Positive Energy Theorems for Charged Spin Initial Data

Simon Raulot

Laboratoire de Mathématiques Raphaël Salem
Université de Rouen Normandie

LICHNEROWICZ CONFERENCE
Journées Relativistes de Tours

Tours, France
June 4, 2026

Einstein equations

$$\text{geometry of } (\mathcal{M}^{n+1}, \mathfrak{g}) \rightsquigarrow \mathrm{Ric}_{\mathfrak{g}} - \frac{1}{2}\mathrm{Scal}_{\mathfrak{g}}\, \mathfrak{g} = \mathcal{T} \leftsquigarrow \text{physical fields}$$

Relate the geometry of spacetime, a Lorentzian manifold $(\mathcal{M}^{n+1}, \mathfrak{g})$, to its matter/energy content encoded by the energy-momentum tensor $\mathcal{T}$.

Einstein equations

$$\text{geometry of } (\mathcal{M}^{n+1}, \mathfrak{g}) \leadsto \mathrm{Ric}_{\mathfrak{g}} - \frac{1}{2}\mathrm{Scal}_{\mathfrak{g}}\, \mathfrak{g} = \mathcal{T} \leftsquigarrow \text{physical fields}$$

Relate the geometry of spacetime, a Lorentzian manifold $(\mathcal{M}^{n+1}, \mathfrak{g})$, to its matter/energy content encoded by the energy-momentum tensor $\mathcal{T}$.

Dominant energy condition

For every future-directed timelike vector field T, the vector field

$$-\mathcal{T}(T, \cdot)^{\sharp}$$

is future-directed causal.

Energy density is nonnegative and energy propagates causally.

- When $\mathcal{T} = 0$, one obtains the vacuum Einstein equations.
  In this case, gravity is the only interaction present.

- When $\mathcal{T} = 0$, one obtains the vacuum Einstein equations.
  In this case, gravity is the only interaction present.

- When
$$\mathcal{T} = \mathcal{T}_F := 2\left(F \circ F - \frac{1}{2}|F|^2\mathfrak{g}\right)$$
  is the electromagnetic energy-momentum tensor, where F is a 2-form on $\mathcal{M}$ (the Faraday tensor), satisfying the Maxwell equations
$$dF = 0, \qquad d(*F) = 0,$$
  one obtains the (source-free) Einstein–Maxwell equations.

A charged initial data set is a quadruple (M^n, g, K, E) where

- (M^n, g) is a Riemannian manifold;

$$g = \mathfrak{g}_{|M} \quad \text{for a spacelike hypersurface } M^n \hookrightarrow (\mathcal{M}^{n+1}, \mathfrak{g}).$$

Mathematical framework

Charged initial data sets

A charged initial data set is a quadruple (M^n, g, K, E) where

- (M^n, g) is a Riemannian manifold;

$$g = \mathfrak{g}_{|M} \quad \text{for a spacelike hypersurface } M^n \hookrightarrow (\mathcal{M}^{n+1}, \mathfrak{g}).$$

- K is a symmetric $(0,2)$-tensor;

$$K = \text{the second fundamental form of } M^n \hookrightarrow (\mathcal{M}^{n+1}, \mathfrak{g}).$$

A charged initial data set is a quadruple (M^n, g, K, E) where

- (M^n, g) is a Riemannian manifold;

$$g = \mathfrak{g}_{|M} \quad \text{for a spacelike hypersurface } M^n \hookrightarrow (\mathcal{M}^{n+1}, \mathfrak{g}).$$

- K is a symmetric $(0,2)$-tensor;

$$K = \text{the second fundamental form of } M^n \hookrightarrow (\mathcal{M}^{n+1}, \mathfrak{g}).$$

- E is a vector field on M; in the purely electric case,

$$F = c_n T^\flat \wedge E^\flat, \qquad c_n = \sqrt{\frac{(n-1)(n-2)}{2}},$$

where T is the future-directed unit timelike normal to M.

Associated with a charged initial data set (M^n, g, K, E), we define

- the energy density: $\mu := \frac{1}{2}\left(R - |K|^2 + (\mathrm{tr}K)^2 - (n-1)(n-2)|E|^2\right)$;

Associated with a charged initial data set (M^n, g, K, E), we define

- the energy density: $\mu := \frac{1}{2}\left(R - |K|^2 + (\mathrm{tr}K)^2 - (n-1)(n-2)|E|^2\right)$;
- the current density: $J := \mathrm{div}(K) - d\mathrm{tr}(K)$;

Associated with a charged initial data set (M^n, g, K, E), we define

- the energy density: $\mu := \frac{1}{2}\left(R - |K|^2 + (\mathrm{tr}K)^2 - (n-1)(n-2)|E|^2\right)$;
- the current density: $J := \mathrm{div}(K) - d\mathrm{tr}(K)$;
- the charge density: $\varpi := (n-1)\mathrm{div}(E)$.

Associated with a charged initial data set (M^n, g, K, E), we define

- the energy density: $\mu := \frac{1}{2}\left(R - |K|^2 + (\mathrm{tr}K)^2 - (n-1)(n-2)|E|^2\right)$;
- the current density: $J := \mathrm{div}(K) - d\mathrm{tr}(K)$;
- the charge density: $\varpi := (n-1)\mathrm{div}(E)$.

Dominant energy condition (DEC)

A charged initial data set satisfies the **DEC** if $\mu \geq \sqrt{|J|^2 + \varpi^2}$.

Associated with a charged initial data set (M^n, g, K, E), we define

- the energy density: $\mu := \frac{1}{2}\left(R - |K|^2 + (\mathrm{tr}K)^2 - (n-1)(n-2)|E|^2\right)$;
- the current density: $J := \mathrm{div}(K) - d\mathrm{tr}(K)$;
- the charge density: $\varpi := (n-1)\mathrm{div}(E)$.

Dominant energy condition (DEC)

A charged initial data set satisfies the **DEC** if $\mu \geq \sqrt{|J|^2 + \varpi^2}$.

In the Einstein–Maxwell case, the Einstein–Maxwell constraint equations imply

$$\mu = 0, \qquad J = 0, \qquad \varpi = 0,$$

so that the DEC is saturated.

Mathematical framework

Asymptotically flat ends

To model isolated gravitating systems, one requires that the gravitational and electromagnetic interactions vanish at infinity.

To model isolated gravitating systems, one requires that the gravitational and electromagnetic interactions vanish at infinity.
A subset $M_{\rm ext} \subset M$ is said to be an asymptotically flat end (**AF end**) if there exists a chart at infinity

$$\psi^{-1} : M_{\rm ext} \longrightarrow \mathbb{R}^n \setminus \overline{B}_R(0)$$

such that, setting $e := \psi^* g - \delta$, one has

$$e = O_2(r^{-\tau}), \qquad \psi^* K = O_1(r^{-\tau-1}), \qquad \psi^* E = O_1(r^{-\tau-1}),$$

for some

$$\tau > \frac{n-2}{2}.$$

Here the decay is measured with respect to the Euclidean metric δ and its Levi–Civita connection ∇^δ.

