos\theta)\,\Pi E + (\sin\theta)\,\Pi H = d \tag{2}$$

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- propagation of the conformal constraints: linear symmetric hyperbolic system, no boundary condition!

- rewriting with meaningful quantities on the boundary

$$\Pi E \sim \Pi' \mathfrak{t} \,, \qquad \Pi H \sim \Pi' \mathfrak{y} \,,$$

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  generate the source terms

- homogeneous Robin (S. '25): gauge-dependent boundary conditions

$$\Pi'(\mathfrak{t} - \mu \mathfrak{y}) = 0 \tag{4}$$

  divergence condition → homogeneous system for other components

Thank you for your attention!