To model isolated gravitating systems, one requires that the gravitational and electromagnetic interactions vanish at infinity.
A subset $M_{\rm ext}\subset M$ is said to be an asymptotically flat end (**AF end**) if there exists a chart at infinity

$$\psi^{-1}:M_{\rm ext}\longrightarrow\mathbb{R}^n\setminus\overline{B}_R(0)$$

such that, setting $e:=\psi^*g-\delta$, one has

$$e=O_2(r^{-\tau}),\qquad \psi^*K=O_1(r^{-\tau-1}),\qquad \psi^*E=O_1(r^{-\tau-1}),$$

for some

$$\tau>\frac{n-2}{2}.$$

Here the decay is measured with respect to the Euclidean metric δ and its Levi–Civita connection ∇^δ. We also assume

$$\mu,\ J,\ \varpi\in L^1(M).$$

For an asymptotically flat charged initial data set, one defines:

- the ADM energy (or ADM mass):

$$m = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \big(\mathrm{div}_\delta e - d\mathrm{tr}_\delta e\big)(\nu_r)\, d\sigma_r;$$

For an asymptotically flat charged initial data set, one defines:

- the ADM energy (or ADM mass):

$$m = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \big(\mathrm{div}_\delta e - d\mathrm{tr}_\delta e\big)(\nu_r)\, d\sigma_r;$$

- the ADM linear momentum: for $i = 1, \cdots, n$,

$$\mathcal{P}_i = \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \big(\psi^* K - (\mathrm{tr}_\delta \psi^* K)\delta\big)(\partial_i, \nu_r)\, d\sigma_r;$$

For an asymptotically flat charged initial data set, one defines:

- the ADM energy (or ADM mass):

$$m = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \big(\mathrm{div}_\delta e - d\mathrm{tr}_\delta e\big)(\nu_r)\, d\sigma_r;$$

- the ADM linear momentum: for $i = 1, \cdots, n$,

$$\mathcal{P}_i = \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \big(\psi^* K - (\mathrm{tr}_\delta \psi^* K)\delta\big)(\partial_i, \nu_r)\, d\sigma_r;$$

- the total charge:

$$Q = \frac{1}{\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \delta\big(\psi^* E, \nu_r\big)\, d\sigma_r.$$

Mathematical framework

ADM energy-momentum and total charge

For an asymptotically flat charged initial data set, one defines:

- the ADM energy (or ADM mass):

$$m = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \big(\mathrm{div}_\delta e - d\mathrm{tr}_\delta e\big)(\nu_r)\, d\sigma_r;$$

- the ADM linear momentum: for $i = 1, \cdots, n$,

$$\mathcal{P}_i = \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \big(\psi^* K - (\mathrm{tr}_\delta \psi^* K)\delta\big)(\partial_i, \nu_r)\, d\sigma_r;$$

- the total charge:

$$Q = \frac{1}{\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \delta\big(\psi^* E, \nu_r\big)\, d\sigma_r.$$

Asymptotic flatness $\Longrightarrow$ these quantities are geometric invariants (Bartnik ('86), Chruściel ('86); see also Michel ('10) for a general geometric formulation).

The positive energy theorem

Positive energy theorem

Let (M^n, g, K) be a **complete** initial data set containing an **AF end** and satisfying the **DEC**. Then its ADM energy-momentum vector $(m, \mathcal{P}) \in \mathbb{R}^{n,1}$ satisfies

$$m \geq |\mathcal{P}|.$$

Moreover, equality holds if and only if (M^n, g, K) can be isometrically embedded into a pp-wave spacetime with second fundamental form K.

The positive energy theorem

Positive energy theorem

Let (M^n, g, K) be a **complete** initial data set containing an **AF end** and satisfying the **DEC**. Then its ADM energy-momentum vector $(m, \mathcal{P}) \in \mathbb{R}^{n,1}$ satisfies

$$m \geq |\mathcal{P}|.$$

Moreover, equality holds if and only if (M^n, g, K) can be isometrically embedded into a pp-wave spacetime with second fundamental form K.

Two main approaches:

- Minimal hypersurfaces + Jang equation: Schoen–Yau ('79), Lohkamp ('06), Eichmair ('13), Huang–Lee ('19), Lesourd–Unger–Yau ('24), Brendle–Wang ('26);

The positive energy theorem

Positive energy theorem

Let (M^n, g, K) be a **complete** initial data set containing an **AF end** and satisfying the **DEC**. Then its ADM energy-momentum vector $(m, \mathcal{P}) \in \mathbb{R}^{n,1}$ satisfies

$$m \geq |\mathcal{P}|.$$

Moreover, equality holds if and only if (M^n, g, K) can be isometrically embedded into a pp-wave spacetime with second fundamental form K.

Two main approaches:

- Minimal hypersurfaces + Jang equation: Schoen–Yau ('79), Lohkamp ('06), Eichmair ('13), Huang–Lee ('19), Lesourd–Unger–Yau ('24), Brendle–Wang ('26);
- Dirac operators: Witten ('81), Parker–Taubes ('82), Bartnik ('86), Chruściel–Maerten ('06), Hirsch–Zhang ('25).

The positive energy theorem with charge for $n = 3$

Positive energy theorem with charge

Let (M^3, g, K, E) be a **complete** charged initial data set containing an **AF end** and satisfying the **DEC**. Then its ADM energy-momentum vector $(m, \mathcal{P}) \in \mathbb{R}^{3,1}$ and total charge Q satisfy

$$m \geq \sqrt{|\mathcal{P}|^2 + Q^2}.$$

Moreover, if equality holds, $\mathrm{tr} K = 0$ and if some additional natural assumptions are satisfied, then (M^3, g, K, E) can be isometrically embedded into a Majumdar–Papapetrou spacetime.

The positive energy theorem with charge for $n = 3$

Positive energy theorem with charge

Let (M^3, g, K, E) be a **complete** charged initial data set containing an **AF end** and satisfying the **DEC**. Then its ADM energy-momentum vector $(m, \mathcal{P}) \in \mathbb{R}^{3,1}$ and total charge Q satisfy

$$m \geq \sqrt{|\mathcal{P}|^2 + Q^2}.$$

Moreover, if equality holds, $\mathrm{tr} K = 0$ and if some additional natural assumptions are satisfied, then (M^3, g, K, E) can be isometrically embedded into a Majumdar–Papapetrou spacetime.

- Witten's method: Gibbons–Hull ('82), Gibbons–Hawking–Horowitz–Perry ('83), Bartnik–Chruściel ('05), Chruściel–Reall–Tod ('06);

The positive energy theorem with charge for $n = 3$

Positive energy theorem with charge

Let (M^3, g, K, E) be a **complete** charged initial data set containing an **AF end** and satisfying the **DEC**. Then its ADM energy-momentum vector $(m, \mathcal{P}) \in \mathbb{R}^{3,1}$ and total charge Q satisfy

$$m \geq \sqrt{|\mathcal{P}|^2 + Q^2}.$$

Moreover, if equality holds, $\mathrm{tr}K = 0$ and if some additional natural assumptions are satisfied, then (M^3, g, K, E) can be isometrically embedded into a Majumdar–Papapetrou spacetime.

- Witten's method: Gibbons–Hull ('82), Gibbons–Hawking–Horowitz–Perry ('83), Bartnik–Chruściel ('05), Chruściel–Reall–Tod ('06);
- Spacetime harmonic functions: Bray–Hirsch–Kazaras–Khuri–Zhang ('23);

But what can be said in higher dimensions?

Positive Energy Theorems with charge for spin initial data

But what can be said in higher dimensions?

Theorem (R.'26, CQG)

Let (M^n, g, K, E) be a **complete spin** charged initial data set containing an **AF end** and satisfying the **DEC**. Then its ADM energy-momentum vector $(m, \mathcal{P}) \in \mathbb{R}^{n,1}$ and total charge Q satisfy

$$m \geq \sqrt{|\mathcal{P}|^2 + Q^2}.$$

Positive Energy Theorems with charge for spin initial data

But what can be said in higher dimensions?

Theorem (R.'26, CQG)

Let (M^n, g, K, E) be a **complete spin** charged initial data set containing an **AF end** and satisfying the **DEC**. Then its ADM energy-momentum vector $(m, \mathcal{P}) \in \mathbb{R}^{n,1}$ and total charge Q satisfy

$$m \geq \sqrt{|\mathcal{P}|^2 + Q^2}.$$

Proof strategy: **Witten's spinorial method**.

We restrict to the time-symmetric case $K = 0$ (so $J = 0$ and $\mathcal{P} = 0$) that is

$$\begin{cases} (M^n, g, E) \text{ complete spin with an AF end} \\ \mu \geq |\varpi| \end{cases} \Longrightarrow \quad m \geq |Q|.$$

We restrict to the time-symmetric case $K = 0$ (so $J = 0$ and $\mathcal{P} = 0$) that is

$$\begin{cases} (M^n, g, E) \text{ complete spin with an AF end} \\ \mu \geq |\varpi| \end{cases} \Longrightarrow \quad m \geq |Q|.$$

Since M is spin (topological assumption), we have:

- the spinor bundle $S_g \longrightarrow M$;

We restrict to the time-symmetric case $K = 0$ (so $J = 0$ and $\mathcal{P} = 0$) that is

$$\begin{cases} (M^n, g, E) \text{ complete spin with an AF end} \\ \mu \geq |\varpi| \end{cases} \implies m \geq |Q|.$$

Since M is spin (topological assumption), we have:

- the spinor bundle $S_g \longrightarrow M$;
- the Clifford multiplication $c : TM \otimes S_g \longrightarrow S_g$;

We restrict to the time-symmetric case $K = 0$ (so $J = 0$ and $\mathcal{P} = 0$) that is

$$\left\{ \begin{array}{c} (M^n, g, E) \text{ complete spin with an AF end} \\ \mu \geq |\varpi| \end{array} \right. \quad \Longrightarrow \quad m \geq |Q|.$$

Since M is spin (topological assumption), we have:

- the spinor bundle $S_g \longrightarrow M$;
- the Clifford multiplication $c : TM \otimes S_g \longrightarrow S_g$;
- a natural Hermitian product $\langle \cdot, \cdot \rangle$ on S_g;

We restrict to the time-symmetric case $K = 0$ (so $J = 0$ and $\mathcal{P} = 0$) that is

$$\left\{ \begin{array}{c} (M^n, g, E) \text{ complete spin with an AF end} \\ \mu \geq |\varpi| \end{array} \right. \quad \Longrightarrow \quad m \geq |Q|.$$

Since M is spin (topological assumption), we have:

- the spinor bundle $S_g \longrightarrow M$;
- the Clifford multiplication $c : TM \otimes S_g \longrightarrow S_g$;
- a natural Hermitian product $\langle \cdot, \cdot \rangle$ on S_g;
- the spin Levi–Civita connection $\nabla : \Gamma(S_g) \longrightarrow \Gamma(T^*M \otimes S_g)$;

Witten's method

Spin geometry

We restrict to the time-symmetric case $K = 0$ (so $J = 0$ and $\mathcal{P} = 0$) that is

$$\begin{cases} (M^n, g, E) \text{ complete spin with an AF end} \\ \mu \geq |\varpi| \end{cases} \Longrightarrow \quad m \geq |Q|.$$

Since M is spin (topological assumption), we have:

- the spinor bundle $S_g \longrightarrow M$;
- the Clifford multiplication $c : TM \otimes S_g \longrightarrow S_g$;
- a natural Hermitian product $\langle \cdot, \cdot \rangle$ on S_g;
- the spin Levi–Civita connection $\nabla : \Gamma(S_g) \longrightarrow \Gamma(T^*M \otimes S_g)$;
- the associated Dirac operator $D = c \circ \nabla$.

Without loss of generality, we may assume $Q \leq 0$.

Without loss of generality, we may assume $Q \leq 0$. Define the modified connection on S_g by

$$\nabla^-_X \varphi := \nabla_X \varphi + \frac{1}{2} c(X) c(E) \varphi + \frac{n-1}{2} g(E, X) \varphi.$$

Witten's method

Charged Schrödinger–Lichnerowicz formula

Without loss of generality, we may assume $Q \leq 0$. Define the modified connection on S_g by

$$\nabla^{-}_{X}\varphi := \nabla_X\varphi + \frac{1}{2}c(X)c(E)\varphi + \frac{n-1}{2}g(E,X)\varphi.$$

The associated modified Dirac operator is

$$D^{-} := \sum_{j=1}^{n} c(e_j)\nabla^{-}_{e_j} = D - \frac{1}{2}c(E).$$

Without loss of generality, we may assume $Q \leq 0$. Define the modified connection on S_g by

$$\nabla^-_X\varphi := \nabla_X\varphi + \frac{1}{2}c(X)c(E)\varphi + \frac{n-1}{2}g(E,X)\varphi.$$

The associated modified Dirac operator is

$$D^- := \sum_{j=1}^{n} c(e_j)\nabla^-_{e_j} = D - \frac{1}{2}c(E).$$

Charged Schrödinger–Lichnerowicz formula

$$(D^-)^*D^- = (\nabla^-)^*\nabla^- + \frac{1}{2}(\mu + \varpi)\mathrm{Id}_{S_g}.$$

Let Φ_0 be a spinor which is constant on the asymptotically flat end

$$\nabla^{\delta}\Phi_0 = 0, \qquad |\Phi_0| = 1.$$

Let Φ_0 be a spinor which is constant on the asymptotically flat end

$$\nabla^\delta \Phi_0 = 0, \qquad |\Phi_0| = 1.$$

Using the analytic framework developed by Bartnik–Chruściel ('05), the operator

$$D^- : \mathbb{H}^- \longrightarrow L^2(S_g)$$

is an isomorphism. Hence there exists a unique spinor Φ such that

$$D^-\Phi = 0, \qquad \Phi - \Phi_0 \in \mathbb{H}^-.$$

Let Φ_0 be a spinor which is constant on the asymptotically flat end

$$\nabla^\delta \Phi_0 = 0, \qquad |\Phi_0| = 1.$$

Using the analytic framework developed by Bartnik–Chruściel ('05), the operator

$$D^- : \mathbb{H}^- \longrightarrow L^2(S_g)$$

is an isomorphism. Hence there exists a unique spinor Φ such that

$$D^- \Phi = 0, \qquad \Phi - \Phi_0 \in \mathbb{H}^-.$$

Integrating the charged Schrödinger–Lichnerowicz formula gives

$$\frac{n-1}{2}\omega_{n-1}(m - |Q|) = \int_M \Big(|\nabla^- \Phi|^2 + \frac{1}{2}(\mu + \varpi)|\Phi|^2 - \underbrace{|D^- \Phi|^2}_{=0}\Big)\, d\mu.$$

Let Φ_0 be a spinor which is constant on the asymptotically flat end

$$\nabla^{\delta}\Phi_0 = 0, \qquad |\Phi_0| = 1.$$

Using the analytic framework developed by Bartnik–Chruściel ('05), the operator

$$D^{-} : \mathbb{H}^{-} \longrightarrow L^2(S_g)$$

is an isomorphism. Hence there exists a unique spinor Φ such that

$$D^{-}\Phi = 0, \qquad \Phi - \Phi_0 \in \mathbb{H}^{-}.$$

Integrating the charged Schrödinger–Lichnerowicz formula gives

$$\frac{n-1}{2}\omega_{n-1}(m - |Q|) = \int_M \Big(\underbrace{|\nabla^{-}\Phi|^2}_{\geq 0} + \frac{1}{2}(\mu + \varpi)|\Phi|^2 - \underbrace{|D^{-}\Phi|^2}_{=0} \Big)\, d\mu.$$

Let Φ_0 be a spinor which is constant on the asymptotically flat end

$$\nabla^\delta \Phi_0 = 0, \qquad |\Phi_0| = 1.$$

Using the analytic framework developed by Bartnik–Chruściel ('05), the operator

$$D^- : \mathbb{H}^- \longrightarrow L^2(S_g)$$

is an isomorphism. Hence there exists a unique spinor Φ such that

$$D^-\Phi = 0, \qquad \Phi - \Phi_0 \in \mathbb{H}^-.$$

Integrating the charged Schrödinger–Lichnerowicz formula gives

$$\frac{n-1}{2}\omega_{n-1}(m - |Q|) = \int_M \Big(\underbrace{|\nabla^-\Phi|^2}_{\geq 0} + \frac{1}{2} \underbrace{(\mu + \varpi)|\Phi|^2}_{\geq 0 \text{ by the DEC}} - \underbrace{|D^-\Phi|^2}_{=0} \Big)\, d\mu.$$

Witten's method

The Witten argument

Let Φ_0 be a spinor which is constant on the asymptotically flat end

$$\nabla^\delta \Phi_0 = 0, \qquad |\Phi_0| = 1.$$

Using the analytic framework developed by Bartnik–Chruściel ('05), the operator

$$D^- : \mathbb{H}^- \longrightarrow L^2(S_g)$$

is an isomorphism. Hence there exists a unique spinor Φ such that

$$D^-\Phi = 0, \qquad \Phi - \Phi_0 \in \mathbb{H}^-.$$

Integrating the charged Schrödinger–Lichnerowicz formula gives

$$\frac{n-1}{2}\omega_{n-1}(m - |Q|) = \int_M \Big(\underbrace{|\nabla^-\Phi|^2}_{\geq 0} + \frac{1}{2} \underbrace{(\mu + \varpi)|\Phi|^2}_{\geq 0 \text{ by the DEC}} - \underbrace{|D^-\Phi|^2}_{=0} \Big)\, d\mu \;\geq\; 0.$$

Charged parallel spinors

Definition

The rigidity statement is governed by special spinors.

The rigidity statement is governed by special spinors.

Definition

Let (M^n, g, E) be a charged spin initial data set. A spinor field $\Phi \in \Gamma(S_g)$ satisfying $\nabla^-_X \Phi = 0$ that is

$$\nabla_X \Phi = -\frac{1}{2} c(X) c(E) \Phi - \frac{n-1}{2} g(E, X) \Phi$$

for all $X \in \Gamma(TM)$ is called a (negative) charged parallel spinor (**CPS**).

Charged parallel spinors
Definition

The rigidity statement is governed by special spinors.

Definition

Let (M^n, g, E) be a charged spin initial data set. A spinor field $\Phi \in \Gamma(S_g)$ satisfying $\nabla^-_X\Phi = 0$ that is

$$\nabla_X\Phi = -\frac{1}{2}c(X)c(E)\Phi - \frac{n-1}{2}g(E,X)\Phi$$

for all $X \in \Gamma(TM)$ is called a (negative) charged parallel spinor (**CPS**).

Example

If $E = 0$, CPS are parallel spinors. In particular, (M^n, g) is Ricci-flat.

Let Φ be a CPS. Set $V := |\Phi|^2$ and assume that $\mathrm{div}(E) = 0$.

Charged parallel spinors
Properties

Let Φ be a CPS. Set $V := |\Phi|^2$ and assume that $\mathrm{div}(E) = 0$. Then

- Φ has no zeros and
$$E^\flat = -\frac{1}{n-2} d \ln V;$$

Let Φ be a CPS. Set $V := |\Phi|^2$ and assume that $\mathrm{div}(E) = 0$. Then

- Φ has no zeros and
$$E^\flat = -\frac{1}{n-2} d \ln V;$$
- the Lorentzian metric $\mathfrak{g} = -V^2 dt^2 + g$ defines a static solution of the Einstein–Maxwell equations with Faraday tensor $F = c_n \, dt \wedge E^\flat$.

Charged parallel spinors

Properties

Let Φ be a CPS. Set $V := |\Phi|^2$ and assume that $\mathrm{div}(E) = 0$. Then

- Φ has no zeros and
$$E^\flat = -\frac{1}{n-2} d \ln V;$$
- the Lorentzian metric $\mathfrak{g} = -V^2 dt^2 + g$ defines a static solution of the Einstein–Maxwell equations with Faraday tensor $F = c_n \, dt \wedge E^\flat$.

(M^n, g, Φ)
Φ CPS
$\Delta(\ln V) = 0$

Charged parallel spinors

Properties

Let Φ be a CPS. Set $V := |\Phi|^2$ and assume that $\mathrm{div}(E) = 0$. Then

- Φ has no zeros and
$$E^\flat = -\frac{1}{n-2} d \ln V;$$
- the Lorentzian metric $\mathfrak{g} = -V^2 dt^2 + g$ defines a static solution of the Einstein–Maxwell equations with Faraday tensor $F = c_n \, dt \wedge E^\flat$.

Conformal correspondence

(M^n, g, Φ)	$\overline{g} = V^{\frac{2}{n-2}} g$	$(M^n, \overline{g}, \overline{\Psi})$
Φ CPS	$\leftrightsquigarrow$	$\overline{\Psi}$ $\overline{g}$-parallel
$\Delta(\ln V) = 0$	$\Psi = V^{-1/2}\Phi$	$\overline{\Delta}(V^{-1}) = 0$

Charged parallel spinors

A general construction

Start with a manifold $(M^n, \overline{g})$ carrying a parallel spinor $\overline{\Psi}$ and let $U > 0$ be a $\overline{g}$-harmonic function.

Start with a manifold $(M^n, \overline{g})$ carrying a parallel spinor $\overline{\Psi}$ and let $U > 0$ be a $\overline{g}$-harmonic function. Define

$$g = U^{\frac{2}{n-2}}\overline{g}, \qquad E^\flat = \frac{1}{n-2} d \ln U.$$

Then (M^n, g, E) carries a charged parallel spinor.

Start with a manifold $(M^n, \overline{g})$ carrying a parallel spinor $\overline{\Psi}$ and let $U > 0$ be a $\overline{g}$-harmonic function. Define

$$g = U^{\frac{2}{n-2}}\overline{g}, \qquad E^\flat = \frac{1}{n-2} d \ln U.$$

Then (M^n, g, E) carries a charged parallel spinor.

What about parallel spinors?

Charged parallel spinors

A general construction

Start with a manifold $(M^n, \overline{g})$ carrying a parallel spinor $\overline{\Psi}$ and let $U > 0$ be a $\overline{g}$-harmonic function. Define

$$g = U^{\frac{2}{n-2}}\overline{g}, \qquad E^\flat = \frac{1}{n-2} d \ln U.$$

Then (M^n, g, E) carries a charged parallel spinor.

Cone construction (Bär '93)

If (Σ^{n-1}, γ) carries a real Killing spinor

$$\nabla^\gamma_X \psi = -\frac{\lambda}{n-1} c_\gamma(X)\psi, \qquad X \in \Gamma(T\Sigma), \qquad \lambda \in \mathbb{R}^*$$

then the Riemannian cone

$$(C(\Sigma), g_C) = \big((0,\infty) \times \Sigma,\ dr^2 + r^2\gamma\big)$$

has a parallel spinor.

For $(\Sigma^{n-1}, \gamma) = (\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$, we get $g_C = \delta$.

For $(\Sigma^{n-1}, \gamma) = (\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$, we get $g_C = \delta$. Choose

$$U_m(r) = 1 + \frac{m}{r^{n-2}}, \qquad m > 0,$$

then the metric $g_m = U_m^{\frac{2}{n-2}} \delta$ on $\mathbb{R}^n \setminus \{0\}$ defines the extremal Reissner–Nordström manifold.

Charged parallel spinors

The extremal Reissner–Nordström manifold

For $(\Sigma^{n-1}, \gamma) = (\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$, we get $g_C = \delta$. Choose

$$U_m(r) = 1 + \frac{m}{r^{n-2}}, \qquad m > 0,$$

then the metric $g_m = U_m^{\frac{2}{n-2}}\delta$ on $\mathbb{R}^n \setminus \{0\}$ defines the extremal Reissner–Nordström manifold. It has

- one AF end with mass m and charge $Q = -m$ that is $m = |Q|$;

For $(\Sigma^{n-1},\gamma)=(\mathbb{S}^{n-1},g_{\mathbb{S}^{n-1}})$, we get $g_C=\delta$. Choose

$$U_m(r)=1+\frac{m}{r^{n-2}},\qquad m>0,$$

then the metric $g_m=U_m^{\frac{2}{n-2}}\delta$ on $\mathbb{R}^n\setminus\{0\}$ defines the extremal Reissner–Nordström manifold. It has

- one AF end with mass m and charge $Q=-m$ that is $m=|Q|$;
- one asymptotically cylindrical end (**AC end**) :

$$g_m \underset{t\to\infty}{\sim} dt^2+h,\qquad h=m^{\frac{2}{n-2}}g_{\mathbb{S}^{n-1}}\qquad t=-m^{\frac{1}{n-2}}\ln r;$$

For $(\Sigma^{n-1},\gamma)=(\mathbb{S}^{n-1},g_{\mathbb{S}^{n-1}})$, we get $g_C=\delta$. Choose

$$U_m(r)=1+\frac{m}{r^{n-2}},\qquad m>0,$$

then the metric $g_m=U_m^{\frac{2}{n-2}}\delta$ on $\mathbb{R}^n\setminus\{0\}$ defines the extremal Reissner–Nordström manifold. It has

- one AF end with mass m and charge $Q=-m$ that is $m=|Q|$;
- one asymptotically cylindrical end (**AC end**) :

$$g_m \underset{t\to\infty}{\sim} dt^2+h,\qquad h=m^{\frac{2}{n-2}}g_{\mathbb{S}^{n-1}}\qquad t=-m^{\frac{1}{n-2}}\ln r;$$

- the CPS is given by $\Phi=U_m^{-\frac{1}{2}}\Psi$ and

$$V_m:=|\Phi|^2=m^{-1}e^{-(n-2)qt}\big(1+O_2(e^{-(n-2)qt})\big),\qquad q=m^{-\frac{1}{n-2}}.$$

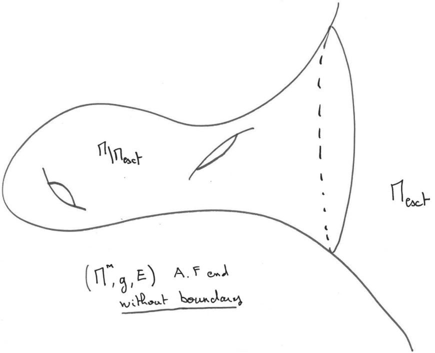
Charged parallel spinors

Non-trivial charged equality cases require additional geometric structure

Charged parallel spinors

Non-trivial charged equality cases require additional geometric structure

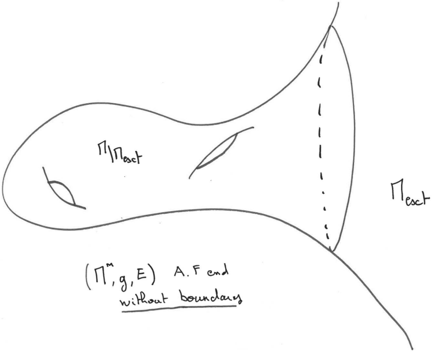
Assume (M^n, g, E) is complete, without boundary, $M \setminus M_{\rm ext}$ is compact, and $\mathrm{div}(E) = 0$.



Charged parallel spinors

Non-trivial charged equality cases require additional geometric structure

Assume (M^n, g, E) is complete, without boundary, $M \setminus M_{\mathrm{ext}}$ is compact, and $\mathrm{div}(E) = 0$.

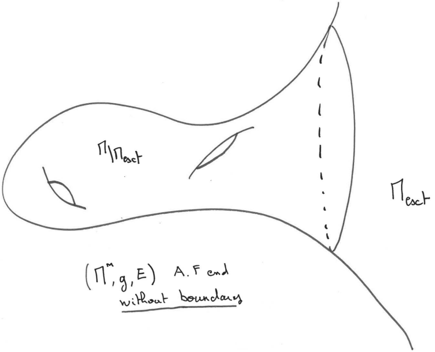


$$0 = \int_{M_r} \mathrm{div}(E)\, d\mu$$

Charged parallel spinors

Non-trivial charged equality cases require additional geometric structure

Assume (M^n, g, E) is complete, without boundary, $M \setminus M_{\mathrm{ext}}$ is compact, and $\mathrm{div}(E) = 0$.

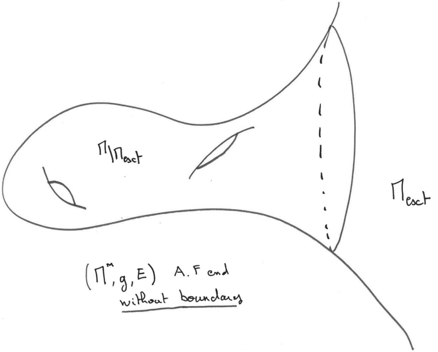


$$0 = \int_{M_r} \mathrm{div}(E)\, d\mu = \int_{S_r} g(E, \nu_r)\, d\sigma_r$$

Charged parallel spinors

Non-trivial charged equality cases require additional geometric structure

Assume (M^n, g, E) is complete, without boundary, $M \setminus M_{\text{ext}}$ is compact, and $\mathrm{div}(E) = 0$.

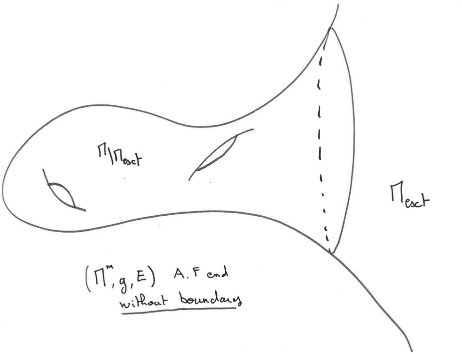


$$0 = \int_{M_r} \mathrm{div}(E)\, d\mu = \int_{S_r} g(E, \nu_r)\, d\sigma_r \underset{r \to +\infty}{\longrightarrow} \omega_{n-1} Q$$

Charged parallel spinors

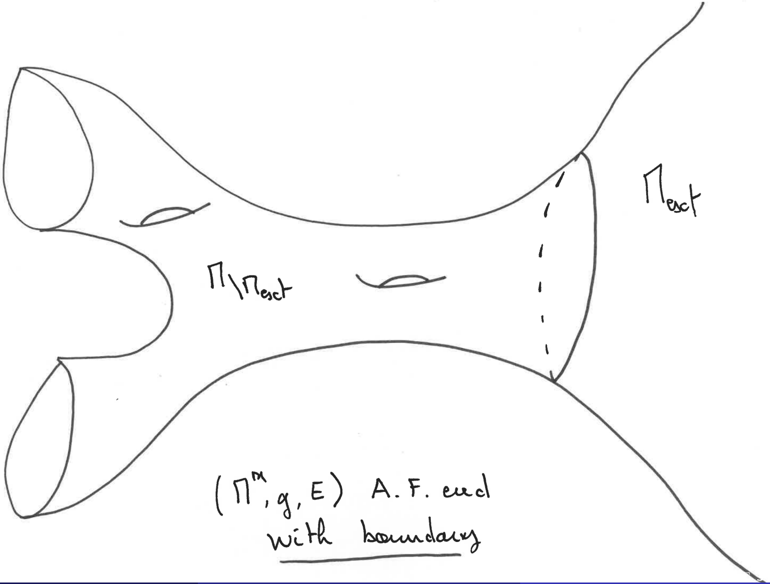
Non-trivial charged equality cases require additional geometric structure

Assume (M^n, g, E) is complete, without boundary, $M \setminus M_{\mathrm{ext}}$ is compact, and $\mathrm{div}(E) = 0$.



$$0 = \int_{M_r} \mathrm{div}(E)\, d\mu = \int_{S_r} g(E, \nu_r)\, d\sigma_r \underset{r \to +\infty}{\longrightarrow} \omega_{n-1} Q \implies m = |Q| = 0.$$

The mass-charge inequality for manifolds with boundary



The mass-charge inequality for manifolds with boundary

For $r_0 > 0$, the domain in the extremal Reissner-Nordström manifold

$$\Omega_{m,r_0} := \mathbb{R}^n \setminus B_{r_0}(0),$$

is a complete manifold with an AF end and a compact inner boundary.

The mass-charge inequality for manifolds with boundary

For $r_0 > 0$, the domain in the extremal Reissner-Nordström manifold

$$\Omega_{m,r_0} := \mathbb{R}^n \setminus B_{r_0}(0),$$

is a complete manifold with an AF end and a compact inner boundary. The restriction of the CPS Φ on $\partial\Omega_{m,r_0}$ satisfies

$$\not{D}\Phi_{|\partial\Omega_{m,r_0}} = -\frac{n-1}{2}(m + r_0^{n-2})^{-\frac{1}{n-2}}\Phi_{|\partial\Omega_{m,r_0}}$$

where $\not{D}$ is the Dirac operator of $(\partial\Omega_{m,r_0}, g_{m|\partial\Omega_{m,r_0}})$

The mass-charge inequality for manifolds with boundary

For $r_0 > 0$, the domain in the extremal Reissner-Nordström manifold

$$\Omega_{m,r_0} := \mathbb{R}^n \setminus B_{r_0}(0),$$

is a complete manifold with an AF end and a compact inner boundary. The restriction of the CPS Φ on $\partial\Omega_{m,r_0}$ satisfies

$$\not{D}\Phi_{|\partial\Omega_{m,r_0}} = -\frac{n-1}{2}(m + r_0^{n-2})^{-\frac{1}{n-2}}\Phi_{|\partial\Omega_{m,r_0}}$$

where $\not{D}$ is the Dirac operator of $(\partial\Omega_{m,r_0}, g_{m|\partial\Omega_{m,r_0}})$ that is

$$\begin{cases} D^-\Phi = 0 & \text{on } \Omega_{m,r_0} \\ P_{>0}\Phi = 0 & \text{on } \partial\Omega_{m,r_0} \rightsquigarrow \text{Atiyah–Patodi–Singer condition} \\ \Phi \to \Phi_0 & \text{at infinity.} \end{cases}$$

The mass-charge inequality for manifolds with boundary

Theorem (R.'26)

Let $n \geq 3$ and let (M^n, g, E) be a complete charged spin initial data set containing an AF end and with a **compact** boundary ∂M with positive scalar curvature. Assume that the DEC holds and that

$$\sup_{\partial M} \left(H + (n-1)|g(E,\nu)|\right) \leq \sqrt{\frac{n-1}{n-2} \inf_{\partial M} R_{\partial M}}.$$

Then the mass and charge satisfy

$$m \geq |Q|.$$

If ∂M is **connected** and $\mathrm{div}(E) = 0$, equality occurs if and only if (M^n, g, E) is isometric to the exterior region of a coordinate sphere in an extremal Reissner–Nordström manifold.

Integrating the charged Schrödinger–Lichnerowicz formula yields

$$\frac{n-1}{2}\omega_{n-1}(m-|Q|)=\int_M\left(|\nabla^-\Phi|^2+\frac{1}{2}(\mu+\varpi)|\Phi|^2\right)d\mu$$

$$-\int_{\partial M}\left\langle D\!\!\!/\Phi+\frac{1}{2}(H-(n-1)g(E,\nu))\Phi,\Phi\right\rangle d\sigma.$$

Integrating the charged Schrödinger–Lichnerowicz formula yields

$$\frac{n-1}{2}\omega_{n-1}(m-|Q|)\geq\int_M\left(|\nabla^-\Phi|^2+\frac{1}{2}(\mu+\varpi)|\Phi|^2\right)d\mu$$

$$+\int_{\partial M}\left(\lambda_1-\frac{1}{2}(H+(n-1)|g(E,\nu)|)\right)|\Phi|^2 d\sigma.$$

Integrating the charged Schrödinger–Lichnerowicz formula yields

$$\frac{n-1}{2}\omega_{n-1}(m-|Q|) \geq \int_M \left(|\nabla^- \Phi|^2 + \frac{1}{2}(\mu+\varpi)|\Phi|^2\right) d\mu$$

$$+ \int_{\partial M} \left(\lambda_1 - \frac{1}{2}(H+(n-1)|g(E,\nu)|)\right) |\Phi|^2 d\sigma.$$

By Friedrich's inequality,

$$\lambda_1 \geq \frac{1}{2}\sqrt{\frac{n-1}{n-2}\inf_{\partial M} R_{\partial M}}$$

Integrating the charged Schrödinger–Lichnerowicz formula yields

$$\frac{n-1}{2}\omega_{n-1}(m-|Q|) \geq \int_M \left(|\nabla^-\Phi|^2 + \frac{1}{2}(\mu+\varpi)|\Phi|^2\right) d\mu$$

$$+ \int_{\partial M} \left(\lambda_1 - \frac{1}{2}(H + (n-1)|g(E,\nu)|)\right) |\Phi|^2 d\sigma.$$

By Friedrich's inequality,

$$\lambda_1 \geq \frac{1}{2}\sqrt{\frac{n-1}{n-2}\inf_{\partial M} R_{\partial M}} \geq \frac{1}{2}\sup_{\partial M}\left(H + (n-1)|g(E,\nu)|\right),$$

hence $m \geq |Q|$.

Assume that equality holds, the boundary is **connected** and $\mathrm{div}(E) = 0$.

Assume that equality holds, the boundary is **connected** and $\mathrm{div}(E) = 0$. The spinor Φ is a CPS with $\Phi \to \Phi_0 \neq 0$, $|\Phi_0| = 1$ then

$$\ln V = \frac{Q}{r^{n-2}} + o_2(r^{2-n}).$$

Assume that equality holds, the boundary is **connected** and $\mathrm{div}(E) = 0$. The spinor Φ is a CPS with $\Phi \to \Phi_0 \neq 0$, $|\Phi_0| = 1$ then

$$V = 1 + \frac{Q}{r^{n-2}} + o_2(r^{2-n}).$$

Assume that equality holds, the boundary is **connected** and $\mathrm{div}(E) = 0$. The spinor Φ is a CPS with $\Phi \to \Phi_0 \neq 0$, $|\Phi_0| = 1$ then

$$V = 1 + \frac{Q}{r^{n-2}} + o_2(r^{2-n}).$$

Consider the conformal metric $\overline{g} = V^{\frac{2}{n-2}} g$ then $(M^n, \overline{g})$

- is Ricci-flat,

Assume that equality holds, the boundary is **connected** and $\mathrm{div}(E) = 0$. The spinor Φ is a CPS with $\Phi \to \Phi_0 \neq 0$, $|\Phi_0| = 1$ then

$$V = 1 + \frac{Q}{r^{n-2}} + o_2(r^{2-n}).$$

Consider the conformal metric $\overline{g} = V^{\frac{2}{n-2}} g$ then $(M^n, \overline{g})$

- is Ricci-flat,
- has an AF end,

The mass-charge inequality for manifolds with boundary

Sketch of the proof: the equality case I

Assume that equality holds, the boundary is **connected** and $\mathrm{div}(E) = 0$. The spinor Φ is a CPS with $\Phi \to \Phi_0 \neq 0$, $|\Phi_0| = 1$ then

$$V = 1 + \frac{Q}{r^{n-2}} + o_2(r^{2-n}).$$

Consider the conformal metric $\overline{g} = V^{\frac{2}{n-2}} g$ then $(M^n, \overline{g})$

- is Ricci-flat,
- has an AF end,
- with total mass $\overline{m} = m + Q = 0$ (and appropriate boundary behaviour).

Assume that equality holds, the boundary is **connected** and $\mathrm{div}(E) = 0$. The spinor Φ is a CPS with $\Phi \to \Phi_0 \neq 0$, $|\Phi_0| = 1$ then

$$V = 1 + \frac{Q}{r^{n-2}} + o_2(r^{2-n}).$$

Consider the conformal metric $\overline{g} = V^{\frac{2}{n-2}} g$ then $(M^n, \overline{g})$

- is Ricci-flat,
- has an AF end,
- with total mass $\overline{m} = m + Q = 0$ (and appropriate boundary behaviour).

By Herzlich's positive mass theorem with boundary,

$$(M^n, \overline{g}) \simeq (\mathbb{R}^n \setminus B_{r_0}(0), \delta).$$

Setting $U := V^{-1}$ we obtain

$$\begin{cases} \Delta_\delta U &= 0 \quad \text{on } \mathbb{R}^n \setminus B_{r_0}(0), \\ U &\to 1 \quad \text{at infinity}, \\ U &= \alpha \quad \text{on } \partial B_{r_0}(0). \end{cases}$$

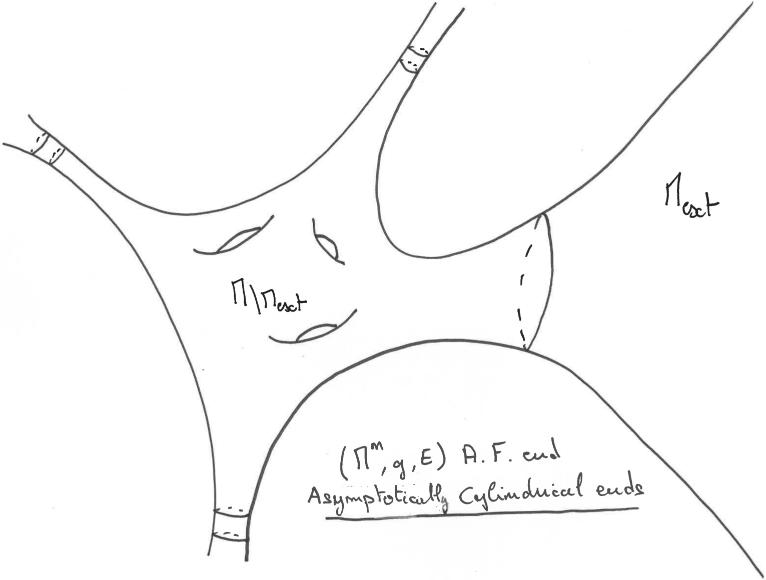
Setting $U := V^{-1}$ we obtain

$$\begin{cases} \Delta_\delta U &= 0 \quad \text{on } \mathbb{R}^n \setminus B_{r_0}(0), \\ U &\to 1 \quad \text{at infinity}, \\ U &= \alpha \quad \text{on } \partial B_{r_0}(0). \end{cases}$$

It follows that $U(r) = U_m(r) = 1 + \frac{m}{r^{n-2}}$ hence

$$g = U_m^{\frac{2}{n-2}}\delta = g_m,$$

so (M^n, g, E) is the exterior region of a sphere in an extremal Reissner –Nordström manifold.



M_{ext}
$M \setminus M_{ext}$
(M^n, g, E) A.F. end
Asymptotically Cylindrical ends

A subset $\mathcal{E} \subset M$ in a charged initial data set (M^n, g, E) is an asymptotically cylindrical end (**AC end**) if

$$\mathcal{E} \simeq (T_0, \infty) \times \Sigma, \qquad g_\infty = dt^2 + h, \qquad E_\infty = Y + q\partial_t$$

where (Σ^{n-1}, h), the limiting cross-section of the AC end, is a $(n-1)$-dimensional Riemannian closed manifold, $Y \in \Gamma(T\Sigma)$ and $q \in C^\infty(\Sigma)$

A subset $\mathcal{E} \subset M$ in a charged initial data set (M^n, g, E) is an asymptotically cylindrical end (**AC end**) if

$$\mathcal{E} \simeq (T_0, \infty) \times \Sigma, \qquad g_\infty = dt^2 + h, \qquad E_\infty = Y + q\partial_t$$

where (Σ^{n-1}, h), the limiting cross-section of the AC end, is a $(n-1)$-dimensional Riemannian closed manifold, $Y \in \Gamma(T\Sigma)$ and $q \in C^\infty(\Sigma)$ such that

$$|(\nabla^\infty)^j (g - g_\infty)|_{g_\infty} = O(e^{-\alpha t}), \qquad j = 0, 1, 2,$$

$$|(\nabla^\infty)^j (E - E_\infty)|_{g_\infty} = O(e^{-\alpha t}), \qquad j = 0, 1.$$

Here $\alpha > 0$, and ∇^∞ is the Levi–Civita connection of g_∞.

The mass-charge inequality for manifolds with AC ends

Theorem (R.'26)

Let $n \geq 3$ and let (M^n, g, E) be a complete charged spin manifold containing at least one AF end and a finite number of AC ends. Assume that the DEC holds. Then

$$m \geq |Q|.$$

If moreover $\mathrm{div}(E) = 0$ and M has exactly **one AC end** *whose limiting cross-section has* **positive scalar curvature**, *then equality holds if and only if (M^n, g, E) is isometric to the extremal Reissner–Nordström manifold.*

The mass-charge inequality for manifolds with AC ends

Theorem (R.'26)

Let $n \geq 3$ and let (M^n, g, E) be a complete charged spin manifold containing at least one AF end and a finite number of AC ends. Assume that the DEC holds. Then

$$m \geq |Q|.$$

If moreover $\mathrm{div}(E) = 0$ and M has exactly **one AC end** *whose limiting cross-section has* **positive scalar curvature***, then equality holds if and only if (M^n, g, E) is isometric to the extremal Reissner–Nordström manifold.*

Main idea of the proof: transform the cylindrical end into an isolated conical singularity and apply the positive mass theorem of Dai–Sun–Wang ('24).

The mass-charge inequality for manifolds with AC ends
Sketch of the proof I

The Witten spinor Φ is a charged parallel spinor and if $V := |\Phi|^2$, we have:

- along the AC end,

$$E^\flat = -\frac{1}{n-2} d \ln V = Y^\flat + q\, dt + O_1(e^{-\alpha t})$$

and then $q = -\omega_{n-1} Q/\mathrm{Vol}(\Sigma, h) \geq 0$ is a **nonnegative constant**;

The Witten spinor Φ is a charged parallel spinor and if $V := |\Phi|^2$, we have:

- along the AC end,

$$E^\flat = -\frac{1}{n-2} d \ln V = Y^\flat + q\, dt + O_1(e^{-\alpha t})$$

  and then $q = -\omega_{n-1} Q/\mathrm{Vol}(\Sigma, h) \geq 0$ is a **nonnegative constant**;
- then integrating yields

$$V(t,y) = e^{-(n-2)qt} v(y)\big(1 + O(e^{-\alpha t})\big), \qquad (t,y) \in (T_0, \infty) \times \Sigma.$$

The Witten spinor Φ is a charged parallel spinor and if $V := |\Phi|^2$, we have:

- along the AC end,

$$E^\flat = -\frac{1}{n-2} d \ln V = Y^\flat + q\, dt + O_1(e^{-\alpha t})$$

  and then $q = -\omega_{n-1} Q / \mathrm{Vol}(\Sigma, h) \geq 0$ is a **nonnegative constant**;

- then integrating yields

$$V(t,y) = e^{-(n-2)qt} v(y) \big(1 + O(e^{-\alpha t})\big), \qquad (t,y) \in (T_0, \infty) \times \Sigma.$$

  After renormalization,

$$\Psi_t = e^{\frac{n-2}{2}qt} \Phi_{|\Sigma_t} \longrightarrow \Psi_\infty \neq 0 \quad \text{in } C^1,$$

  and the limiting spinor satisfies

$$D\!\!\!/_h \Psi_\infty = -\frac{n-1}{2} q\, \Psi_\infty.$$

- Friedrich's inequality and the Gauss equation yield

$$\frac{(n-1)^2}{4}q^2 \geq \lambda_1^2 \geq \frac{n-1}{4(n-2)}\inf_{\Sigma} R_{\Sigma}$$

- Friedrich's inequality and the Gauss equation yield

$$\frac{(n-1)^2}{4}q^2 \geq \lambda_1^2 \geq \frac{n-1}{4(n-2)}\inf_{\Sigma} R_{\Sigma} = \frac{(n-1)^2}{4}\inf_{\Sigma}(|Y|_h^2 + q^2)$$

The mass-charge inequality for manifolds with AC ends

Sketch of the proof II

- Friedrich's inequality and the Gauss equation yield

$$\frac{(n-1)^2}{4}q^2 \geq \lambda_1^2 \geq \frac{n-1}{4(n-2)}\inf_{\Sigma} R_{\Sigma} = \frac{(n-1)^2}{4}\inf_{\Sigma}(|Y|_h^2 + q^2)$$

so Ψ_∞ is a real Killing spinor on (Σ^{n-1}, h) with $q > 0$. In particular, R_Σ is constant and so $Y \equiv 0$.

- Friedrich's inequality and the Gauss equation yield

$$\frac{(n-1)^2}{4}q^2 \geq \lambda_1^2 \geq \frac{n-1}{4(n-2)}\inf_\Sigma R_\Sigma = \frac{(n-1)^2}{4}\inf_\Sigma(|Y|_h^2 + q^2)$$

so Ψ_∞ is a real Killing spinor on (Σ^{n-1}, h) with $q > 0$. In particular, R_Σ is constant and so $Y \equiv 0$.

- Set $\overline{g} = V^{\frac{2}{n-2}} g$ with $V = |\Phi|^2$ then $(M, \overline{g})$ is Ricci-flat with an AF end with zero mass.

- Friedrich's inequality and the Gauss equation yield

$$\frac{(n-1)^2}{4}q^2 \geq \lambda_1^2 \geq \frac{n-1}{4(n-2)}\inf_{\Sigma} R_\Sigma = \frac{(n-1)^2}{4}\inf_{\Sigma}(|Y|_h^2 + q^2)$$

  so Ψ_∞ is a real Killing spinor on (Σ^{n-1}, h) with $q > 0$. In particular, R_Σ is constant and so $Y \equiv 0$.

- Set $\overline{g} = V^{\frac{2}{n-2}} g$ with $V = |\Phi|^2$ then $(M, \overline{g})$ is Ricci-flat with an AF end with zero mass.

What happens to the AC end after the conformal change?

The mass-charge inequality for manifolds with AC ends

Sketch of the proof II

- Friedrich's inequality and the Gauss equation yield

$$\frac{(n-1)^2}{4}q^2 \geq \lambda_1^2 \geq \frac{n-1}{4(n-2)}\inf_{\Sigma} R_{\Sigma} = \frac{(n-1)^2}{4}\inf_{\Sigma}(|Y|_h^2 + q^2)$$

  so Ψ_∞ is a real Killing spinor on (Σ^{n-1}, h) with $q > 0$. In particular, R_Σ is constant and so $Y \equiv 0$.

- Set $\overline{g} = V^{\frac{2}{n-2}}g$ with $V = |\Phi|^2$ then $(M, \overline{g})$ is Ricci-flat with an AF end with zero mass.

- Since $Y \equiv 0$ on the AC end:

$$V(t,y) = Ce^{-(n-2)qt}\big(1 + O_2(e^{-\alpha t})\big)$$

- Friedrich's inequality and the Gauss equation yield

$$\frac{(n-1)^2}{4}q^2 \geq \lambda_1^2 \geq \frac{n-1}{4(n-2)}\inf_{\Sigma} R_{\Sigma} = \frac{(n-1)^2}{4}\inf_{\Sigma}(|Y|_h^2 + q^2)$$

so Ψ_∞ is a real Killing spinor on (Σ^{n-1}, h) with $q > 0$. In particular, R_Σ is constant and so $Y \equiv 0$.

- Set $\overline{g} = V^{\frac{2}{n-2}} g$ with $V = |\Phi|^2$ then $(M, \overline{g})$ is Ricci-flat with an AF end with zero mass.
- Since $Y \equiv 0$ on the AC end:

$$V(t,y) = Ce^{-(n-2)qt}\big(1 + O_2(e^{-\alpha t})\big)$$

then, setting $\rho = C^{1/(n-2)} q^{-1} e^{-qt}$, one obtains

$$\overline{g} = d\rho^2 + \rho^2 \widetilde{h} + \kappa, \quad \widetilde{h} = q^2 h, \quad |(\nabla^C)^j \kappa|_{g_C} = O(\rho^{\beta - j}), \qquad j = 0, 1, 2,$$

that is, an isolated conical singularity.

Rigidity in the positive mass theorem with isolated conical singularities of Dai–Sun–Wang ('24) gives

$$(M, \overline{g}) \simeq (\mathbb{R}^n \setminus \{0\}, \delta).$$

Rigidity in the positive mass theorem with isolated conical singularities of Dai–Sun–Wang ('24) gives

$$(M, \overline{g}) \simeq (\mathbb{R}^n \setminus \{0\}, \delta).$$

Setting $U := V^{-1}$, we obtain

$$\Delta_\delta U = 0, \qquad U \to 1 \quad \text{at infinity},$$

and U has an isolated pole at 0.

Rigidity in the positive mass theorem with isolated conical singularities of Dai–Sun–Wang ('24) gives

$$(M, \overline{g}) \simeq (\mathbb{R}^n \setminus \{0\}, \delta).$$

Setting $U := V^{-1}$, we obtain

$$\Delta_\delta U = 0, \qquad U \to 1 \quad \text{at infinity},$$

and U has an isolated pole at 0. By Bôcher's theorem,

$$U(x) = U_m(x) = 1 + \frac{m}{|x|^{n-2}},$$

hence

$$g = U_m^{\frac{2}{n-2}}\delta = g_m \qquad \text{and} \qquad E^\flat = \frac{1}{n-2} d \ln U_m = E_m^\flat.$$

Therefore (M^n, g, E) is the extremal Reissner–Nordström manifold of mass m.

Perspectives

- Extend the rigidity statement to manifolds with multiple AC ends and recover the higher-dimensional Majumdar–Papapetrou manifolds

$$\left(\mathbb{R}^n \setminus \{p_1, \dots, p_N\},\ U^{\frac{2}{n-2}}\delta\right), \qquad U(x) = 1 + \sum_{j=1}^{N} \frac{m_j}{|x - p_j|^{n-2}},$$

by introducing suitable weighted spaces.

Perspectives

- Extend the rigidity statement to manifolds with multiple AC ends and recover the higher-dimensional Majumdar–Papapetrou manifolds

$$\left(\mathbb{R}^n \setminus \{p_1, \ldots, p_N\},\ U^{\frac{2}{n-2}}\delta\right), \qquad U(x) = 1 + \sum_{j=1}^{N} \frac{m_j}{|x - p_j|^{n-2}},$$

  by introducing suitable weighted spaces.
- Study the general (non time-symmetric) setting (M^n, g, K, E).

Perspectives

- Extend the rigidity statement to manifolds with multiple AC ends and recover the higher-dimensional Majumdar–Papapetrou manifolds

$$\left(\mathbb{R}^n \setminus \{p_1, \dots, p_N\},\ U^{\frac{2}{n-2}}\delta\right), \qquad U(x) = 1 + \sum_{j=1}^{N} \frac{m_j}{|x - p_j|^{n-2}},$$

  by introducing suitable weighted spaces.
- Study the general (non time-symmetric) setting (M^n, g, K, E).
- Include the magnetic field.

Perspectives

- Extend the rigidity statement to manifolds with multiple AC ends and recover the higher-dimensional Majumdar–Papapetrou manifolds

$$\left(\mathbb{R}^n \setminus \{p_1, \dots, p_N\},\ U^{\frac{2}{n-2}}\delta\right), \qquad U(x) = 1 + \sum_{j=1}^{N} \frac{m_j}{|x - p_j|^{n-2}},$$

  by introducing suitable weighted spaces.
- Study the general (non time-symmetric) setting (M^n, g, K, E).
- Include the magnetic field.
- Applications to uniqueness theorems for extremal black holes and photon spheres.

Perspectives

- Extend the rigidity statement to manifolds with multiple AC ends and recover the higher-dimensional Majumdar–Papapetrou manifolds

$$\left(\mathbb{R}^n \setminus \{p_1, \dots, p_N\},\ U^{\frac{2}{n-2}}\delta\right), \qquad U(x) = 1 + \sum_{j=1}^{N} \frac{m_j}{|x - p_j|^{n-2}},$$

  by introducing suitable weighted spaces.
- Study the general (non time-symmetric) setting (M^n, g, K, E).
- Include the magnetic field.
- Applications to uniqueness theorems for extremal black holes and photon spheres.
- Generalize the theory to the asymptotically hyperbolic setting (Preprint '26).

Thank you for your attention!